Partial coherence with application to the monotonicity problem of coherence involving skew information

Shunlong Luo* and Yuan Sun

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China

(Received 29 April 2017; published 28 August 2017)

Quantifications of coherence are intensively studied in the context of completely decoherent operations (i.e., von Neuamnn measurements, or equivalently, orthonormal bases) in recent years. Here we investigate partial coherence (i.e., coherence in the context of partially decoherent operations such as Lüders measurements). A bona fide measure of partial coherence is introduced. As an application, we address the monotonicity problem of K-coherence (a quantifier for coherence in terms of Wigner-Yanase skew information) [Girolami, Phys. Rev. Lett. **113**, 170401 (2014)], which is introduced to realize a measure of coherence as axiomatized by Baumgratz, Cramer, and Plenio [Phys. Rev. Lett. **113**, 140401 (2014)]. Since K-coherence fails to meet the necessary requirement of monotonicity under incoherent operations, it is desirable to remedy this monotonicity problem. We show that if we modify the original measure by taking skew information with respect to the spectral decomposition of an observable, rather than the observable itself, as a measure of coherence, then the problem disappears, and the resultant coherence measure satisfies the monotonicity. Some concrete examples are discussed and related open issues are indicated.

DOI: 10.1103/PhysRevA.96.022136

I. INTRODUCTION

In recent years, inspired by the seminal work of Baumgratz *et al.* [1], there is increasing interest in axiomatic and quantitative studies of coherence [2–12]. In a remarkable study, Girolami introduced K-coherence, an intuitive quantifier for coherence, based on the Wigner-Yanase skew information [2]. This quantifier satisfies all desirable properties of a coherence measure as postulated by Baumgratz *et al.* [2], with the exception of monotonicity under incoherent operations. However, monotonicity is a key feature for any reasonable coherence measure, and it is desirable to remedy this problem.

On reconsideration, the failure of monotonicity of K-coherence is just reasonable since the monotonicity axiom is formulated with respect to an orthonormal base (von Neumann measurement), while the K-coherence is based on an observable K, which is quite different from the associated orthonormal base due to the involvement of eigenvalues. Once this is recognized, the recipe to remedying the problem is immediate: One simply replaces the skew information based on an observable by that based on the corresponding spectral decomposition. Since spectral decompositions of observables in general yield Lüders measurements which can only extract partial information of coherence, we are led naturally to the notion of partial coherence and its quantifier in terms of summation of the skew information involving measurement operators. This is indeed a bona fide, as well as more general, measure of coherence, as will be established in this paper.

Before elaborating on the results, let us first recall the axiomatic approach to coherence by Baumgratz *et al.* [1], which is formulated as a resource theory with three basic ingredients:

(i) a fixed orthonormal base $\{|i\rangle\}$ (alternatively, a von Neumann measurement $\{|i\rangle\langle i|\}$), which serves as a reference;

(ii) incoherent states

$$\mathcal{I} = \left\{\sigma = \sum_i p_i|i\rangle\langle i| : p_i \geqslant 0, \sum_i p_i = 1\right\},$$

which are mathematically described by diagonal matrices in the fixed orthonormal base $\{|i\rangle\}$;

(iii) incoherent operations, which cannot create coherence from incoherent states. More specifically, a quantum operation Φ with Kraus operators $\{E_k\}$ is called incoherent according to Ref. [1] if $\frac{1}{\lambda_k}E_k\sigma E_k^\dagger \in \mathcal{I}$ for any $\sigma \in \mathcal{I}$ and any k, here $\lambda_k = \mathrm{tr}E_k\sigma E_k^\dagger$. We note that there are other, different, notions of incoherent operations [9], and we will not pursue them here.

A functional $C(\rho)$ on the space of quantum states is regarded as a coherence measure (with respect to the orthonormal base $\{|i\rangle\}$), if it satisfies the following requirements.

C1: $C(\rho) \geqslant 0$, and $C(\sigma) = 0$ for any incoherent state $\sigma \in \mathcal{I}$.

C2: $C(\rho)$ is convex in ρ.

C3: Monotonicity under incoherent operations: for any quantum state ρ and any incoherent operation Φ, it holds that $C(\Phi(\rho)) \leqslant C(\rho)$. For simplicity, we do not treat the strong monotonicity here.

