Weak measurements, quantum-state collapse, and the Born rule

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Projective measurement is used as a fundamental axiom in quantum mechanics, even though it is discontinuous and cannot predict which measured operator eigenstate will be observed in which experimental run. The probabilistic Born rule gives it an ensemble interpretation, predicting proportions of various outcomes over many experimental runs. Understanding gradual weak measurements requires replacing this scenario with a dynamical evolution equation for the collapse of the quantum state in individual experimental runs. We revisit the quantum trajectory framework that models quantum measurement as a continuous nonlinear stochastic process. We describe the ensemble of quantum trajectories as noise fluctuations on top of geodesics that attract the quantum state towards the measured operator eigenstates. In this effective theory framework for the ensemble of quantum trajectories, the measurement interaction can be specific to each system-apparatus pair, a context necessary for understanding weak measurements. Also in this framework, the constraint to reproduce projective measurement as per the Born rule in the appropriate limit requires that the magnitudes of the noise and the attraction are precisely related, in a manner reminiscent of the fluctuation-dissipation relation. This relation implies that both the noise and the attraction have a common origin in the underlying measurement interaction between the system and the apparatus. We analyze the quantum trajectory ensemble for the scenarios of quantum diffusion and binary quantum jump, and show that the ensemble distribution is completely determined in terms of a single evolution parameter. This trajectory ensemble distribution can be tested in weak measurement experiments. We also comment on how the required noise may arise in the measuring apparatus.

I. BACKGROUND

The axiomatic formulation of quantum mechanics has two distinct dynamical mechanisms for evolving a state. One is unitary evolution, specified by the Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle, \ i\frac{d}{dt}\rho = [H,\rho]. \quad (1)$$

It is continuous, reversible, and deterministic. The other is the von Neumann projective measurement, which gives one of the eigenvalues of the measured observable as the measurement outcome and collapses the state to the corresponding eigenvector. With P_i denoting the projection operator for the eigenstate $|i\rangle$,

$$|\psi\rangle \longrightarrow P_i|\psi\rangle/|P_i|\psi\rangle|, \quad (2)$$

$$P_i = P_i^{\dagger}, \ P_i P_j = P_i \delta_{ij}, \ \sum_i P_i = I. \quad (3)$$

This change is discontinuous, irreversible, and probabilistic in the choice of "i." It is consistent on repetition, i.e., a second measurement of the same observable on the same system gives the same result as the first one.

Both these evolutions, not withstanding their dissimilar properties, take pure states to pure states. They have been experimentally verified so well that they are accepted as basic axioms in the theoretical formulation of quantum mechanics. Nonetheless, the formulation misses something: *While the complete set of orthogonal projection operators $\{P_i\}$ is fixed by the measured observable, only one "i" occurs in a particular*

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experimental run, and there is no prediction for which "i" that would be.

What appears instead in the formulation is the probabilistic Born rule, requiring an ensemble interpretation for the outcomes. Measurement of an observable on a collection of identically prepared quantum states gives

$$\text{prob}(i) = \langle\psi|P_i|\psi\rangle = \text{Tr}(P_i\rho), \ \rho \longrightarrow \sum_i P_i \rho P_i. \quad (4)$$

This rule evolves pure states to mixed states. All predicted quantities are expectation values obtained as averages over many experimental runs. The mixed state also necessitates a density matrix description, instead of a ray in the Hilbert space description for a pure state.

A. Environmental decoherence

Over the years, many attempts have been made to combine the two distinct quantum evolution rules in a single framework. Although the problem of which "i" will occur in which experimental run has remained unsolved, understanding of the "ensemble evolution" of a quantum system has been achieved in the framework of environmental decoherence. This framework assumes that both the system and its environment (which includes the measuring apparatus) are governed by the same set of basic quantum rules. The essential difference between the system and the environment is that the degrees of freedom of the system are observed while those of the environment are not. Consequently, all the unobserved degrees of freedom need to be "summed over" to determine how the remaining observed degrees of freedom evolve.

Interactions between the system and its environment, with a unitary evolution for the whole universe, entangles the observed system degrees of freedom with the unobserved environmental degrees of freedom. The extent of this entanglement

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can be controlled somewhat by designing experiments where the system mostly interacts with the measuring apparatus and has little direct interaction with the rest of the environment. When the unobserved degrees of freedom are summed over, a pure but entangled state for the whole universe reduces to a mixed density matrix for the system:

$$|\psi\rangle_{SE} \longrightarrow U_{SE}|\psi\rangle_{SE}, \ \rho_S = \text{Tr}_E(\rho_{SE}), \ \rho_S^2 \neq \rho_S. \quad (5)$$

In general, the evolution of a reduced density matrix is linear, Hermiticity preserving, trace preserving, and positive, but not unitary. Such a superoperator evolution can be expressed in the Kraus decomposition form

$$\rho_S \longrightarrow \sum_\mu M_\mu \rho_S M_\mu^\dagger,$$

$$M_\mu = {}_E\langle\mu|U_{SE}|0\rangle_E, \ \sum_\mu M_\mu^\dagger M_\mu = I, \quad (6)$$

using a complete basis for the environment $\{|\mu\rangle_E\}$. Since the reduced density matrix has the same structure as the probabilistic ensemble of classical statistical mechanics, it can be described in the same language. But, a "quantum jump" mechanism is still needed to explain how an entangled system-environment state collapses to an unentangled system eigenstate in every single experimental run.

Generically, the environment has a much larger number of degrees of freedom than the system. Then, in the Markovian approximation which assumes that information leaked from the system does not return, the evolution of the reduced density matrix for the system can be expressed by the Lindblad master equation [1,2]

$$\frac{d}{dt}\rho_S = i[\rho_S, H] + \sum_\mu \mathcal{L}[L_\mu]\rho_S, \quad (7)$$

$$\mathcal{L}[L_\mu]\rho_S = L_\mu \rho_S L_\mu^\dagger - \frac{1}{2}\rho_S L_\mu^\dagger L_\mu - \frac{1}{2}L_\mu^\dagger L_\mu \rho_S. \quad (8)$$

The terms on the right-hand side involving sum over μ modify the unitary Schrödinger evolution, while $\text{Tr}(d\rho_S/dt) = 0$ preserves the total probability. When $H = 0$, the fixed point of the evolution is a diagonal ρ_S, in the basis that diagonalizes $\{L_\mu\}$. This preferred basis is determined by the system-environment interaction. (When there is no diagonal basis for $\{L_\mu\}$, the evolution leads to equipartition, i.e., $\rho_S \propto I$.) Furthermore, the off-diagonal elements of ρ_S decay, due to destructive interference among environmental contributions with varying phases (arising from a large number of random elastic scattering events), which is known as decoherence.

