Quantum coherence versus quantum uncertainty

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The notion of measurement is of both foundational and instrumental significance in quantum mechanics, and coherence destroyed by measurements (decoherence) lies at the very heart of quantum to classical transition. Qualitative aspects of this spirit have been widely recognized and analyzed ever since the inception of quantum theory. However, axiomatic and quantitative investigations of coherence are attracting great interest only recently with several figures of merit for coherence introduced [Baumgratz, Cramer, and Plenio, Phys. Rev. Lett. **113**, 140401 (2014)]. While these resource theoretic approaches have many appealing and intuitive features, they rely crucially on various notions of incoherent operations which are sophisticated, subtle, and not uniquely defined, as have been critically assessed [Chitambar and Gour, Phys. Rev. Lett. **117**, 030401 (2016)]. In this paper, we elaborate on the idea that coherence and quantum uncertainty are dual viewpoints of the same quantum substrate, and address coherence quantification by identifying coherence of a state (with respect to a measurement) with quantum uncertainty of a measurement (with respect to a state). Consequently, coherence measures may be set into correspondence with measures of quantum uncertainty. In particular, we take average quantum Fisher information as a measure of quantum uncertainty, and introduce the corresponding measure of coherence, which is demonstrated to exhibit desirable properties. Implications for interpreting quantum purity as maximal coherence, and quantum discord as minimal coherence, are illustrated.

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I. INTRODUCTION

Both foundations of quantum mechanics and practices of quantum technologies are based on measurements, which have evolved from the original notion of observables [1], to von Neumann measurements [2], to Lüders measurements [3], and to quantum operations [channels, or positive-operator-valued measures (POVMs)] [4–7]. Intimately related to measurements are uncertainties of measuring results and coherence of quantum states [8–10], which play basic roles in applications of quantum theory and gain even more significance with the emergence of quantum information theory [7]. The Heisenberg uncertainty relations, which synthesize trade-off between measurement uncertainties, are a hallmark of quantum mechanics, while coherence is among the most prominent features of quantumness. In this context, a basic and natural question arises: What are the interrelations between uncertainty and coherence? The aim of this paper is to establish an intrinsic and quantitative link between quantum uncertainty and coherence. More precisely, we make the following identification (see Fig. 1): Coherence = Quantum Uncertainty, and reveal basic features of the corresponding coherence measures. Furthermore, the results are applied to illustrate that quantum purity is actually maximal coherence, while quantum discord arises from minimal coherence.

II. COHERENCE

In recent years, inspired by the work of Åberg [11], Levi and Mintert [12], and more influentially, Baumgratz *et al.* [13], there is a flurry of interest in the quantification issues of coherence and, consequently, several important quantifiers of coherence, including the relative entropy of coherence (coherence cost), coherence formation, robustness of coherence, l^p-norm coherence, distance-based coherence, etc., have been introduced and assessed [14–22]. In particular, the resource theoretic perspective of coherence, in analogy to that of entanglement, has been established [23–30].

All these approaches are based on the notions of incoherent operations, which have many species such as maximally incoherent operations, incoherent operations, physically incoherent operations, strictly incoherent operations, genuinely incoherent operations, dephase covariant operations, translation invariant operations, and energy preserving operations [29]. These diversities complicate the issue, and most of them are actually not free of coherence when implemented via ancillaries [31,32].

Another severe, neither necessary nor desirable, restriction of existent coherence measures lies in the reference bases, which are always taken as orthonormal bases, or equivalently, von Neumann measurements. In both theoretical and practical investigations involving correlations and decoherence-free

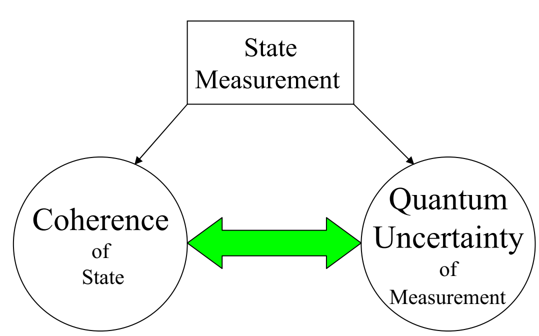

FIG. 1. From the state-measurement duality, the coherence of the state ρ (with respect to the measurement M) is identified with the quantum uncertainty of the measurement M (with respect to the state ρ).

