

Scaling maps of s -ordered quasiprobabilities are either nonpositive or completely positive

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Continuous-variable systems in quantum theory can be fully described through any one of the s -ordered family of quasiprobabilities $\Lambda_s(\alpha)$, $s \in [-1, 1]$. We ask for what values of (s, a) is the scaling map $\Lambda_s(\alpha) \rightarrow a^{-2}\Lambda_s(a^{-1}\alpha)$ a positive map? Our analysis based on a duality we establish settles this issue: (i) the scaling map generically fails to be positive, showing that there is no useful entanglement witness of the scaling type beyond the transpose map, and (ii) in the two particular cases ($s = 1, |a| \leq 1$) and ($s = -1, |a| \geq 1$), and only in these two nontrivial cases, the map is not only positive but also completely positive as seen through the noiseless attenuator and amplifier channels. We also present a “phase diagram” for the behavior of the scaling maps in the $s - a$ parameter space with regard to its positivity, obtained from the viewpoint of symmetric-ordered characteristic functions. This also sheds light on similar diagrams for the practically relevant attenuation and amplification maps with respect to the noise parameter, especially in the range below the complete-positivity (or quantum-limited) threshold.

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I. INTRODUCTION

Completely positive maps mathematically describe physical processes or quantum channels [1–5]. Positive maps which fail to be completely positive cannot represent physical processes, but they play a key role in the study of inseparability of mixed states as entanglement witnesses [6–8]. The desirability of studying maps represented by uniform scaling of phase space variables of continuous-variable systems, namely, $\hat{q} \rightarrow a\hat{q}$, $\hat{p} \rightarrow a\hat{p}$, $a \in \mathbb{R} \setminus \{0\}$ was suggested in Ref. [9] in the hope that *such a map might be positive but not completely positive*, and hence partial scaling (i.e., uniform phase space scaling on one party of a bipartite state followed or preceded by the transpose map) could prove useful as an entanglement witness, “generalizing” [9] the partial momentum reversal or partial transpose criterion [10] for separability. These authors pointed to the interesting construct that this scaling of phase space could, alternatively, be viewed as scaling of the Planck constant: $\hbar \rightarrow a^2\hbar$. (We shall comment on this toward the end of the paper.) Note that the signature of the scale parameter a can be changed simply through a natural unitary evolution of the mode through half a period.

Although this uniform scaling is a linear transformation at the operator level, it is not canonical for $|a| \neq 1$, and hence cannot be implemented as a linear, unitary transformation on Hilbert space vectors. We therefore introduce abstract scaling maps $\Gamma_{s,a}$ whose action on the density operators can be represented at the level of s -ordered quasiprobabilities $\Lambda_s(q, p)$ in the following manner:

$$\Gamma_{s,a} : \Lambda_s(q, p; \Gamma_{s,a}(\hat{\rho})) = a^{-2}\Lambda_s(a^{-1}q, a^{-1}p; \hat{\rho}), \quad (1)$$

with $s \in [-1, 1]$, $a \in \mathbb{R} \setminus \{0\}$.

The Wigner distribution ($s = 0$) was the choice of Ref. [9] to implement the scaling transformation through

$$W(q, p; \hat{\rho}) \rightarrow W(q, p; \hat{\rho}') = a^{-2}W(a^{-1}q, a^{-1}p; \hat{\rho}). \quad (2)$$

That this map preserves Hermiticity and normalization of density operators is transparent. It turns out that it does not preserve the non-negativity property of density operators (see Proposition 2 and also Refs. [11,12]), showing that the scaling map defined in this manner through the Wigner quasiprobability is not positive. The original expectation of the authors of Ref. [9] thus turns out to be unfounded.

If a positive map is obtained for any other value of order parameter s , then there are two possibilities: (a) the positive map may not be completely positive, in which case it will be useful as an entanglement witness; and (b) if it turns out to be completely positive it will correspond to a quantum channel. Indeed, since any scaling map $\Gamma_{s,a}$ transforms Gaussian quasiprobability distributions into Gaussian distributions, any completely positive scaling map will correspond to a bosonic Gaussian channel, a topic of considerable current interest [12–27].