This modification of a quantum system's evolution, due to its interaction with the unobserved environmental degrees of freedom, provides a proper ensemble description, and a quantitative understanding of how the off-diagonal elements of ρ_S decay [3,4]. Still, the "measurement problem" is not fully solved until we find the quantum jump process that can predict outcomes of individual experimental runs, and that forces us to go beyond the dynamics of Eq. (1).

B. Continuous stochastic measurement

The von Neumann interaction is usually taken to be the first step of the measurement process. It is continuous and deterministic, and creates perfectly entangled "Schrödinger cat" states between the measurement eigenstates of the system and the pointer states of the apparatus (which is part of the system's environment):

$$|\psi\rangle_S = \sum_i c_i |i\rangle_S, \ |\psi\rangle_S|0\rangle_E \longrightarrow \sum_i c_i |i\rangle_S|\tilde{i}\rangle_E. \quad (9)$$

To complete the measurement process, it needs to be supplemented by the probabilistic quantum jump that selects a particular $|i\rangle$ and collapses the reduced ρ_S to a projection operator. Although the physical mechanism behind quantum jump is not understood, it is common to attribute quantum jump to interactions of a system with its surroundings. With this postulate, we can define which system-environment interactions are measurement interactions: *a measurement interaction is the one in which the apparatus does not remain, for whatever reasons, in a superposition of pointer states.*

The quantum jump can be realized via a continuous stochastic process, while retaining the ensemble interpretation. The familiar method is to add noise to a deterministic evolution, converting it into a Langevin equation [5–7]. Such a realization is strongly constrained by the properties of quantum measurement. To ensure repeatability of measurement outcomes, the measurement eigenstates need to be fixed points of the evolution. The attraction towards the eigenstates as well as the noise have to vanish at the fixed points, which requires the evolution dynamics to be either nonlinear or nonunitary. Furthermore, lack of simultaneity in special relativity must not conflict with outcome probabilities in multipartite measurements. For instance, one can consider pausing (or even abandoning) measurement part of the way along, and that must not conflict with the consistency of the results. Since the Born rule is fully consistent with special relativity, a solution is to demand that the Born rule be satisfied at every instant of the measurement process, when one averages over the noise. It is indeed remarkable that such a continuous stochastic process for quantum measurement exists [6]. It uses a precise combination of the attraction towards the eigenstates and unbiased white noise to reproduce the Born rule. (Some variations of the stochastic process away from this specific form have been studied [5], but they fail to satisfy all the constraints [8].)

The relation between the attraction towards the eigenstates and the noise, needed to make the Born rule a constant of evolution in Ref. [6], implies that the environmental degrees of freedom contribute to either both the attraction and the noise (these degrees of freedom would be considered the apparatus) or to neither of them (these degrees of freedom can be ignored). This division strongly indicates a common origin for the deterministic and the stochastic contributions to the measurement evolution, quite reminiscent of the fluctuation-dissipation theorem of statistical mechanics which is a consequence of both diffusion and viscous damping arising from the same underlying molecular scattering. Such an intrinsic property of quantum measurement dynamics would be an important clue to figuring out what may lie beyond, an underlying theory of which quantum measurement would be an effective process.

Understanding quantum measurement as a stochastic process is the focus of our investigation, and we analyze its ingredients in detail in Sec. II. With the technological progress

in making quantum devices, such an analysis is not just a formal theoretical curiosity, but is also a necessity for increasing accuracy of quantum control and feedback [4]. A practical situation is that of the weak measurement [9], where information about the measured observable is extracted from the system at a slow rate. Such a stretching out of the time scale would allow one to monitor how the system state collapses to an eigenstate of the measured observable, and to track properties of the intermediate states created along the way by an incomplete measurement. Knowledge of what really happens in a particular experimental run (and not the ensemble average) would be invaluable in making quantum devices more efficient and stable.

C. Beyond quantum mechanics

The stochastic measurement process does provide a continuous interpolation of projective measurement. But, its nonlinear dynamics is distinct from the unitary Schrödinger evolution, and one wonders how it may arise as an effective description from a theory more fundamental than quantum mechanics. Over the years, a variety of theoretical approaches have been proposed to either solve or bypass the quantum measurement problem.

Some of the approaches that go beyond quantum mechanics are physical, e.g., introduction of hidden variables with novel dynamics or ignored interactions with known dynamics. Examples include Bohmian mechanics [10], Ghirardi-Rimini-Weber (GRW) and Continuous-Spontaneous-Localization (CSL) spontaneous collapse mechanisms [11,12], and modification of quantum rules due to gravity [13]. Some others philosophically question what is real and what is observable, in principle as well as by human beings with limited capacity. Examples include the "many worlds" interpretation [14] that assigns a distinct world (i.e., an evolutionary branch) to each probabilistic outcome, and the consistent histories formalism [15]. Although these attempts are not theoretically inconsistent, none of them have been positively verified by experiments; only bounds exist on their parameters.

In this work, we reanalyze the quantum trajectory formalism for state collapse (earlier reviews can be found in Refs. [16,17]) to achieve a deeper understanding of its dynamics. It is a particular case of the class of stochastic collapse models that add a measurement driving term and a random noise term to the Schrödinger evolution of Eq. (1) [12]. We treat these terms in an effective theory approach, without assuming a specific collapse basis (e.g., energy or position basis) or a specific collapse interaction (e.g., gravity or some other universal interaction). This approach allows us to address the possibility that the collapse process is nonuniversal, and the signal amplification during the system-apparatus measurement interaction may be responsible for it. The GRW and CSL models have not explored this possibility much, and have typically focused on a particular collapse basis with a particular collapse interaction.

Our approach is motivated by recent experimental advances in monitoring quantum evolution during weak measurements on superconducting transmon qubits [18,19], where the collapse basis as well as the system-apparatus interaction strength can be varied by changing the circuit parameters

and without changing the apparatus size. With suitable choice of parameters, quantum trajectories interpolating from the initial state to the final projected state have been observed [19], setting up a stage where the validity of the stochastic collapse paradigm during measurement can be experimentally tested in detail. Such effective theory tests would then impose restrictions on any extension of the standard quantum theory, and that is what we aim for.