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subspaces, it is necessary to consider coherence with respect to general POVMs. In this general context, there are no simple notions of incoherent states and incoherent operations. Nevertheless, there are certainly still intrinsically coherent issues.

In this paper, we will take a direct approach to coherence quantification via quantum uncertainty, which in turn is quantified via average quantum Fisher information [33–35]. We will elucidate that this viewpoint captures the essence of, leads to interesting implications for, and sheds considerable lights on, coherence.

III. QUANTUM UNCERTAINTY

Given an observable A and a quantum state ρ, the variance $V(\rho,A) := \mathrm{tr}\rho A^2 - (\mathrm{tr}\rho A)^2$ is a fundamental quantity representing the total uncertainty of A in ρ, which may be formally decomposed into a classical part and a quantum part as: $V(\rho,A) = C(\rho,A) + Q(\rho,A)$. These two kinds of uncertainties are postulated to satisfy the following intuitive requirements [34]:

(1) The quantum uncertainty $Q(\rho,A)$ is convex in ρ. In contrast, the classical uncertainty $C(\rho,A)$ is concave in ρ.

(2) When ρ is pure, $V(\rho,A) = Q(\rho,A)$ and $C(\rho,A) = 0$. There is no classical mixing and all uncertainties are quantum for any pure state.

(3) When ρ commutes with A, $Q(\rho,A) = 0$ and $C(\rho,A) = V(\rho,A)$ because, in this situation, ρ and A can be diagonalized simultaneously and thus behave like classical variables. Consequently there is no quantum uncertainty of A in ρ.

There is no unique choice of $Q(\rho,A)$, and depending on the context and problems, one may make different choices. Based on the quantum estimation theory, it is natural to take quantum Fisher information as a measure of quantum uncertainty [34].

Now, for any measurement mathematically represented by a POVM $M = \{M_i : i = 1,2,\cdots,m\}$ with $M_i \geqslant 0$, $\sum_i M_i = \mathbf{1}$, its action on a quantum state results in a postmeasurement state $M(\rho) = \sum_i \sqrt{M_i}\rho\sqrt{M_i}$ in the nonselective case, and a postmeasurement ensemble $\{\rho_i = \frac{1}{p_i}\sqrt{M_i}\rho\sqrt{M_i}, p_i = \mathrm{tr}\rho M_i\}$ in the selective case. We define the total uncertainty of the measurement M in ρ as $V(\rho,M) := \sum_i V(\rho,M_i)$. To extract the quantum part, we define the quantum uncertainty of the measurement M in ρ as $Q(\rho,M) := \sum_i F(\rho,M_i)$, which indeed meets the above requirements. Here $F(\rho,M_i)$ is a version of quantum Fisher information of ρ with respect to M_i. There are infinite versions of quantum Fisher information [36–43], among which, two prominent ones are defined via symmetric logarithmic derivative and commutator, respectively. The latter corresponds to the Wigner-Yanase skew information [36,42,43]

$$I(\rho,A) := -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho},A]^2,$$

which has already been regarded as quantum uncertainty of the observable A in the state ρ [34]. There are several interpretations of the skew information:

(1) Information content of ρ skew to A [36].

(2) Noncommutativity between A and ρ [44].

(3) Quantum Fisher information of ρ with respect to a parameter conjugate to A [42].

(4) Quantum uncertainty of A in ρ [34,43].

(5) Coherence of ρ with respect to A [14].

(6) Asymmetry of ρ with respect to A [45,46].

All these consistent and different manifestations of the same quantum subject indicate that skew information is a significant and versatile quantity.

IV. COHERENCE AS QUANTUM UNCERTAINTY

Now we have a measure of quantum uncertainty of $M = \{M_i\}$ in ρ as

$$Q(\rho,M) := \sum_{i=1}^{m} I(\rho,M_i),$$

in which the measurement M plays an active role, while the state ρ plays a passive role (i.e., serves as a background reference). Taking a dual point of view, we regard the state ρ as active, and the measurement M as passive, and interpret this quantity as coherence of ρ with respect to M. It turns out that $Q(\rho,M)$ is indeed a bona fide measure for coherence, as consolidated by the following properties.