Several measures of coherence, including the relative entropy of coherence, l^1-norm coherence, robustness of coherence, etc., are constructed and verified to satisfy the above requirements. In an informational approach to coherence [2], Girolami proposed to use the Wigner-Yanase skew information [13–15]

$$I(\rho,K) := -\mathrm{tr}\tfrac{1}{2}[\sqrt{\rho},K]^2$$

to quantify coherence, and called it K-coherence. Here K is diagonal in the base $\{|i\rangle\}$. More precisely, this quantity should be considered as a quantifier for coherence of ρ with respect to the observable K rather than the associated orthonormal base.

*luosl@amt.ac.cn

This quantifier satisfies C1 and C2, but not C3, as pointed out in Refs. [16,17].

A question arises immediately: How to reinstall the monotonicity, while still employing the skew information to quantify coherence? We present a solution here by showing that the K-coherence can be simply modified to a bona fide measure of coherence satisfying all the above requirements C1–C3 (including the monotonicity under incoherent operations). The key observation is to use the spectral decomposition of the observable K rather than the observable K itself.

II. PARTIAL COHERENCE

We will work in a more general framework. Suppose that the observable K has the spectral decomposition $K = \sum_i a_i \Pi_i$ with Π_i orthogonal projections (not necessarily one dimensional) satisfying $\sum_i \Pi_i = \mathbf{1}$, then it induces a corresponding Lüders measurement $\Pi = \{\Pi_i\}$ [18,19], which includes von Neumann measurements as special cases. In this context, the basic ingredients for partial coherence are

(i) A fixed Lüders measurement $\Pi = \{\Pi_i\}$, or equivalently, an orthogonal decomposition of the system Hilbert space $H = \bigoplus_i H_i$ with $H_i = \Pi_i H$, which serves as reference for coherence. Two extreme cases are (1) there is only a single component with $H_1 = H$ (corresponding to the trivial measurement $\Pi = \{I\}$), and (2) there are $d = \mathrm{dim}H$ components with $H_i = \{c|i\rangle : c \in \mathbb{C}\}$, $i = 1,2,\ldots,d$, where $\{|i\rangle\}$ constitutes an orthonormal base for H (corresponding to the von Neumann measurement $\Pi = \{\Pi_i\}$).

(ii) Incoherent states $\mathcal{I}_{\mathrm{L}} = \{\sigma : \Pi(\sigma) = \sigma\}$ consisting of all block-diagonal states σ (with respect to the previous space decomposition), here $\Pi(\sigma) = \sum_i \Pi_i \sigma \Pi_i$. We emphasize that coherence here is with respect to the Lüders measurement Π, and consequently, the incoherent states here may have coherence with respect to other measurements, which are refinements of Π. This means we are actually talking about partial coherence.

(iii) Incoherent operations Φ satisfying $\Phi(\sigma) \in \mathcal{I}_{\mathrm{L}}$ for any $\sigma \in \mathcal{I}_{\mathrm{L}}$.

Now in analogy to C1–C3, the corresponding reasonable requirements for a coherence measure $C(\rho|\Pi)$ can be reformulated as:

L1: $C(\rho|\Pi) \geqslant 0$, and $C(\sigma|\Pi) = 0$ if and only if $\sigma \in \mathcal{I}_{\mathrm{L}}$.

L2: $C(\rho|\Pi)$ is convex in ρ.

L3: Monotonicity under incoherent operations: for any quantum state ρ, and for any incoherent operation Φ, it holds that

$$C(\Phi(\rho)|\Pi) \leqslant C(\rho|\Pi).$$

In this setting, it is natural to define a coherence measure of a state ρ with respect to the Lüders measurement $\Pi = \{\Pi_i\}$ as

$$C(\rho|\Pi) := \sum_i I(\rho,\Pi_i),$$

which indeed satisfies the above three requirements L1–L3, as shown in the following.

Item L1 follows simply from $I(\rho,\Pi_i) = 0$ if and only if ρ and Π_i commute, i.e., $[\rho,\Pi_i] = 0$. As for item L2, the convexity of the skew information guarantees that $C(\rho|\Pi)$ is convex in ρ [13]. For item L3, we first note that the coherence measure can be equivalently expressed as