We formulate our model in the next section, separating the quantum collapse trajectories into a geodesic evolution part and a fluctuating part on top of it. This separation allows us to point out that the Born rule is equivalent to a fluctuation-dissipation type of relation between the two parts, and we demonstrate that for two different types of noise in Secs. III and IV. We conclude with a discussion on the implications of this property and the possible physical origin of the noise.

II. QUANTUM GEODESIC COLLAPSE

In what follows, to keep the analysis simple, we concentrate only on the evolution due to the effective system-apparatus interaction that leads to measurement; other contributions to the system evolution can be added later when needed. It is also convenient to abbreviate ρ_S as ρ. We now proceed to construct the complete quantum collapse dynamics in steps.

A. A single geodesic trajectory

Assuming that the projective measurement results from a continuous geodesic evolution of an initially pure quantum state to an eigenstate $|i\rangle$ of the measured observable, i.e., a great circle on the unit sphere in the Hilbert space, one arrives at the nonlinear evolution equation [20]

$$\frac{d}{dt}\rho = g[\rho P_i + P_i \rho - 2\rho \operatorname{Tr}(P_i \rho)]. \tag{10}$$

Here, t is the "measurement time," the coupling g represents the strength of the system-apparatus interaction, while gt is dimensionless. This simple nonlinear evolution equation describing an individual quantum trajectory has several noteworthy properties [21].

(a) In addition to maintaining $\operatorname{Tr}(\rho) = 1$, the evolution takes pure states to pure states. $\rho^2 = \rho$ implies

$$\frac{d}{dt}(\rho^2 - \rho) = \rho\frac{d}{dt}\rho + \left(\frac{d}{dt}\rho\right)\rho - \frac{d}{dt}\rho = 0. \tag{11}$$

Thus, the component of the state along P_i grows at the expense of the other orthogonal components.

(b) Each projective measurement outcome is the fixed point of the deterministic evolution

$$\frac{d}{dt}\rho = 0 \quad \text{at} \quad \rho_i^* = P_i \rho P_i / Tr(P_i \rho). \tag{12}$$

The fixed-point nature of the evolution makes the measurement outcome consistent on repetition.

(c) In a bipartite setting (which includes the decoherence scenario), the complete set of projection operators can be selected as $\{P_i\} = \{P_{i_1} \otimes P_{i_2}\}$, with $\sum_i P_{i_k} = I_k$. Since the evolution is linear in the projection operators, a sum over the unobserved projection operators and a partial trace over the unobserved degrees of freedom produces the same

equation (and hence the same fixed point) for the reduced density matrix for the system, as long as g is independent of the environment. This decoupling from the environment forbids any possibility of superluminal signaling. Moreover, the evolution purifies the state; for example, a qubit state in the interior of the Bloch sphere evolves to a fixed point on its surface.

(d) For pure states, the geodesic evolution equation is just [using the notation of Eq. (8)]

$$\frac{d}{dt}\rho = -2g\mathcal{L}[\rho]P_i. \tag{13}$$

Compared to Eq. (7), here the Lindblad operator acts on the pointer state, the density matrix plays the role of L_μ, and the sign is reversed. This structure hints at an action-reaction relationship between the processes of decoherence and collapse. Note here that both ρ and P_i are projection operators, and after the von Neumann interaction creates a symmetric entangled state of the system and the apparatus as in Eq. (9), it is a matter of subjective choice to consider whether the system decoheres the apparatus or the apparatus decoheres the system. In particular, P_i can be looked upon as the apparatus state influenced by the system operator ρ. Adding $0 = \mathcal{L}[\rho]P_i - \mathcal{L}[\rho]P_i$ to the joint system-apparatus evolution equation, one can then envision the following breakup during the measurement process: $\mathcal{L}[\rho]P_i$ combined with the apparatus dynamics decoheres the apparatus state P_i (it cannot remain in superposition by definition), and the equal and opposite $-\mathcal{L}[\rho]P_i$ combined with the system dynamics collapses the system state ρ. Details of such a scenario remain to be worked out.

(e) The limit $gt \to \infty$ corresponds to projective measurement, while small gt values describe weak measurements. Asymptotic convergence to the fixed point is exponential, $||\rho - P_i|| \sim e^{-2gt}$ as $t \to \infty$, similar to the charging of a capacitor.

These properties make Eq. (10) a legitimate candidate for describing the collapse of a quantum state during measurement, modeling the single quantum trajectory specific to a particular experimental run. It represents a superoperator that preserves Hermiticity, trace, and positivity, but is nonlinear.

B. Ensemble of geodesic trajectories

We next need a separate criterion for selection of P_i to reproduce the stochastic ensemble interpretation of quantum measurement. This choice of "quantum jump" requires a particular P_i to be picked with probability $\text{Tr}[P_i\rho(t = 0)]$ as per the Born rule. Picking one of the P_i at the start of the measurement, and leaving it unaltered thereafter, is unsuitable for gradual weak measurements, and we look for other ways to combine the evolution trajectories for different P_i.

Let w_i be the weight of the evolution trajectory for P_i, with $\sum_i w_i = 1$. We want to find real $w_i(t)$, as some functions of $\rho(t)$, that reproduce the well-established quantum behavior. The geodesic trajectory averaged evolution of the density matrix during measurement is

$$\frac{d}{dt}\rho = \sum_i w_i g[\rho P_i + P_i \rho - 2\rho \, \text{Tr}(P_i\rho)]. \tag{14}$$

Irrespective of the choice for w_i, this evolution maintains the properties (a)–(d) described in the previous subsection, i.e., preservation of purity, fixed-point nature of all P_i, decoupling from environment, and a role-reversal relation with Lindblad operators.