(1) The coherence is nonnegative, and vanishes if and only if ρ commutes with every M_i, i.e., $Q(\rho,M) \geqslant 0$, and the minimal value 0 is achieved if and only if $[\rho,M_i] = 0$.

(2) The coherence $Q(\rho,M)$ is convex in ρ, that is,

$$Q\Big(\sum_j c_j\rho_j, M\Big) \leqslant \sum_j c_j Q(\rho_j,M),$$

where $c_j \geqslant 0$, $\sum_j c_j = 1$, and ρ_j are quantum states.

(3) The coherence $Q(\rho,M)$ is unitarily covariant in the sense that $Q(U\rho U^\dagger, UMU^\dagger) = Q(\rho,M)$ for any unitary operator U. Here $UMU^\dagger = \{UM_iU^\dagger\}$.

(4) The coherence $Q(\rho,M)$ is decreasing under partial trace in the sense that

$$Q(\rho^{ab}, M^a \otimes \mathbf{1}^b) \geqslant Q(\rho^a, M^a).$$

Here, ρ^{ab} is a bipartite state shared between parties a and b, $M^a \otimes \mathbf{1}^b = \{M_i^a \otimes \mathbf{1}^b\}$ is a measurement on the composite system, while $M^a = \{M_i^a\}$ is a measurement on party a, $\mathbf{1}^b$ is the identity operator on party b, $\rho^a = \mathrm{tr}_b\rho^{ab}$ is the reduced state on party a.

(5) The coherence $Q(\rho,M)$ is decreasing, i.e., $Q(\rho,M) \geqslant Q(\Phi(\rho),M)$ under any quantum operation Φ which does not disturb the measurement M (in the technical sense that $\Phi^\dagger(\sqrt{M_i}) = \sqrt{M_i}$, $\Phi^\dagger(M_i) = M_i$ for all i). Noting that the adjoint operation $\Phi^\dagger$ is defined as $\Phi^\dagger(A) = \sum_j E_j^\dagger A E_j$ if $\Phi(\rho) = \sum_j E_j\rho E_j^\dagger$ in the Kraus representation.

We now sketch the reasons of the above properties.

Item (1) is clear since $I(\rho,M_i) \geqslant 0$ for any i. The vanishing condition is equivalent to $I(\rho,M_i) = 0$ for all i, i.e., ρ commutes with every M_i.

Item (2) follows readily from the well-known convex property of the skew information [36,37].

Item (3) follows from simple manipulation of the expression of the skew information.

Item (4) follows from the decreasing property of the skew information under partial trace, as established by Lieb [37]: $I(\rho^{ab}, M_i^a \otimes \mathbf{1}^b) \geqslant I(\rho^a, M_i^a)$.

Item (5) is subtle. First, under the nondisturbance conditions, it follows that $I(\Phi(\rho),M_k) \leqslant I(\rho,M_k)$. To establish this, define the affinity [47]: $A(\rho,\tau) := \mathrm{tr}\sqrt{\rho}\sqrt{\tau}$. Consider the von Neumann-Landau equation

$$i\frac{\partial}{\partial t}\rho_t = [M_k,\rho_t], \quad \rho_0 = \rho,$$

then $A(\rho_t,\rho) = 1 - I(\rho,M_k)t^2 + o(t^2)$ for sufficiently small t. Similarly, due to the nondisturbance condition, $\Phi(\rho_t)$ satisfies the von Neumann-Landau equation with generator M_k and initial condition $\Phi(\rho_t)|_{t=0} = \Phi(\rho)$, and thus

$$A(\Phi(\rho_t),\Phi(\rho)) = 1 - I(\Phi(\rho),M_k)t^2 + o(t^2)$$

for sufficiently small t. Now, by the monotonicity of affinity, we have $A(\Phi(\rho_t),\Phi(\rho)) \geqslant A(\rho_t,\rho)$ which implies $I(\Phi(\rho),M_k) \leqslant I(\rho,M_k)$. Summing these inequalities with respect to k yields the desired result.

We remark that Item (5) actually implies Item (4), and in the present context, there are no natural notions for incoherent states or incoherent operations.