$$C(\rho|\Pi) = 1 - \mathrm{tr}\sqrt{\rho}\,\Pi(\sqrt{\rho}),$$

which follows from $\Pi_i^2 = \Pi_i$, $\sum_i \Pi_i = \mathbf{1}$. Now since

$$\Pi(\sqrt{\rho}) = \sum_i \Pi_i \sqrt{\rho}\,\Pi_i$$

is block diagonal, we have

$$[\Pi(\sqrt{\rho})]^2 = \sum_i (\Pi_i \sqrt{\rho}\,\Pi_i)^2 = a\gamma$$

where $a := \mathrm{tr}\sqrt{\rho}\,\Pi(\sqrt{\rho})$, and $\gamma := \frac{1}{a}\sum_i (\Pi_i\sqrt{\rho}\,\Pi_i)^2$ is a block-diagonal state (incoherent state). It follows that both $\Phi(\gamma)$ and $\sqrt{\Phi(\gamma)}$ are block-diagonal operators for any incoherent operation Φ. By the monotonicity of affinity and the Schwarz inequality, we have

$$\begin{aligned}
\mathrm{tr}\sqrt{\rho}\,\Pi(\sqrt{\rho}) &= \frac{1}{a}[\mathrm{tr}\sqrt{\rho}\,\Pi(\sqrt{\rho})]^2 \\
&= (\mathrm{tr}\sqrt{\rho}\sqrt{\gamma})^2 \\
&\leqslant (\mathrm{tr}\sqrt{\Phi(\rho)}\sqrt{\Phi(\gamma)})^2 \\
&= (\mathrm{tr}\sqrt{\Phi(\rho)}\,\Pi[\sqrt{\Phi(\gamma)}])^2 \\
&= (\mathrm{tr}\Pi[\sqrt{\Phi(\rho)}]\sqrt{\Phi(\gamma)})^2 \\
&\leqslant \mathrm{tr}(\Pi[\sqrt{\Phi(\rho)}])^2\mathrm{tr}\Phi(\gamma) \\
&= \mathrm{tr}\sqrt{\Phi(\rho)}\,\Pi[\sqrt{\Phi(\rho)}],
\end{aligned}$$

which implies the desired property L3.

We consider two special cases: (i) If the Lüders measurement Π is trivial in the sense that $\Pi = \{\mathbf{1}\}$ (identity operator), i.e., Π constitutes only the identity operator, then $C(\rho|\Pi) = 0$ for any state ρ. This is intuitive since here the measurement amounts to do nothing. (ii) If $\rho = |\psi\rangle\langle\psi|$ is a pure state with $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_i |\psi_i\rangle$, $|\psi_i\rangle = \sum_{j=1}^{d_i} |\psi_{ij}\rangle \in \Pi_i H = H_i$, $\mathrm{dim}H_i = d_i, d = \sum_i d_i$, then we have $I(\rho,\Pi_i) = \frac{d_i}{d} - (\frac{d_i}{d})^2$ and $C(\rho|\Pi) = \sum_i I(\rho,\Pi_i) = 1 - \frac{\sum_i d_i^2}{d^2}$. The maximal value $\max_\Pi C(\rho|\Pi) = 1 - \frac{1}{d}$ is achieved when $d_i = 1$ for all i, that is, when Π is a von Neumann measurement.

By the way, we remark that the discord $D_{\mathrm{H}}(\rho^{ab})$ introduced in Ref. [20] is precisely the minimal coherence:

$$D_{\mathrm{H}}(\rho^{ab}) = \min_{\Pi^a} C(\rho^{ab}|\Pi),$$

where ρ^{ab} is a bipartite state shared by parties a and b, and $\Pi = \{\Pi_i^a \otimes \mathbf{1}\}$ is the Lüders measurement extension of the local von Neumann measurement $\Pi^a = \{\Pi_i^a\}$ on party a. This instance also shows the necessary to go beyond von Neumann measurements in studying the interplay between coherence and quantum correlations.

III. ILLUSTRATIVE EXAMPLES

To illustrate the different behaviors in the monotonicity problem between the coherence measure $C(\rho|\Pi)$ and the K-coherence $I(\rho,K)$ in a more concrete way, we revisit two counterexamples to the monotonicity for the K-coherence, and show that the problem disappears for the coherence measure $C(\rho|\Pi)$ (i.e., the partial coherence is indeed monotonically nonincreasing under incoherent operations): The first is a numerical counterexample found in Ref. [16], the second is a general argument proposed in Ref. [17].