The main purpose of the present work is to address the issue of positivity of scaling maps, a generalization of the one raised in Ref. [9], in a definitive manner in Sec. II. In Sec. III we introduce a “phase diagram” for the behavior of the scaling maps $\Gamma_{s,a}$ over the $s - a$ parameter space with regard to its positivity. As an almost unintended fallout of our analysis, we present a related phase diagram for attenuation and amplification maps with respect to the noise parameter of the map, especially depicting the behavior of these maps below the complete-positivity or quantum-limited noise threshold. We conclude in Sec. IV.

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II. QUASIPROBABILITIES AND SCALING TRANSFORMATIONS

Quasiprobabilities are defined more conveniently in terms of the non-Hermitian operators \hat{a}, \hat{a}^\dagger obeying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, than in terms of their Hermitian parts \hat{q}, \hat{p} obeying $[\hat{q}, \hat{p}] = i$. In correspondence with the relation $\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}$ we will associate to every point (q, p) in the phase space plane the complex number $\alpha = (q + ip)/\sqrt{2}$. The characteristic function $\chi_s(\xi; \hat{\rho})$ of the s -ordered quasiprobability $\Lambda_s(\alpha; \hat{\rho})$ associated with a density operator $\hat{\rho}$ is defined through [28]

$$\begin{aligned} \chi_s(\xi; \hat{\rho}) &= \exp\left[\frac{1}{2}s|\xi|^2\right]\chi_0(\xi; \hat{\rho}), \quad -1 \leq s \leq 1; \\ \chi_0(\xi; \hat{\rho}) &= \text{Tr}[\hat{\rho}\mathcal{D}(\xi)], \quad \mathcal{D}(\xi) = \exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}]. \end{aligned} \quad (3)$$

For every $-1 \leq s \leq 1$, the quasiprobability $\Lambda_s(\alpha; \hat{\rho})$ and the associated characteristic function $\chi_s(\xi; \hat{\rho})$ are related through the Fourier pair

$$\begin{aligned} \Lambda_s(\alpha; \hat{\rho}) &= \pi^{-1} \int \exp[\alpha\xi^* - \alpha^*\xi] \chi_s(\xi; \hat{\rho}) d^2\xi, \\ \chi_s(\xi; \hat{\rho}) &= \pi^{-1} \int \exp[\xi\alpha^* - \xi^*\alpha] \Lambda_s(\alpha; \hat{\rho}) d^2\alpha. \end{aligned} \quad (4)$$

The familiar diagonal ‘‘weight’’ function ϕ [29] or P distribution [30], Wigner distribution W , and Husimi or Q distribution correspond, respectively, to $s = 1, 0, -1$ [28]. It is clear that the association between operators $\hat{\rho}$ and quasiprobabilities $\Lambda_s(\alpha; \hat{\rho})$ [or characteristic functions $\chi_s(\xi; \hat{\rho})$] is one-to-one invertible for any $-1 \leq s \leq 1$. While Hermiticity of $\hat{\rho}$ translates into reality of $\Lambda_s(\alpha; \hat{\rho})$ and the normalization $\text{Tr} \hat{\rho} = 1$ into the normalization $\int \Lambda_s(\alpha; \hat{\rho}) d^2\alpha = 1$ [or equivalently, $\chi_s(\xi; \hat{\rho})|_0 = 1$], positivity of $\hat{\rho}$ gets encoded in $\Lambda_s(\alpha; \hat{\rho})$ in a more subtle manner [28].