With the decomposition $\rho = \sum_{jk} P_j\rho P_k$, the projected components of the density matrix evolve as

$$\frac{d}{dt}(P_j\rho P_k) = P_j\rho P_k g\left[w_j + w_k - 2\sum_i w_i Tr(P_i\rho)\right]. \tag{15}$$

Independent of the choice of $\{w_i\}$, we have the identity

$$\frac{2}{P_j\rho P_k}\frac{d}{dt}(P_j\rho P_k) = \frac{1}{P_j\rho P_j}\frac{d}{dt}(P_j\rho P_j)$$
$$+ \frac{1}{P_k\rho P_k}\frac{d}{dt}(P_k\rho P_k), \tag{16}$$

with the consequence that the diagonal projections of ρ completely determine the evolution of all the off-diagonal projections. For an n-dimensional quantum system, therefore, the evolution has only $n - 1$ independent degrees of freedom. For one-dimensional projections, $P_j\rho(t)P_j = d_j(t)P_j$ with $d_j \geqslant 0$, we obtain

$$P_j\rho(t)P_k = P_j\rho(0)P_k\left[\frac{d_j(t)d_k(t)}{d_j(0)d_k(0)}\right]^{1/2}. \tag{17}$$

In particular, phases of the off-diagonal projections $P_j\rho P_k$ do not evolve, in sharp contrast to what happens during decoherence. Also, their asymptotic values, i.e., $P_j\rho(t \to \infty)P_k$, may not vanish, whenever more than one diagonal $P_j\rho(t \to \infty)P_j$ remains nonzero.

It is easily seen that when all the w_i are equal, no information is extracted from the system by the measurement and ρ does not evolve. More generally, the diagonal projections evolve according to

$$\frac{d}{dt}d_j = 2g d_j\left(w_j - \sum_i w_i d_i\right). \tag{18}$$

Here, with $\sum_i d_i = 1$, $\sum_i w_i d_i \equiv w_{\text{av}}$ is the weighted average of $\{w_i\}$. Clearly, the diagonal projections with $w_j > w_{\text{av}}$ grow and the ones with $w_j < w_{\text{av}}$ decay. Any d_j that is zero initially does not change, and the evolution is therefore restricted to the subspace spanned by all the $P_j\rho(t = 0)P_j \neq 0$. These features are stable under small perturbations of the density matrix.

A naive guess for the trajectory weights is the "instantaneous Born rule," i.e., $w_j = w_j^{IB} \equiv \text{Tr}[\rho(t)P_j]$ throughout the measurement process. It avoids logical inconsistency in weak measurement scenarios, where one starts the measurement, pauses somewhere along the way, and then restarts the measurement. In this situation, the geodesic trajectory averaged evolution is

$$\frac{d}{dt}(P_j\rho P_k) = P_j\rho P_k g\left[w_j^{IB} + w_k^{IB} - 2\sum_i \left(w_i^{IB}\right)^2\right]. \tag{19}$$

This evolution converges towards the subspace specified by the largest diagonal projections of the initial $\rho(t = 0)$, i.e., the closest fixed points. It is deterministic too, and differs from Eq. (4). So, $w_j = w_j^{IB}$ is unphysical, and we need to find w_i with stochastic behavior that would reproduce the Born rule.

C. Addition of noise

Instead of heading towards the nearest fixed point, quantum trajectories can be made to wander around and explore other possibilities by adding noise to their dynamics. The combination of geodesics and fluctuations generically appears in variational calculus, easily seen in the path-integral framework for instance. Noisy fluctuations are also expected to contribute to the measurement process [18,19,22]. So, we search for a suitable noise, which when combined with the geodesics already described would reproduce Eq. (4). The existence of such a noise is a hypothesis, to be verified by its explicit construction and evaluation of its consequences. In order to not lose the handsome features of the geodesic trajectories, we make the noise part of the trajectory weights w_i, while retaining $\sum_i w_i = 1$. In describing quantum measurement as a stochastic process, two commonly considered situations are "white noise" and "shot noise," with the corresponding evolution dynamics labeled "quantum diffusion" and "binary quantum jump," respectively [16,17], and we analyze them in turn in the next two sections. It should be noted that our formalism allows us to freely vary the size of the noise, unlike the fixed specific values considered in earlier works, and explore the consequences.

III. QUANTUM DIFFUSION

In the quantum diffusion model, unbiased and uncorrelated noise (i.e., white noise) is added to the geodesic evolution. With a gradual addition of the noise, the quantum trajectories remain continuous but become nondifferentiable. The deterministic evolution equation in the Hilbert space gets converted to a stochastic Langevin-type equation, and we need to find the magnitude of the frequency-independent noise that makes the measurement process consistent with the Born rule.

A. Constraint on white noise

Results of the previous section take a considerably simpler form in case of the smallest quantum system, i.e., the two-dimensional qubit with $|0\rangle$ and $|1\rangle$ as the measurement eigenstates. Evolution of the density matrix during the measurement, Eqs. (18) and (17), is then given by

$$\frac{d}{dt}\rho_{00} = 2g(w_0 - w_1)\rho_{00}\rho_{11}, \tag{20}$$

$$\rho_{01}(t) = \rho_{01}(0)\left[\frac{\rho_{00}(t)\rho_{11}(t)}{\rho_{00}(0)\rho_{11}(0)}\right]^{1/2}. \tag{21}$$

Because of $\rho_{11}(t) = 1 - \rho_{00}(t)$ and $w_1(t) = 1 - w_0(t)$, only one independent variable describes the evolution of the system. Selecting the trajectory weights as addition of real white noise to the "instantaneous Born rule", we have

$$w_0 - w_1 = \rho_{00} - \rho_{11} + \sqrt{S_\xi}\xi. \tag{22}$$
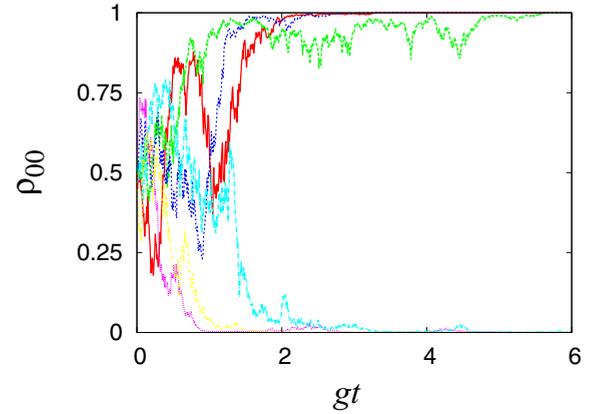
FIG. 1. Individual quantum evolution trajectories for the initial state $\rho_{00} = 0.5$, the measurement eigenstates $\rho_{00} = 0, 1$, and in presence of measurement noise satisfying $gS_\xi = 1$.

Here, $\langle\!\langle\xi(t)\rangle\!\rangle = 0$ is unbiased, $\langle\!\langle\xi(t)\xi(t')\rangle\!\rangle = \delta(t - t')$ fixes the normalization of $\xi(t)$, and S_ξ is the spectral density of the noise.