To illustrate quantum coherence with respect to a general measurement, consider a qubit system with standard base $\{|0\rangle,|1\rangle\}$. Let $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, which is maximally coherent with respect to the standard base in an intuitive sense. Now consider the measurement $M = \{M_i : i = 1,2\}$ with

$$M_1 = \gamma|0\rangle\langle 0| + (1-\gamma)|1\rangle\langle 1|,$$
$$M_2 = (1-\gamma)|0\rangle\langle 0| + \gamma|1\rangle\langle 1|,$$

where $\gamma \in [0,1]$ is a parameter. Then direct evaluation shows that

$$Q(\rho,M) = 2\left(\gamma - \tfrac{1}{2}\right)^2,$$

which vanishes when $\gamma = \frac{1}{2}$, and achieves the maximum $\frac{1}{2}$ when $\gamma = 0$ or 1. Hence, by adjusting γ, quantum coherence can take any value between 0 and $\frac{1}{2}$. In many situations of probing (measuring) a quantum state, it is crucial to take a judicious trade-off between extracting information (which causes decoherence) and maintaining coherence, and general measurements beyond von Neumann measurements are necessary.

We recall that the most general state changes can be described by quantum operations with Kraus representations:

$$\Phi : \rho \to \{(p_i,\rho_i) : i \in I\},$$

which send an initial state to a quantum ensemble with $\rho_i = \frac{1}{p_i}\sum_k A_{ik}\rho A_{ik}^\dagger$, $p_i = \mathrm{tr}\sum_k A_{ik}\rho A_{ik}^\dagger$, $\sum_{ik} A_{ik}^\dagger A_{ik} = \mathbf{1}$, i.e., the measurement yields the outcome labeled by i with probability p_i, with the resulting postmeasurement state ρ_i. In this situation, how do we define quantum uncertainty of this state change? Equivalently, how do we define coherence of a state with respect to the most general measurement Φ? This is an important and subtle issue. First, there is a corresponding POVM $M = \{M_i : i \in I\}$ with $M_i = \sum_k A_{ik}^\dagger A_{ik}$, and if we employ this measurement M to define the quantum uncertainty of Φ, we are reduced to the POVM case. However, it seems that such an approach misses intrinsic characteristics of Φ since the Kraus operators A_{ik} may not be Hermitian. It is desirable to extend the previous formalism and results to this general case, which is left as an open issue for further investigation.

V. SPECIFYING TO LÜDERS MEASUREMENTS

For concreteness, and for the purpose of gaining further intuitive understanding of the coherence $Q(\rho,M)$, we now specify the measurement M to Lüder measurements (including von Neumann measurements as special cases). Consider a quantum system described by a Hilbert space H. Let $\Pi = \{\Pi_i : i = 1,2,\cdots,m\}$ be a Lüders measurement, that is, Π_i are mutually orthogonal projections constituting a resolution of the identity operator: $\sum_i \Pi_i = \mathbf{1}$. This is equivalent to a direct sum decomposition of the system Hilbert space $H = \bigoplus_i H_i$, with Π_i corresponding to the orthogonal projection onto the subspace $H_i = \Pi_i H$. In particular, when all Π_i are one-dimensional, we have a von Neumann measurement, which is equivalent to an orthonormal base for H. While, in most previous studies, coherence measures are usually taken with respect to a fixed orthonormal base (von Neumann measurement), Marvian and Spekkens explicitly suggested to quantify and characterize coherence not only with respect to one-dimensional subspaces, but also with respect to subspaces of arbitrary dimension, i.e., coherence with respect to Lüders measurements (which are phrased in terms of measuring a degenerate observable) [23]. The restriction to von Neumann measurements is unnecessary, and in many situations, unacceptable.

For the Lüders measurement $\Pi = \{\Pi_i\}$, the coherence has the following further nice properties.

(a) $Q(\rho,\Pi)$ can be alternatively expressed as $Q(\rho,\Pi) = \sum_{i\neq j}\mathrm{tr}\sqrt{\rho}\Pi_i\sqrt{\rho}\Pi_j$ which is reminiscent of off-diagonal elements and interference, the characteristic features of coherence.

(b) $Q(\rho,\Pi)$ has the direct sum property in the sense that

$$Q\left(\bigoplus_i \lambda_i\sigma_i,\Pi\right) = \sum_i \lambda_i Q(\sigma_i,\Pi).$$

where $\lambda_i \geqslant 0$, $\sum_i \lambda_i = 1$, and σ_i are quantum states on H_i (thus *a priori* are quantum states on H).