In light of Eqs. (1) and (4), we can rewrite the action of the scaling maps $\Gamma_{s,a}$ as

$$\Lambda_s(\alpha; \Gamma_{s,a}(\hat{\rho})) = a^{-2} \Lambda_s(a^{-1}\alpha; \hat{\rho}). \quad (5)$$

It is clear from the Fourier transform pair (4) that the scaling transformation $\Gamma_{s,a}$ described through its action on s -ordered quasiprobability [Eq. (5)] reads, when transcribed to the associated characteristic function, as the transformation

$$\chi_s(\xi; \Gamma_{s,a}(\hat{\rho})) = \chi_s(a\xi; \hat{\rho}). \quad (6)$$

Also, it turns out to be useful for succeeding sections to introduce a ‘‘dual’’ scaling map $\tilde{\Gamma}_{s,a}$ in the following way:

$$\tilde{\Gamma}_{s,a} : \Lambda_{-s}(\alpha; \tilde{\Gamma}_{s,a}(\hat{\rho})) = a^2 \Lambda_{-s}(a\alpha; \hat{\rho}). \quad (7)$$

Note that while the action of the map $\Gamma_{s,a}$ is described at the level of Λ_s its dual map $\tilde{\Gamma}_{s,a}$ is described at the level of Λ_{-s} with the scaling a replaced by its inverse.

We begin our analysis of scaling maps by establishing an important duality between the pair of scaling maps $\Gamma_{s,a}$ and $\tilde{\Gamma}_{s,a}$ with regard to positivity.

Proposition 1 (Duality). The scaling map $\Gamma_{s,a^{-1}}$ is positive if and only if the dual map $\tilde{\Gamma}_{s,a^{-1}}$ is positive.

Proof. Positivity of entities are often defined or described through the ‘‘company they keep’’. Thus an operator $\hat{\rho}$ is positive if and only if $\text{Tr}(\hat{\rho}\hat{\rho}') \geq 0$ for all positive operators $\hat{\rho}'$. Transcription of this statement to the language of s -ordered

quasiprobabilities involves a dual pair of quasiprobabilities $\Lambda_s(\alpha; \hat{\rho})$ and $\Lambda_{-s}(\alpha; \hat{\rho})$ and reads : $\Lambda_s(\alpha; \hat{\rho})$ is a s -ordered quasiprobability (i.e., it corresponds to a positive operator) if and only if

$$\int \Lambda_s(\alpha; \hat{\rho}) \Lambda_{-s}(\alpha; \hat{\rho}') d^2\alpha \geq 0, \quad (8)$$

for every $(-s)$ -ordered quasiprobability $\Lambda_{-s}(\alpha; \hat{\rho}')$. It is in this sense that the s -ordered quasiprobabilities $\Lambda_s(\alpha; \hat{\rho})$ and the $(-s)$ -ordered quasiprobabilities $\Lambda_{-s}(\alpha; \hat{\rho})$ are mutually dual. In particular, the quasiprobability of Wigner is self-dual, and this is the only self-dual s -ordered quasiprobability, while the quasiprobabilities Q and P are mutually dual.

Now, by definition, the map $\Gamma_{s,a^{-1}}$ is positive if and only if $a^2 \Lambda_s(a\alpha; \hat{\rho})$ is an s -ordered quasiprobability for every s -ordered quasiprobability $\Lambda_s(\alpha; \hat{\rho})$. In view of (8), the necessary and sufficient condition for $\Gamma_{s,a^{-1}}$ to be positive is that

$$\int \Lambda_s(a\alpha; \hat{\rho}) \Lambda_{-s}(\alpha; \hat{\rho}') d^2\alpha \geq 0, \quad (9)$$

for every s -ordered quasiprobability $\Lambda_s(\alpha; \hat{\rho})$ and $(-s)$ -ordered quasiprobability $\Lambda_{-s}(\alpha; \hat{\rho}')$. Since $a \neq 0$, the last stipulation (9) is the same as the condition that

$$\int \Lambda_s(\alpha; \hat{\rho}) \Lambda_{-s}(a^{-1}\alpha; \hat{\rho}') d^2\alpha \geq 0, \quad (10)$$

for every s -ordered quasiprobability $\Lambda_s(\alpha; \hat{\rho})$ and $[(-s)$ -ordered quasiprobability $\Lambda_{-s}(\alpha; \hat{\rho}')$. But this condition precisely constitutes the assertion that the scaling map $\tilde{\Gamma}_{s,a^{-1}}$ is a positive map, and thus completes proof of the proposition.