Equations (20) and (22) define a stochastic differential process on the interval [0,1]. The fixed points at $\rho_{00} = 0, 1$ are perfectly absorbing boundaries where the evolution stops. In general, a quantum trajectory would zigzag through the interval before ending at one of the two boundary points. Some examples of such trajectories are shown in Fig. 1.

Let $P(x)$ be the probability that the initial state with $\rho_{00} = x$ evolves to the fixed point at $\rho_{00} = 1$. Obviously, $P(0) = 0$, $P(0.5) = 0.5$, $P(1) = 1$. Two extreme situations are easy to figure out. When there is no noise, the evolution is governed by the sign of $\rho_{00} - \rho_{11}$ and the trajectory monotonically approaches the fixed point closest to the starting point:

$$S_\xi = 0: \quad P(x) = \theta(x - 0.5). \tag{23}$$

Also, when $\rho_{00} - \rho_{11}$ is negligible compared to the noise, symmetry of the evolution makes both eigenstates equiprobable, i.e., $P(x) = 0.5$ for $S_\xi \to \infty$.

The stochastic evolution equations (20) and (22) are in the Stratonovich form. For further insight into the evolution, and for numerical simulations, it is instructive to convert them into the Itô form that specifies forward evolutionary increments [23]:

$$dp_{00} = 2g\rho_{00}\rho_{11}(\rho_{00} - \rho_{11})(1 - gS_\xi)dt$$
$$+ 2g\sqrt{S_\xi}\rho_{00}\rho_{11}dW. \tag{24}$$

Here, the stochastic Wiener increment $dW = \xi dt$ obeys $\langle\!\langle dW(t)\rangle\!\rangle = 0$, $\langle\!\langle(dW(t))^2\rangle\!\rangle = dt$, and can be modeled as a random walk. The first term on the right-hand side produces drift in the evolution, while the second term gives rise to diffusion.

The evolution with no drift, i.e., the pure Wiener process, is particularly interesting. In that case, after averaging over the stochastic noise, the Born rule is a constant of evolution [5,6]:

$$\langle\!\langle d\rho_{00}\rangle\!\rangle = 0 \Longleftrightarrow gS_\xi = 1. \tag{25}$$

More explicitly, starting at x, one moves forward to $x + \epsilon$ with some probability, moves backward to $x - \epsilon$ with the same probability, and stays put otherwise. On
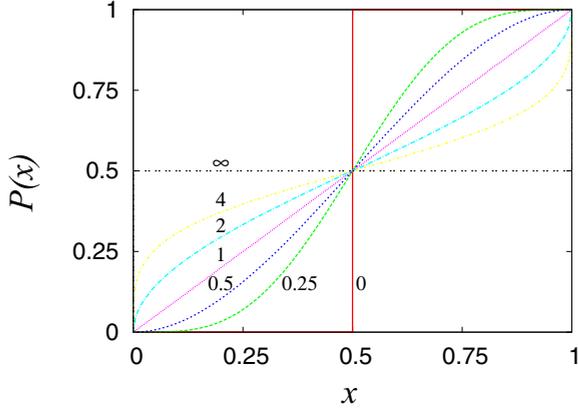
FIG. 2. Probability that the initial qubit state $\rho_{00} = x$ evolves to the measurement eigenstate $\rho_{00} = 1$, for different magnitudes of the measurement noise. The gS_ξ values label the curves.

balancing the probabilities, $P(x) = \alpha[P(x + \epsilon) + P(x - \epsilon)] + (1 - 2\alpha)P(x)$, and we get

$$gS_\xi = 1: \quad P(x + \epsilon) - 2P(x) + P(x - \epsilon) = 0. \quad (26)$$

The general solution, independent of the choice of ϵ, is that $P(x)$ is a linear function of x. Imposing the boundary conditions $P(0) = 0$ and $P(1) = 1$, we obtain $P(x) = x$, which is the Born rule. Note that specific choices of g, ϵ, and α only alter the rate of evolution, but not this final outcome.

Going further, we performed numerical simulations of the stochastic evolution for several values of gS_ξ, and the results are presented in Fig. 2. We used the integrated form of Eq. (20) over a short time step $g\Delta t \ll 1$:

$$\frac{\rho_{00}(t + \Delta t)}{\rho_{11}(t + \Delta t)} = \frac{\rho_{00}(t)}{\rho_{11}(t)} e^{2g\Delta t \overline{w}}, \quad (27)$$

$$\overline{w} = \frac{1}{\Delta t} \int_t^{t+\Delta t} (w_0 - w_1) dt. \quad (28)$$

$\overline{w}$ was generated as a Gaussian random number with mean $\rho_{00}(t) - \rho_{11}(t)$ and variance $S_\xi / \Delta t$. We averaged the results over a million trajectories at each simulation point. The data clearly show the crossover from evolution with no noise to evolution with only noise, and the Born rule behavior appears for $gS_\xi = 1$.

We point out that with $gS_\xi = 1$, Eq. (24) is the same as the corresponding result of Ref. [6]. But, our strategy of breaking up the evolution into geodesic and fluctuating parts allows us to analyze the two contributions separately, e.g., in the fluctuation-dissipation relation described later in Sec. III C, and explore the implications. Also, we can easily extend the result to n-dimensional orthogonal measurements as in Eq. (29).

The preceding results are valid for binary orthogonal measurements on any quantum system, with the replacement $\rho_{ii} \to \mathrm{Tr}(\rho P_i)$. One way to extend them to a larger set of P_i, is to express nonbinary orthogonal projection operators as a product of mutually commuting binary projection operators, and then treat each binary projection as per the preceding analysis with its own stochastic noise [6]. An alternate way to implement n-dimensional orthogonal quantum measurements is to observe that $P_0 - P_1$ is one of the Cartan generators

of SU(n), and it can be rotated to any of the other Cartan generators of SU(n) by the unitary symmetry. Such a rotation of Eq. (22) allows us to fix all the orthonormal set of weights as ($k = 1, 2, \ldots, n - 1$)

$$\sum_{i=0}^{k-1} w_i - k w_k = \sum_{i=0}^{k-1} \rho_{ii} - k\rho_{kk} + \sqrt{\frac{k(k+1)S_\xi}{2}} \xi_k. \quad (29)$$

Here, $\xi_k(t)$ are independent white-noise terms. The condition for the evolution to be a pure Wiener process, and consequently satisfy the Born rule, remains $gS_\xi = 1$. With this condition, the evolution equation in the Stratonovich form is Eq. (18), while in the Itô form it is given by

$$d(d_j) = 2\sqrt{g}d_j\left(\tilde{w}_j - \sum_i \tilde{w}_i d_i\right), \quad (30)$$

$$\sum_{i=0}^{k-1} \tilde{w}_i - k\tilde{w}_k = \sqrt{\frac{k(k+1)}{2}} dW_k. \quad (31)$$