(c) $Q(\rho,\Pi)$ has the following tensor product property:

$$1 - Q(\rho^a \otimes \rho^b,\Pi^{ab}) = [1 - Q(\rho^a,\Pi^a)][1 - Q(\rho^b,\Pi^b)],$$

where ρ^{ab} is a bipartite state shared by parties a and b, $\Pi^a = \{\Pi_i^a\}$ and $\Pi^b = \{\Pi_j^b\}$ are Lüders measurements on parties a and b, respectively, and $\Pi^{ab} = \{\Pi_i^a \otimes \Pi_j^b\}$.

To establish (a), noting $\sum_i \Pi_i = \mathbf{1}$, we have

$$\begin{aligned} 0 &= \mathrm{tr}\rho\left(\sum_i \Pi_i\right)^2 - \mathrm{tr}\sqrt{\rho}\left(\sum_i \Pi_i\right)\sqrt{\rho}\left(\sum_j \Pi_j\right) \\ &= \sum_i \mathrm{tr}\rho\Pi_i^2 - \sum_{i,j}\mathrm{tr}\sqrt{\rho}\Pi_i\sqrt{\rho}\Pi_j \\ &= \sum_i I(\rho,\Pi_i) - \sum_{i\neq j}\mathrm{tr}\sqrt{\rho}\Pi_i\sqrt{\rho}\Pi_j \end{aligned}$$

from which the desired result follows.

Item (b) follows readily from $I(\bigoplus_i \lambda_i\sigma_i,\Pi_j) = \lambda_j I(\sigma_j,\Pi_j)$.

Item (c) can be established as follows:

$$\begin{aligned} &Q(\rho^a\otimes\rho^b,\Pi^{ab}) \\ &\quad = \sum_{ij} I\left(\rho^a\otimes\rho^b,\Pi_i^a\otimes\Pi_j^b\right) \\ &\quad = \sum_{ij}\left(\mathrm{tr}\rho^a\Pi_i^a\cdot\mathrm{tr}\rho^b\Pi_j^b - \mathrm{tr}\left(\sqrt{\rho^a}\Pi_i^a\right)^2\cdot\mathrm{tr}\left(\sqrt{\rho^b}\Pi_j^b\right)^2\right) \\ &\quad = 1-[1-Q(\rho^a,\Pi^a)][1-Q(\rho^b,\Pi^b)]. \end{aligned}$$

VI. MAXIMAL, MINIMAL, AND AVERAGE COHERENCE

In general, $Q(\rho,\Pi)$ should be regarded as a functional of both the state ρ and the measurement Π. In this context, it is interesting to consider the maximal coherence of ρ, when Π varies over all von Neumann measurements. Consequently, we introduce $Q_{\max}(\rho) := \max_\Pi Q(\rho,\Pi)$, which is the maximally possible value of coherence of ρ with respect to von Neumann measurements. Clearly, $Q_{\max}(\rho) = \max_U Q(\rho,U\Pi U^\dagger)$ where U is unitary. It is natural to expect this should be a measure of quantum information content, or quantum purity [30], of ρ. Indeed, we have

$$Q_{\max}(\rho) = \frac{1}{n}\sum_{j=1}^{n^2} I(\rho,X_j) = 1-\frac{1}{n}(\mathrm{tr}\sqrt{\rho})^2,$$

with the last equality follows from Refs. [33–35]. Here $\{X_i\}$ is an orthonormal base for the Hilbert space $L(H)$ of operators on H with the scalar product $\langle A|B\rangle = \mathrm{tr}A^\dagger B$, and $n = \dim H$. This leads to the observation that quantum purity may be interpreted as maximal coherence.