It is well known that the Gaussian function $\exp[-\frac{1}{2}b|\xi|^2]$ qualifies to be the characteristic function of some s -ordered quasiprobability if and only if $b \geq 1 - s$. This is basically a statement of the Heisenberg uncertainty principle. Indeed, saturation of this inequality corresponds to the ground state of the oscillator $\hat{\rho} = |0\rangle\langle 0|$, and this is true for every $-1 \leq s \leq 1$. The case $b > 1 - s$ corresponds to thermal states with the temperature of the state being a monotone increasing function of the difference $b - (1 - s) = b + s - 1$.

Subjecting the ground state to the transformation $\Gamma_{s,a}$ one readily concludes, for all $s \neq 1$, that a necessary condition for this map to be positive is that $|a| \geq 1$. Note that this requirement is independent of the value of s (excluding $s = 1$). Applying this requirement on the vacuum state to both the cases s and $-s$ we conclude, in view of the duality established in Proposition 1, the following.

Proposition 2. The scaling map $\Gamma_{s,a}$ is not a positive map for $(-1 < s < 1, |a| \neq 1)$, $(s = -1, |a| < 1)$, and $(s = 1, |a| > 1)$.

It is interesting that all quasiprobabilities, other than possibly P and Q corresponding to $s \pm 1$, respectively, are on the same footing as far as positivity of the scaling transformation is concerned. That Proposition 2 is silent on P (with $|a| < 1$) and hence on its dual Q (with $|a| > 1$) is due to the fact that in the case $s = 1$ the s -ordered characteristic function of the ground state is a constant. These two cases therefore need closer examination.

It turns out that help is at hand from the detailed study of the properties of bosonic Gaussian channels and their respective

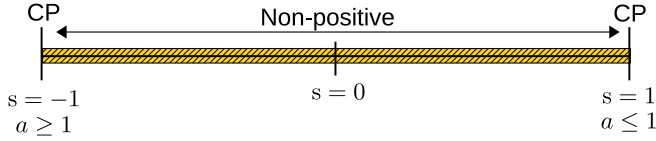


FIG. 1. The positivity properties of the scaling maps as depicted on the s axis. For $s \in (-1, 1)$ and $|a| \neq 1$, the scaling maps are nonpositive. This is also true for $(s = -1, |a| < 1)$ and $(s = 1, |a| > 1)$. The scaling maps are completely positive for the cases $(s = -1, |a| > 1)$, $(s = 1, |a| < 1)$, including the trivial situation of $|a| = 1$ for all s .

operator-sum representations [16], and we have the following proposition.

Proposition 3 (Theorem 10 of [16]). The scaling map $\Gamma_{-1,a} : \mathcal{Q}(\alpha; \Gamma_{-1,a}(\hat{\rho})) = a^{-2} \mathcal{Q}(a^{-1}\alpha; \hat{\rho})$, $a > 1$, is a trace-preserving completely positive map and corresponds to the quantum-limited amplifier channel $\mathcal{C}_2(a)$.

This leaves the final case of the scaling map $\Gamma_{1,a}$ with $a < 1$ and this is addressed in the following proposition.

Proposition 4 (Theorem 6 of [16]). The scaling map $\Gamma_{1,a} : P(\alpha; \Gamma_{1,a}(\hat{\rho})) = a^{-2} P(a^{-1}\alpha; \hat{\rho})$, $0 < a < 1$, is a completely positive trace-preserving map and corresponds to the quantum-limited attenuation channel $\mathcal{C}_1(a)$.