Two important properties, arising from the parametric freedom of the stochastic evolution analyzed here are worth stressing: (a) The equations need specification of only the first two moments of the stochastic noise, whether $\xi(t)$ or $dW(t)$. We can use the remaining freedom in the specification of the noise to simplify our analysis as much as possible. The binary or Z_2 noise is the simplest choice, producing two stochastic possibilities at every time step. (b) The solution to the equations implies that a complete measurement formally takes infinite time, and that may not be a desirable feature [8]. The formal "measurement duration" can be made finite by making the coupling g time dependent. Such a stochastic differential process does not have a time translation symmetry, but the change of variables does not alter the measurement outcomes because the Born rule is satisfied at every instant during the measurement process. With such a modification, the only change required in the results described here is to replace gt by $\tau \equiv \int_0^t g(t')dt'$. The choice of $g(t)$ is not unique, and its physical meaning would depend on the nature of the system-apparatus interaction. For example, with $g(t) = 1/(1 - t^2)$, the measurement interval becomes $t \in [0,1]$ and Eq. (20) can be written in coupling-free form as

$$\frac{d}{ds}\rho_{00} = 2(\rho_{00} - \rho_{11} + \tilde{\xi})\rho_{00}\rho_{11}, \quad (32)$$

where $s = \tanh^{-1} t \in [0, \infty]$ and $\langle\langle \tilde{\xi}(s)\tilde{\xi}(s') \rangle\rangle = \delta(s - s')$.

B. Born rule satisfying trajectory ensemble

Henceforth, we impose $gS_\xi = 1$, and focus on the set of the Born rule satisfying quantum trajectories. For a qubit, the map

$$\tanh(z) = 2\rho_{00} - 1 = \rho_{00} - \rho_{11}, \quad (33)$$

between $\rho_{00} \in [0,1]$ and $z \in [-\infty, \infty]$, simplifies the evolution equations. Both the Stratonovich and the Itô forms, Eqs. (20) and (24), then have the same structure:

$$\frac{dz}{dt} = g\tanh(z) + \sqrt{g}\xi, \quad (34)$$

$$dz = g\tanh(z)dt + \sqrt{g}dW. \quad (35)$$
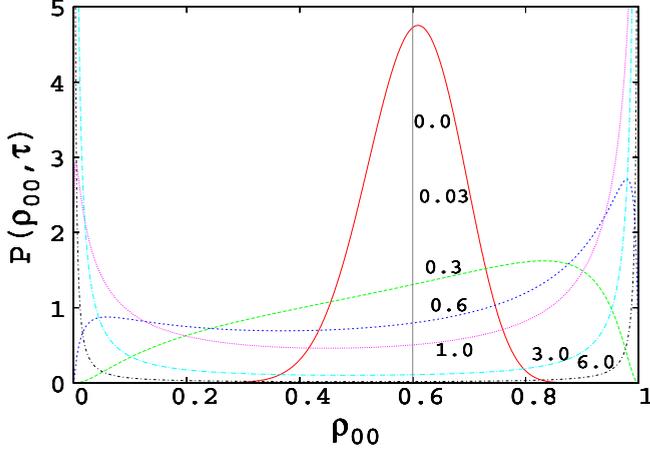
FIG. 3. Evolving distribution of the quantum measurement trajectories for a qubit state initialized to $\rho_{00} = \delta(0.6)$. The curves are labeled by values of the evolution parameter $\tau \equiv \int_0^t g\, dt$, and they accumulate on the eigenstates as $\tau \to \infty$.

In terms of $z(t)$, the density matrix has the form

$$\rho[z(t)] = \begin{pmatrix} \frac{1+\tanh[z(t)]}{2} & \frac{\rho_{01}[z(0)]\mathrm{sech}[z(t)]}{\mathrm{sech}[z(0)]} \\ \frac{\rho_{10}[z(0)]\mathrm{sech}[z(t)]}{\mathrm{sech}[z(0)]} & \frac{1-\tanh[z(t)]}{2} \end{pmatrix}$$

$$= \mathrm{sech}[z(t)] \begin{pmatrix} \frac{1}{2e^{z(t)}} & \frac{\rho_{01}[z(0)]}{\mathrm{sech}[z(0)]} \\ \frac{\rho_{10}[z(0)]}{\mathrm{sech}[z(0)]} & \frac{1}{2}e^{-z(t)} \end{pmatrix}, \qquad (36)$$

and average over the stochastic noise provides the Born rule constraint $\langle\langle\tanh[z(t)]\rangle\rangle = \tanh[z(0)]$.

The stochastic Langevin evolution can be converted to the Fokker-Planck equation to obtain the collective behavior of the quantum trajectories. For measurement of a single qubit with $g S_\xi = 1$, the probability distribution of trajectories $p(\rho_{00}, t)$ or $p(z, t)$ satisfies

$$\frac{\partial p(\rho_{00}, t)}{\partial t} = 2g \frac{\partial^2}{\partial^2 \rho_{00}} \left[\rho_{00}^2 (1 - \rho_{00})^2 p(\rho_{00}, t) \right],$$

$$\frac{\partial p(z, t)}{\partial t} = -g \frac{\partial}{\partial z} [\tanh(z) p(z, t)] + \frac{g}{2} \frac{\partial^2}{\partial z^2} p(z, t). \quad (37)$$

With the initial condition $p(\rho_{00}, 0) = \delta(x)$, this equation can be solved exactly [5,6]. The solution consists of two noninterfering peaks with areas x and $1 - x$, monotonically traveling to the boundaries at $\rho_{00} = 1$ and 0, respectively. In terms of the variable z, the two peaks are diffusing Gaussians, with centers at $z_\pm(t) = \tanh^{-1}(2x - 1) \pm gt$ and common variance gt,

$$p(z, t) = \frac{1}{\sqrt{2\pi gt}} \left\{ x \exp\left[-\frac{(z - z_+)^2}{2gt} \right] \right.$$

$$\left. + (1 - x) \exp\left[-\frac{(z - z_-)^2}{2gt} \right] \right\}. \quad (38)$$

The two peaks reach the boundaries only asymptotically:

$$p(\rho_{00}, \infty) = x\delta(\rho_{00} - 1) + (1 - x)\delta(\rho_{00}). \quad (39)$$

A particular case of how a narrow initial distribution splits into two components that evolve to the measurement eigenstates is illustrated in Fig. 3. For $gt > 10$, 99% of the probability is

within 1% of the two fixed points. Subsequent convergence to projective measurement is exponential, e.g., 99.9% of the probability is within 0.1% of the two fixed points for $gt > 15$.