When the worst cases are relevant, one may be interested in the minimal coherence $Q_{\min}(\rho) := \min_\Pi Q(\rho,\Pi)$. In particular, quantum discord may be interpreted as the minimal coherence [19], with minimization over all local von Neumann measurements. To illustrate this, consider a bipartite state ρ^{ab} shared by two parties a and b, and let $\Pi^a = \{\Pi_i^a\}$ be a von Neumann measurement on party a, then $\Pi^a\otimes\mathbf{1}^b = \{\Pi_i^a\otimes\mathbf{1}^b\}$ is a Lüders measurement on the combined system ab. The geometric discord $D_{\mathrm{H}}(\rho^{ab}) := \min_{\Pi^a}\mathrm{tr}[\sqrt{\rho^{ab}} - (\Pi^a\otimes\mathbf{1}^b)(\sqrt{\rho^{ab}})]^2$ quantifies quantum correlations (with respect to party a) in ρ^{ab} [48], where $(\Pi^a\otimes\mathbf{1}^b)(\sqrt{\rho^{ab}}) = \sum_i(\Pi_i^a\otimes\mathbf{1}^b)\sqrt{\rho^{ab}}(\Pi_i^a\otimes\mathbf{1}^b)$. On the other hand, in view of the coherence measure, we have the coherence of ρ^{ab} with respect to the Lüders measurement $\Pi^a\otimes\mathbf{1}^b$ as $Q(\rho^{ab},\Pi^a\otimes\mathbf{1}^b)$. This quantity of course depends on the local von Neumann measurement Π^a. If we optimize over Π^a, then $\min_{\Pi^a} Q(\rho^{ab},\Pi^a\otimes\mathbf{1}^b) = D_{\mathrm{H}}(\rho^{ab})$. Consequently, the geometric discord is precisely the minimal coherence in this context.

Intermediate between maximal and minimal coherence is the average coherence $Q_{\mathrm{ave}}(\rho) := \int_{\mathcal{U}} Q(U\rho U^\dagger,\Pi)dU$, which can also be regarded as the average coherence of a fixed state ρ with respect to the unitary orbit of the measurement Π in the sense that $Q_{\mathrm{ave}}(\rho) = \int_{\mathcal{U}} Q(\rho,U^\dagger\Pi U)dU$. Here the integration is with respect to the Haar measure on the group of unitary operators. The explicit evaluation of the integral remains to be investigated.

VII. COMPARISONS

The coherence measure $Q(\rho,\Pi)$ should be compared with the K-coherence $I(\rho,K)$ introduced in Ref. [14], which violates the important axiom for monotonicity [49]. The measure $Q(\rho,\Pi)$ satisfies the monotonicity. When the measurement Π is a von Neumann measurement $\Pi = \{|i\rangle\langle i|\}$ with $\{|i\rangle\}$ an orthonormal base for the system Hilbert space, the measure reduces to the one studied in Ref. [50].

We emphasize that the present coherence measure is fundamentally different from existent measures of coherence in several aspects: First, while previous coherence measures are with respect to a fixed orthonormal base (equivalently, von Neumann measurement), here the coherence measure is more general, since it is constructed with respect to any quantum measurement. This generalization has wide and important implications because it is necessary to go beyond von Nuemann measurements in many situations, such as in the considerations of decoherence-free subspaces and error correcting codes [51–54]. Second, we do not rely on the notions of the so-called incoherent operations, which are very subtle and complicated [31,32]. Third, unlike many other measures of coherence, here optimization is not involved.

VIII. DISCUSSIONS

In contrast to the resource theoretic approach to coherence, we have introduced a direct and intuitive approach to coherence by identifying quantum uncertainty and coherence, and have defined the corresponding coherence measure via quantum Fisher information. The coherence measure is defined, without reference to incoherent operations, in a more broad framework involving general POVMs rather than orthonormal bases corresponding to von Neumann measurements, and it enjoys several desirable and intuitive properties. In particular, we are led to corroborate the following formal identifications:

(1) Quantum Uncertainty $\sim$ Coherence
(2) Quantum purity $\sim$ Maximal Coherence
(3) Quantum Discord $\sim$ Minimal Coherence

A lot of important questions call for further investigation, including the foundational implications, operational significance, resource theoretic connection, and experimental usage of the ideas and results illustrated here.