The example due to Du and Bai [16] is as follows. Let the system Hilbert space be three dimensional with orthonormal base $\{|i\rangle\}$, and

$$K = |1\rangle\langle 1| + 10|2\rangle\langle 2| + 5|3\rangle\langle 3|,$$

$$\rho = |\psi\rangle\langle\psi|, \quad |\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle).$$

The operation Φ is defined by $\Phi(\rho) = \sum_{i=1}^{3} A_i\rho A_i^\dagger$ with

$$A_1 = \frac{1}{\sqrt{2}}(|1\rangle\langle 1| + |2\rangle\langle 2|),$$

$$A_2 = \frac{1}{\sqrt{2}}(|1\rangle\langle 2| + |2\rangle\langle 3|),$$

$$A_3 = \frac{1}{\sqrt{2}}(|1\rangle\langle 3| + |2\rangle\langle 1|).$$

Clearly, $\sum_i A_i^\dagger A_i = \mathbf{1}$, Φ is an incoherent operation, and $\Phi(\rho) = |\phi\rangle\langle\phi|$ with $|\phi\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$. Direct evaluation shows that [16]

$$I(\Phi(\rho),K) = \tfrac{81}{4} > I(\rho,K) = \tfrac{122}{9}.$$

Consequently, the K-coherence $I(\rho,K)$ does not satisfy the monotonicity under incoherent operations, contrary to the claim in Ref. [2].

However, for this example, we have $\Pi = \{\Pi_i\}$ with $\Pi_i = |i\rangle\langle i|$, $i = 1,2,3$, and

$$I(\rho,\Pi_i) = \tfrac{1}{3} - \tfrac{1}{9} = \tfrac{2}{9}, \quad i = 1,2,3.$$

Therefore

$$C(\rho|\Pi) = \sum_{i=1}^{3} I(\rho,\Pi_i) = \frac{2}{3}.$$

On the other hand,

$$I(\Phi(\rho),\Pi_i) = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}, \quad i = 1,2; \;\; I(\Phi(\rho),\Pi_3) = 0,$$

and thus

$$C(\Phi(\rho)|\Pi) = \sum_{i=1}^{3} I(\Phi(\rho),\Pi) = \frac{1}{2}.$$

Consequently, we have

$$C(\Phi(\rho)|\Pi) = \tfrac{1}{2} < C(\rho|\Pi) = \tfrac{2}{3},$$

and the monotonicity is indeed satisfied by the partial coherence.

Next, we elaborate on a remarkable argument leading to many intuitive counterexamples to monotonicity of K-coherence, as proposed by Marvian *et al.* [17]. Consider a d-dimensional system Hilbert space with orthonormal base $\{|i\rangle\}$. Let

$$K = \sum_i \lambda_i |i\rangle\langle i|, \quad \lambda_i \text{ are real numbers},$$

$$\rho = |\psi\rangle\langle\psi|, \quad |\psi\rangle = \sum_i a_i|i\rangle.$$

The operation Φ is defined as

$$\Phi(\rho) = U_\sigma \rho U_\sigma^\dagger, \quad U_\sigma = \sum_i |\sigma(i)\rangle\langle i|,$$

where σ is a permutation, and U_σ is actually a unitary operator. Clearly, Φ is an incoherent operation, and

$$\Phi(\rho) = |\psi_\sigma\rangle\langle\psi_\sigma|, \quad |\psi_\sigma\rangle = \sum_i a_i|\sigma(i)\rangle.$$

It can be readily evaluated that

$$I(\rho,K) = \sum_i |a_i|^2\lambda_i^2 - \left(\sum_i |a_i|^2\lambda_i\right)^2,$$

$$I(\Phi(\rho),K) = \sum_i |a_i|^2\lambda_{\sigma(i)}^2 - \left(\sum_i |a_i|^2\lambda_{\sigma(i)}\right)^2.$$

Clearly, the monotonicity inequality $I(\Phi(\rho),K) \leqslant I(\rho,K)$ cannot be satisfied by all K (if we vary λ_i) simply because σ is a permutation. More precisely, if $I(\Phi(\rho),K) > I(\rho,K)$, then this is already a counterexample to monotonicity and we are done. If $I(\Phi(\rho),K) < I(\rho,K)$ (we assume that $I(\Phi(\rho,K) \neq I(\rho,K)$ without loss of generality since we are considering only special counterexamples), we simply take $K' = \sum_i \lambda_{\sigma^{-1}(i)}|i\rangle\langle i|$ and $\sigma^2 = 1$, then we have $I(\Phi(\rho),K') = I(\rho,K)$, $I(\Phi(\rho),K) = I(\rho,K')$, which imply that $I(\Phi(\rho),K') > I(\rho,K')$, a counterexample to the monotonicity of K-coherence. For a more concrete illustration, we may take $d = 3$, $K = 2|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|$, $|\psi\rangle = (|1\rangle + \sqrt{2}|2\rangle + |3\rangle)/2, U_\sigma = |1\rangle\langle 2| + |2\rangle\langle 1| + |3\rangle\langle 3|$, then $\rho = |\psi\rangle\langle\psi|$, $\Phi(\rho) = |\phi\rangle\langle\phi|$ with $|\phi\rangle = (\sqrt{2}|1\rangle + |2\rangle + |3\rangle)/2$. Simple calculation yields