Upon taking the ensemble average,

$$\int_\infty^\infty \tanh[z(t)] p(z, t) dz = 2x - 1, \qquad (40)$$

$$\int_\infty^\infty \mathrm{sech}[z(t)] p(z, t) dz = e^{-gt/2} \mathrm{sech}[z(0)]. \qquad (41)$$

The resultant expectation value of the density matrix is [cf. Eq. (36)]

$$\langle\langle\rho(t)\rangle\rangle = \begin{pmatrix} x & e^{-gt/2}\rho_{01}(0) \\ e^{-gt/2}\rho_{10}(0) & 1 - x \end{pmatrix}, \qquad (42)$$

where the diagonal elements do not evolve and the off-diagonal elements decay exponentially. Directly, the constraint of Eq. (21), and Eq. (24), also give

$$d\rho_{01} = \rho_{01}\left(1 + \frac{\rho_{11} - \rho_{00}}{2\rho_{00}\rho_{11}}d\rho_{00} - \frac{1}{8\rho_{00}^2\rho_{11}^2}d\rho_{00}^2\right)$$

$$\implies \langle\langle d\rho_{01}\rangle\rangle = \rho_{01}(-g\,dt/2). \qquad (43)$$

We observe that this mixed state results from averaging individual pure-state fluctuating trajectories. Note that in the conventional ensemble interpretation [cf. Eq. (4)], all the expectation values are linear functions of the density matrix and so depend only on $\langle\langle\rho(t)\rangle\rangle$.

The Lindblad master equation for the same system gives evolution identical to Eq. (42), with the single decoherence operator $L_\mu = \sqrt{\gamma}\sigma_3$ and $\gamma = g/4$:

$$\frac{d}{dt}\rho = \gamma(\sigma_3\rho\sigma_3 - \rho), \qquad (44)$$

$$\rho(t) = \frac{1 + e^{-2\gamma t}}{2}\rho(0) + \frac{1 - e^{-2\gamma t}}{2}\sigma_3\rho(0)\sigma_3. \qquad (45)$$

Amazingly, the nonlinear stochastic evolution of the density matrix, after averaging over the noise, becomes linear evolution described by a completely positive trace-preserving map.

The result can also be expressed in the Kraus decomposed orthogonal form as

$$\rho(t) = M_0\rho(0)M_0 + M_3\rho(0)M_3, \qquad (46)$$

$$M_0^2 + M_3^2 = I, \quad \mathrm{Tr}(M_0 M_3) = 0, \qquad (47)$$

where, with $\cosh\epsilon = e^{2\gamma t} = e^{gt/2}$,

$$M_0 = \sqrt{e^{-\gamma t}\cosh(\gamma t)}\,I = \frac{\cosh(\epsilon/2)}{\sqrt{\cosh\epsilon}}I, \qquad (48)$$

$$M_3 = \sqrt{e^{-\gamma t}\sinh(\gamma t)}\,\sigma_3 = \frac{\sinh(\epsilon/2)}{\sqrt{\cosh\epsilon}}\sigma_3. \qquad (49)$$

The Kraus decomposition can also be performed in a symmetric but nonorthogonal form as

$$\rho(t) = M_+\rho(0)M_+ + M_-\rho(0)M_-, \qquad (50)$$

$$M_+^2 + M_-^2 = I, \quad \mathrm{Tr}(M_+^2) = \mathrm{Tr}(M_-^2), \qquad (51)$$

$$M_\pm = \frac{e^{\pm\epsilon/2}}{\sqrt{2\cosh\epsilon}}\left(\frac{I + \sigma_3}{2}\right) + \frac{e^{\mp\epsilon/2}}{\sqrt{2\cosh\epsilon}}\left(\frac{I - \sigma_3}{2}\right).$$

$$(52)$$

This is the form used to describe binary weak measurement evolution in Ref. [24], with ϵ as the evolution parameter. Then, the two evolution possibilities can be expressed as a biased walk,

$$
\begin{aligned}
\rho(z,\epsilon) &= M_+\rho(z)M_+ + M_-\rho(z)M_- \\
&= p_+\rho(z+\epsilon) + p_-\rho(z-\epsilon), \quad (53)
\end{aligned}
$$

with the parametrization of Eq. (36), and

$$
p_\pm = \mathrm{Tr}(M_\pm\rho M_\pm) = [1 \pm \tanh(z)\tanh(\epsilon)]/2. \quad (54)
$$

Note that when $\rho(z)$ is a pure state, so are $\rho(z\pm\epsilon)$. So the two contributions on the right-hand side of Eq. (53) can be considered two possible trajectories with unequal weights $p_\pm$; it is indeed the finite duration integral of Eq. (35) with Z_2 noise.

C. Salient features

The evolution constraint that produces the Born rule, $gS_\xi = 1$, relates the strength of the geodesic evolution g to the magnitude of the noise S_ξ. So, it is sensible to express it as a fluctuation-dissipation relation.

For the white-noise measurement, the geodesic parameter is $\rho_{00} - \rho_{11}$. From Eq. (24), the size of the fluctuations is, dropping the subleading $o(dt)$ terms,

$$
\langle\!\langle (d\rho_{00} - d\rho_{11})^2 \rangle\!\rangle = 16g^2 S_\xi \rho_{00}^2 \rho_{11}^2 dt. \quad (55)
$$

The geodesic evolution term is, from Eq. (20) with w_j replaced by its average w_j^{IB},

$$
(d\rho_{00} - d\rho_{11})_{\mathrm{geo}} = 4g(\rho_{00} - \rho_{11})\rho_{00}\rho_{11}dt. \quad (56)
$$

Hence, $gS_\xi = 1$ amounts to the coupling-free relation

$$
\langle\!\langle (d\rho_{00} - d\rho_{11})^2 \rangle\!\rangle = 4\rho_{00}\rho_{11}\frac{(d\rho_{00} - d\rho_{11})_{\mathrm{geo}}}{\rho_{00} - \rho_{11}}. \quad (57)
$$

The proportionality factor between the noise and the damping term is not a constant because of the nonlinearity of the evolution, but it becomes independent of (gdt) when the Born rule is satisfied.