We have thus found the complete answer to the problem we set out to study : (1) for none of the $s \in (-1, 1)$ is $\Gamma_{s,a}$ a positive map for any value of the scale parameter a other than the trivial values $a = \pm 1$; (2) for the special case $s = -1$, the map is not positive if $|a| < 1$, and completely positive if $|a| \geq 1$; and (3) for the dual special case $s = 1$, the map is not positive if $|a| > 1$, and completely positive if $|a| \leq 1$. In other words, there is no positive but not completely positive map (entanglement witness) of the scaling type. The results of this section are summarized in the following theorem, and also depicted pictorially in Fig. 1.

Theorem 1. Scaling maps $\Gamma_{s,a}$ are completely positive for $(s \in [-1, 1], |a| = 1)$, $(s = -1, |a| > 1)$, and $(s = 1, |a| < 1)$. For all other values of (s, a) the scaling maps $\Gamma_{s,a}$ are not even positive.

One may apply the scaling map followed (or preceded) by the transpose map [10] $(q, p) \rightarrow (q, -p)$, i.e., $\alpha \rightarrow \alpha^*$, on one subsystem of a bipartite state to obtain what is called the partial scaling map in Ref. [9]. From Theorem 1 and the invertibility of the transpose map we have that partial scaling, with nontrivial scale parameter $|a| \neq 1$, is not a positive map for any $s \in (-1, 1)$; for $s = -1$, the map is not positive if $|a| < 1$, and positive but not completely positive if $|a| \geq 1$; and for $s = 1$, the map is not positive if $|a| > 1$, and positive but not completely positive if $|a| \leq 1$. But in both the cases $s = -1, |a| > 1$ and $s = 1, |a| < 1$ the partial scaling map can be shown to be weaker than the transpose map in its capacity to witness entanglement. Further implications of Theorem 1 will be developed in the following section.

III. SCALING MAPS FROM THE VIEWPOINT OF CHARACTERISTIC FUNCTIONS

By Theorem 1 we thus have two families of completely positive maps of the scaling type as detailed in Propositions 3

and 4. These bosonic Gaussian channels are traditionally described by the transformation the Wigner characteristic function χ_0 suffers through the channel [13,16,18]. So, rewriting Eq. (6) at the level of the symmetric-ordered characteristic function ($s = 0$) we have

$$\chi_0(\xi; \Gamma_{s,a}[\hat{\rho}]) = \chi_0(a\xi; \hat{\rho}) \exp[s(a^2 - 1)|\xi|^2/2]. \quad (11)$$

Note that we now include $a = 0$ in the analysis as this can be interpreted as the constant map with a one-dimensional trivial output space. It turns out that working at the level of the symmetric-ordered characteristic functions helps to identify the reason the scaling maps fail to be positive in the corresponding parameter ranges. It is useful to introduce what we call the classical noise map given by $\mathcal{B}_2(b), b \in \mathbb{R}$ with its action at the level of the characteristic function given by

$$\mathcal{B}_2(b) : \chi_0(\xi; \mathcal{B}_2(b)[\hat{\rho}]) = \chi_0(\xi; \hat{\rho}) \exp[-b|\xi|^2/2], \quad (12)$$

with the map being completely positive for $b \geq 0$ and nonpositive for $b < 0$. Note that one can replace χ_0 with χ_s throughout in Eq. (12) to describe the map $\mathcal{B}_2(b)$. In terms of the classical noise map we can rewrite the scaling maps using Eq. (11):

$$\Gamma_{s,a} = \mathcal{B}_2(s(1 - a^2)) \circ \Gamma_{0,a}. \quad (13)$$

We now consider the case $|a| < 1$. For $s = 0$, we showed that the scaling map is not positive as can be checked by the action of the map on the vacuum state. For $s < 0$ we have that the map in Eq. (11) decomposes into $\chi_0(a\xi; \hat{\rho}) \times \exp[|s|(1 - a^2)|\xi|^2/2]$, i.e., a composition of two nonpositive maps $\mathcal{B}_2(-|s|(1 - a^2)) \circ \Gamma_{0,a}$. For the case $s \in [0, 1)$ we have that the map in Eq. (11) decomposes into $\chi_0(a\xi; \hat{\rho}) \exp[-s(1 - a^2)|\xi|^2/2]$ which is the composition of $\mathcal{B}_2(s(1 - a^2)) \circ \Gamma_{0,a}$, a nonpositive map and a completely positive map. However, the product results in a nonpositive map as can be checked on the vacuum state. In other words, the Gaussian noise term is not sufficient to compensate for the scaling of the characteristic function. The situation changes abruptly, however, for the case $s = 1$ when the entire map now corresponds to a completely positive map as given in Proposition 4.