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[1] W. H. Heisenberg, *The Physical Principles of Quantum Theory* (Univ. Chicago Press, Chicago, 1930).
[2] J. von Neuamnn, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, 1955).
[3] G. Lüders, Annalen der Physik **443**, 322 (1950); for an English translation, see arXiv:quant-ph/0403007.
[4] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton Univ. Press, Princeton, 1983).
[5] B. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).
[6] P. Busch, M. Grabowski, and P. J. Lahti, *Operational Quantum Physics* (Springer, Berlin, 1995).
[7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge, 2000).
[8] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).
[9] E. Joos, H. D. Zeh, C. Kiefer, D. J. W. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, Berlin, 2003), 2nd ed.
[10] M. A. Schlosshauer, Rev. Mod. Phys. **76**, 1267 (2005).
[11] J. Åberg, arXiv:quant-ph/0612146.
[12] F. Levi and F. Mintert, New J. Phys. **16**, 033007 (2014).
[13] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[14] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[15] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[16] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5**, 021001 (2015).
[17] L.-H. Shao, Z. Xi, H. Fan, and Y. Li, Phys. Rev. A **91**, 042120 (2015).
[18] D. P. Pires, L. C. Céleri, and D. O. Soares-Pinto, Phys. Rev. A **91**, 042330 (2015).
[19] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, Phys. Rev. A **92**, 022112 (2015).
[20] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Phys. Rev. Lett. **116**, 150502 (2016).
[21] S. Rana, P. Parashar, and M. Lewenstein, Phys. Rev. A **93**, 012110 (2016).
[22] A. E. Rastegin, Phys. Rev. A **93**, 032136 (2016).
[23] I. Marvian and R. W. Spekkens, Phys. Rev. A **94**, 052324 (2016).
[24] X. D. Yu, D. J. Zhang, G. F. Xu, and D. M. Tong, Phys. Rev. A **94**, 060302(R) (2016).
[25] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Phys. Rev. A **92**, 022124 (2015).
[26] A. Winter and D. Yang, Phys. Rev. Lett. **116**, 120404 (2016).
[27] J. Ma, B. Yadin, D. Girolami, V. Vedral, and M. Gu, Phys. Rev. Lett. **116**, 160407 (2016).
[28] E. Chitambar and M. H. Hsieh, Phys. Rev. Lett. **117**, 020402 (2016).
[29] A. Streltsov, G. Adesso, and M. B. Plenio, arXiv:1609.02439 [Rev. Mod. Phys. (to be published)].
[30] A. Streltsov, H. Kampermann, S. Wölk, M. Gessner, and D. Bruß, arXiv:1612.07570.
[31] E. Chitambar and G. Gour, Phys. Rev. A **94**, 052336 (2016).
[32] E. Chitambar and G. Gour, Phys. Rev. Lett. **117**, 030401 (2016).
[33] S. Luo, Phys. Rev. A **73**, 022324 (2006).
[34] S. Luo, Theor. Math. Phys. **151**, 693 (2007).
[35] S. Luo, S. Fu, and C. H. Oh, Phys. Rev. A **85**, 032117 (2012).
[36] E. P. Wigner and M. M. Yanase, Proc. Nat. Acad. Sci. USA. **49**, 910 (1963).
[37] E. H. Lieb, Adv. Math. **11**, 267 (1973).
[38] H. P. Yuen and M. Lax, IEEE Trans. Inform. Theory **19**, 740 (1973).
[39] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
[40] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[41] D. Petz, Linear Algebra Appl. **244**, 81 (1996).
[42] S. Luo, Phys. Rev. Lett. **91**, 180403 (2003).
[43] S. Luo, Phys. Rev. A **72**, 042110 (2005).
[44] A. Connes and E. Størmer, J. Funct. Anal. **28**, 187 (1978).
[45] I. Marvian and R. W. Spekkens, Nat. Commun. **5**, 3821 (2014).
[46] I. Marvian, R. W. Spekkens, and P. Zanardi, Phys. Rev. A **93**, 052331 (2016).
[47] S. Luo and Q. Zhang, Phys. Rev. A **69**, 032106 (2004).
[48] L. Chang and S. Luo, Phys. Rev. A **87**, 062303 (2013).
[49] S. Du and Z. Bai, Ann. Phys. **359**, 136 (2015).
[50] C. Yu, Phys. Rev. A **95**, 042337 (2017).
[51] D. Kribs, R. Laflamme, and D. Poulin, Phys. Rev. Lett. **94**, 180501 (2005).
[52] D. W. Kribs, R. Laflamme, D. Poulin, and M. Lesosky, Quant. Info. Comp. **6**, 382 (2006).
[53] R. Blume-Kohout, H. K. Ng, D. Poulin, and L. Viola, Phys. Rev. Lett. **100**, 030501 (2008).
[54] S. Luo and S. Fu, Int. J. Mod. Phys. B **27**, 1345026 (2013).