$$I(\Phi(\rho),K) = 1/4 > I(\rho,K) = 3/16,$$

which exhibits a very simple instance for violating the monotonicity of K-coherence.

However, for this example, we have $\Pi = \{\Pi_i\}$ with $\Pi_i = |i\rangle\langle i|$, and for any i,

$$I(\rho,\Pi_i) = |a_i|^2 - |a_i|^4, \quad I(\Phi(\rho),\Pi_i) = |a_{\sigma(i)}|^2 - |a_{\sigma(i)}|^4,$$

Consequently,

$$C(\Phi(\rho)|\Pi) = C(\rho|\Pi)$$

and the monotonicity is indeed satisfied for the coherence measure $C(\rho|\Pi)$. This is consistent with our intuition since Φ in this example is actually a unitary mapping, and the coherence should be preserved.

Coherence is intimately related to asymmetry, as extensively studied in Refs. [17,21–23]. The following example illustrates a fundamental difference between asymmetry and coherence. Consider a system Hilbert space with an orthonormal base $\{|i\rangle\}$. Let

$$\rho = |\psi\rangle\langle\psi|, \quad |\psi\rangle = a|1\rangle + b|m\rangle,$$

and let Φ be an operation taking ρ to

$$\Phi(\rho) = |\phi\rangle\langle\phi|, \quad |\phi\rangle = a|1\rangle + b|m+n\rangle.$$

This operation increases the asymmetry, and from an intuitive viewpoint, it should not increase coherence since both $|\psi\rangle$ and

$|\phi\rangle$ are superposition of two basis states with the same sets of amplitudes. Let us evaluate the coherence. Clearly,

$$I(\rho,\Pi_i) = \begin{cases} |a|^2 - |a|^4, & \text{if } i = 1 \\ |b|^2 - |b|^4, & \text{if } i = m \\ 0, & \text{otherwise;} \end{cases}$$

$$I(\Phi(\rho),\Pi_i) = \begin{cases} |a|^2 - |a|^4, & \text{if } i = 1 \\ |b|^2 - |b|^4, & \text{if } i = m+n \\ 0, & \text{otherwise.} \end{cases}$$

Consequently,

$$C(\rho|\Pi) = \sum_i I(\rho,\Pi_i) = |a|^2 - |a|^4 + |b|^2 - |b|^4,$$

$$C(\Phi(\rho)|\Pi) = \sum_i I(\Phi(\rho),\Pi_i) = |a|^2 - |a|^4 + |b|^2 - |b|^4,$$

and indeed the coherence is preserved, i.e., $C(\rho|\Pi) = C(\Phi(\rho)|\Pi)$, as the intuition requires.

IV. COHERENCE VIA QUANTUM FISHER INFORMATION

In defining the partial coherence, we have employed the Wigner-Yanase skew information, which is a particular version of general quantum Fisher information [14,15]. However, since there are infinitely many versions of quantum Fisher information [24], one is led naturally to ask if we replace the skew information by other quantum Fisher information, will we still obtain a bona fide measure for coherence? A particularly significant version of quantum Fisher information is

$$I_F(\rho,K) = \tfrac{1}{4}\mathrm{tr}\rho L^2, \quad \tfrac{1}{2}(L\rho + \rho L) = i[\rho,K],$$

defined via symmetric logarithmic derivative, which has (both quantitatively and qualitatively) intimate relations with the skew information [25]. If one defines the following measure

$$C_F(\rho|\Pi) = \sum_i I_F(\rho,\Pi_i)$$

in terms of the above quantum Fisher information, then this measure possesses the following properties:

F1: $C_F(\rho|\Pi) \geqslant 0$, and $C_F(\sigma|\Pi) = 0$ is and only if $\sigma \in \mathcal{I}_L$.