Similarly, for the case $|a| > 1$ we have that for $s > 0$ Eq. (11) reduces to $\mathcal{B}_2(-s(a^2 - 1)) \circ \Gamma_{0,a}$, both being nonpositive maps. For the case $s \in (-1, 0]$ the transformation in Eq. (11) can be viewed as a product of maps corresponding to $\mathcal{B}_2(|s|(a^2 - 1)) \circ \Gamma_{0,a}$, which is a product of a nonpositive map and a completely positive map. However, the combined map is nonpositive due to Theorem 1. Similar to the earlier case we have that at $s = -1$ there is an abrupt transition into a completely positive map as detailed in Proposition 3.

So we have presented a complete description of the decomposition of the scaling maps at the level of the symmetric-ordered characteristic function. This brings out in a transparent manner the way in which the scaling maps fail to be positive in the corresponding $s - a$ parameter space and this is depicted in Fig. 2.

Amplification and attenuation maps below complete positivity threshold

The attenuation map is defined as $\mathcal{C}_1(a; b) := \mathcal{B}_2(b) \circ \Gamma_{0,a}$, $|a| < 1, b > 0$. It is well known that the attenuation map

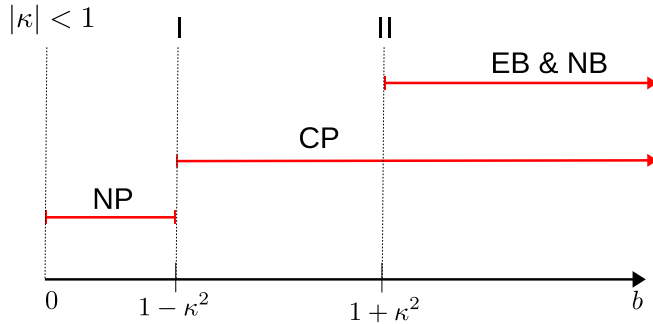


FIG. 4. Phase diagram for the attenuation map $C_1(\kappa, b)$. For a given κ ($|\kappa| < 1$), the map is nonpositive for $b < 1 - \kappa^2$, completely positive for $b \geq 1 - \kappa^2$, and entanglement breaking as well as nonclassicality breaking for $b \geq 1 + \kappa^2$. In a sense, phase transitions occur at $b = 1 - \kappa^2$ (labeled as I) and at $b = 1 + \kappa^2$ (labeled as II).

ranges for the scale parameter are mutually reciprocal; and (iii) the Kraus operators of the two families are mutually dual, being simply related by Hermitian conjugation.

IV. DISCUSSION

We have presented a definitive analysis of uniform phase space scaling maps $\Gamma_{s,a}$ with regard to positivity and summarized the result compactly in Fig. 2 resembling a phase diagram. We find that apart from a measure-zero set of points in the $a - s$ parameter space where the maps are completely positive, the scaling maps are nonpositive. There are no scaling maps of the positive but not completely positive type, and thus cannot be used as an entanglement witness. These properties were obtained by studying the induced action of the scaling maps at the level of s -ordered quasiprobabilities and a certain duality we established among the scaling maps.

As an almost unintended fallout, we also studied the behavior of the amplification and attenuation maps with respect to their classical Gaussian noise parameter taken below the complete positivity threshold. For this purpose, it was useful to view the scaling maps from their induced action at the level of the characteristic functions. Our main finding is that below the complete positivity threshold, the attenuation and amplification maps are nonpositive. As noted earlier, this dual pair of noiseless amplifier and attenuator channels are of fundamental importance due to their practical relevance as in modeling fiber optical communication. A natural question that follows is the behavior of other single-mode Gaussian channels below their corresponding complete positivity or quantum-limited noise threshold. The extension to the multimode case is entirely open.

It may be seen that nothing more interesting can be achieved by replacing the uniform scaling by a more general scaling matrix K acting on the vector $(q, p)^T$. For every nonsingular 2×2 matrix K , there exist symplectic matrices $S_1, S_2 \in \text{Sp}(2, \mathbb{R})$ such that

$$S_1 K S_2 = a \mathbb{I} \quad \text{or} \quad a \sigma_3 \quad (14)$$

according as $\det K$ is positive or negative. This simply corresponds to pre- and postprocessing the given uniform scaling map by unitary (metaplectic) transformations $U(S_1), U(S_2)$,

corresponding to symplectic matrices S_1, S_2 . But unitary processes do not alter the positivity properties of the map.

There is currently considerable interest in non-Gaussian states as potentially advantageous resources in quantum information processing tasks [17,31–40] and one measure of non-Gaussianity based on the departure of the Q distribution of a given state from the closest Gaussian was proposed in [41]. The measure has the property that a Fock state $|m\rangle$ and the m -photon added thermal state [42] possess the same value of non-Gaussianity for every m , since their respective Q functions are related by a uniform scaling of the phase space variables [41] that preserves the shape of the distribution (the temperature being a monotone function of the scale parameter). There are also other possible applications of scaling maps such as to the study of nonclassicality of optical fields like in [43].

We conclude with a remark on Ref. [9] with respect to scaling of the Planck constant, already referred to in the Introduction. That uniform scaling of the Wigner distribution corresponds to scaling of the Planck constant, $\hbar \rightarrow \hbar' = a^2 \hbar$ [44], is true: one can define Wigner distributions for any chosen numerical value of the Planck constant. Let Ω^{\hbar} denote the convex set of Wigner distributions corresponding to a chosen numerical value of the Planck constant \hbar . Now, expecting the scaling map to be a positive map is to expect that the union $\Omega^{(\hbar_1)} \cup \Omega^{(\hbar_2)}$ of $\Omega^{(\hbar_1)}$ and $\Omega^{(\hbar_2)}$ is also a valid set of Wigner distribution. Indeed, this expectation extends to the convex hull of $\Omega^{(\hbar_1)} \cup \Omega^{(\hbar_2)}$, but its untenability can be settled without going that far. Let $\hbar^{(1)}$ be the larger of \hbar_1, \hbar_2 . Let $W^{\hbar_1}(q, p; |1\rangle\langle 1|)$ be the Wigner distribution of the first excited state corresponding to Planck constant \hbar_1 , and let $W^{\hbar_2}(q, p; |0\rangle\langle 0|)$ be that of the ground state corresponding to \hbar_2 . Since $W^{\hbar_1}(q, p; |1\rangle\langle 1|)$ is negative over a circle around the origin of area proportional to \hbar_1 , and since $W^{\hbar_2}(q, p; |0\rangle\langle 0|)$ centered at the origin is narrower than $W^{\hbar_1}(q, p; |1\rangle\langle 1|)$, it is clear that

$$\int dq dp W^{\hbar_1}(q, p; |1\rangle\langle 1|) W^{\hbar_2}(q, p; |0\rangle\langle 0|) < 0. \quad (15)$$

This shows that while any numerical value of Planck constant is acceptable, two distinct values of Planck's constant cannot coexist in the Wigner scheme of things. In a sense, it is fortunate that the Wigner scheme protects itself against a perhaps awkward question of the following type: if $W(q, p; \hat{\rho}) = \lambda W_1(q, p; \hat{\rho}) + (1 - \lambda) W_2(q, p; \hat{\rho})$, $0 < \lambda < 1$ with $W_j(q, p; \hat{\rho}) \in \Omega^{(\hbar_j)}$ is a Wigner distribution, to what numerical value of \hbar would $W(q, p; \hat{\rho})$ correspond to?

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