F2: $C_F(\rho|\Pi)$ is convex in ρ.

We also conjecture that F3: monotonicity under incoherent operations: for any quantum state ρ, and for any incoherent operation Φ, it holds that

$$C_F(\Phi(\rho)|\Pi) \leqslant C_F(\rho|\Pi).$$

To establish item F1, noting that according to Theorem 2 in Ref. [25], we have $I(\rho,K) \leqslant I_F(\rho,K) \leqslant 2I(\rho,K)$, which implies that

$$C(\rho|\Pi) \leqslant C_F(\rho|\Pi) \leqslant 2C(\rho|\Pi).$$

Consequently, $C_F(\rho|\Pi) = 0$ if and only if $C(\rho|\Pi) = 0$. The desired result follows from L1, the corresponding property for $C(\rho|\Pi)$. Item F2 follows from the convexity of quantum Fisher information.

Although we do not have a proof of item F3, the relations between quantum Fisher information and fidelity (equivalently, the Bures distance) discussed in Ref. [15], as well as the monotonicity of fidelity under any operations, may be helpful in establishing the monotonicity. Since quantum Fisher information plays a fundamental role in parameter estimation, it is desirable to investigate whether the coherence measure provide any insight into phase estimation and more generally, quantum metrology. We leave these issues for further investigation.

V. CONCLUSION

To summarize, we have shown that the K-coherence can be readily adapted to a bona fide measure for coherence after simple manipulation. In this process, we are led to consider coherence with respect to Lüders measurements, which are more general and versatile than von Neumann measurements. Of course, when the Lüders measurement reduces to a von Neumann measurement, we recover the conventional setup. It is desirable to explore further applications and implications of this coherence measure, which is expected to play an interesting role in foundational study of coherence and quantum measurements.

Note added in proof. Zhao and Yu addressed the monotonicity in terms of Tsallis relative entropy [26].

ACKNOWLEDGMENTS

The authors are grateful to the referee for helpful suggestions. This work was supported by the National Natural Science Foundation of China, Grant No. 11375259.

[1] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[2] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[3] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5**, 021001 (2015).
[4] L.-H. Shao, Z. Xi, H. Fan, and Y. Li, Phys. Rev. A **91**, 042120 (2015).
[5] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Phys. Rev. A **92**, 022124 (2015).
[6] A. Winter and D. Yang, Phys. Rev. Lett. **116**, 120404 (2016).
[7] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Phys. Rev. Lett. **116**, 150502 (2016).
[8] E. Chitambar and M. H. Hsieh, Phys. Rev. Lett. **117**, 020402 (2016).
[9] E. Chitambar and G. Gour, Phys. Rev. Lett. **117**, 030401 (2016).
[10] S. Rana, P. Parashar, and M. Lewenstein, Phys. Rev. A **93**, 012110 (2016).
[11] I. Marvian and R. W. Spekkens, Phys. Rev. A **94**, 052324 (2016).
[12] A. Streltsov, G. Adesso, and M. B. Plenio, arXiv:1609.02439.

[13] E. P. Wigner and M. M. Yanase, Proc. Nat. Acad. Sci. USA **49**, 910 (1963).
[14] S. Luo, Phys. Rev. Lett. **91**, 180403 (2003).
[15] S. Luo and Q. Zhang, Phys. Rev. A **69**, 032106 (2004).
[16] S. Du and Z. Bai, Ann. Phys. (NY) **359**, 136 (2015).
[17] I. Marvian, R. W. Spekkens, and P. Zanardi, Phys. Rev. A **93**, 052331 (2016).
[18] G. Lüders, Annalen der Physik **443**, 322 (1950).
[19] S. Luo and S. Fu, Int. J. Mod. Phys. B **27**, 1345026 (2013).
[20] L. Chang and S. Luo, Phys. Rev. A **87**, 062303 (2013).
[21] G. Gour, I. Marvian, and R. W. Spekkens, Phys. Rev. A **80**, 012307 (2009).
[22] I. Marvian, Ph.D. thesis, University of Waterloo, 2012, https://uwspace.uwaterloo.ca/handle/10012/7088.
[23] I. Marvian and R. W. Spekkens, Phys. Rev. A **90**, 014102 (2014).
[24] D. Petz, Linear Algebra Appl. **244**, 81 (1996).
[25] S. Luo, Proc. Am. Math. Soc. **132**, 885 (2003).
[26] H. Zhao and C.-S. Yu, arXiv:1704.04876.