In addition, our analysis has revealed the following notable aspects of the quantum diffusion model:

(1) Individual quantum trajectories maintain purity, even in the presence of noise. Mixed states arise only when multiple quantum trajectories with different evolutionary weights are combined.

(2) Although the trajectory weights $w_i(t)$ are real and add up to one, they are not restricted to the interval [0,1], and so cannot be interpreted as probabilities.

(3) The measurement outcomes are independent of $\rho_{i\neq j}$, and so are not affected by decoherence. In general, a different noise can be added to the phases of $\rho_{i\neq j}$, without spoiling the described evolution of ρ_{ii}. The Born rule imposes no constraint on that off-diagonal noise. Measurement and decoherence can therefore be looked upon as independent and complementary processes.

(4) When the Born rule is satisfied, the measurement dynamics allows free reparametrization of the "measurement time" but no other freedom. The choice of measurement

is local between the system and the apparatus; different interacting system-apparatus pairs can have different couplings governing their collapse time scales.

(5) The quantum trajectory distribution, given by Eq. (38) and illustrated in Fig. 3, is fully determined in terms of the single evolution parameter $\tau \equiv \int_0^t g(t')dt'$. In weak measurement experiments on superconducting qubits [18,19], the coupling g is a tunable parameter and τ can be gradually varied, e.g., in the range [0,10]. Such experiments can observe quantum trajectory distributions in detail, and so can verify the theoretical predictions.

IV. BINARY QUANTUM JUMP

In the quantum jump model, a large but infrequent noise (i.e., shot noise) is added to the geodesic evolution. The quantum trajectories are smooth most of the time, except for the instances where sudden addition of the noise makes them discontinuous. The measurement is often a binary process in the Fock space, and sudden jumps terminate it, e.g., by emission of a photon. Again, we need to find the magnitude of the noise that makes the measurement process consistent with the Born rule.

A. Constraint on shot noise

Consider the binary measurement scenario, where the eigenstate P_0 is reached by continuous geodesic evolution, while the eigenstate P_1 is reached by a sudden jump. Then, the density matrix evolution during measurement is specified, with trajectory weights $w_i = \delta_{i0}$ and binary shot noise $dN \in \{0,1\}$, as

$$
d\rho = g[\rho P_0 + P_0\rho - 2\rho\,\mathrm{Tr}(P_0\rho)]dt + (P_1 - \rho)dN. \quad (58)
$$

The shot-noise contribution is not infinitesimal; the density matrix instantaneously jumps to P_1, whenever $dN = 1$. Of course, the probability of occurrence of $dN = 1$ is an infinitesimal function of ρ, and it has to vanish at the measurement eigenstate $\rho = P_0$.

For a single qubit, Eq. (58) reduces to

$$
d\rho_{00} = 2g\rho_{00}\rho_{11}dt - \rho_{00}dN, \quad (59)
$$

$$
d\rho_{01} = g\rho_{01}(\rho_{11} - \rho_{00})dt - \rho_{01}dN. \quad (60)
$$

Once again, evolution of the off-diagonal elements is completely determined in terms of that for the diagonal elements. The Born rule can be implemented as a constant of evolution, constraining how often the jumps occur:

$$
\langle\!\langle d\rho_{00} \rangle\!\rangle = 0 \Longleftrightarrow \langle\!\langle dN \rangle\!\rangle = 2g\rho_{11}dt. \quad (61)
$$

From these evolution equations, an ensemble of quantum trajectories can be constructed, allowing for two possibilities for dN at every instance. The $dN = 0$ branch gradually keeps moving towards $\rho_{00} = 1$ as a function of time, while the $dN = 1$ branch stops evolving immediately after the jump to $\rho_{00} = 0$.
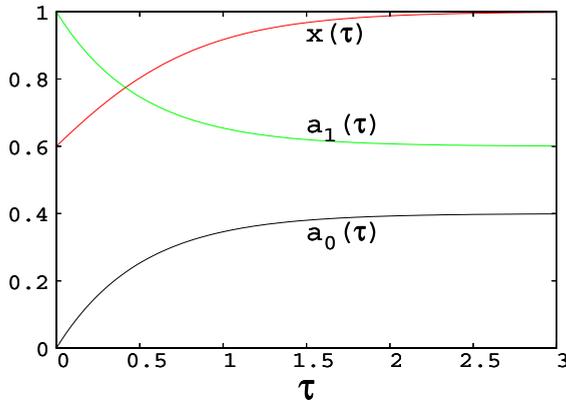
FIG. 4. Properties of the quantum measurement trajectories for quantum jump evolution of a qubit. The initial state is $\rho_{00}(\tau = 0) = 0.6$, and the evolution parameter is $\tau \equiv \int_0^t g(t')dt'$. The initial distribution splits into a monotonically moving component $a_1(\tau)\delta[x(\tau)]$ and a stationary component $a_0(\tau)\delta(0)$, which respectively move to the measurement eigenstates $\rho_{00} = 1$ and 0 as $\tau \to \infty$.

B. Born rule satisfying trajectory ensemble

Even though Eqs. (59) and (60) are not differential equations in the usual sense, due to finite dN, they can be solved exactly as a biased random walk process.

Let the initial condition be $p(\rho_{00}, 0) = \delta(x)$. Because the $dN = 1$ evolution branch terminates at $\rho_{00} = 0$, the solution consists of two δ functions at any instant. The δ function at $\rho_{00} = 0$ steadily grows in size, while the δ function slowly moving to $\rho_{00} = 1$ gradually reduces in size. Explicitly,

$$p(\rho_{00}, t) = [x + (1-x)e^{-2gt}]\delta\left[\frac{x}{x + (1-x)e^{-2gt}}\right]$$
$$+ (1-x)(1 - e^{-2gt})\delta(0). \quad (62)$$

A particular case of how the variables in this distribution evolve is shown in Fig. 4.

The corresponding distribution for the off-diagonal element also consists of two δ functions, given by

$$p(\rho_{01}, t) = [x + (1-x)e^{-2gt}]\delta\left[\frac{\rho_{01}(0)}{xe^{gt} + (1-x)e^{-gt}}\right]$$
$$+ (1-x)(1 - e^{-2gt})\delta(0). \quad (63)$$

Upon taking the ensemble average, the expectation value of the density matrix becomes

$$\langle\!\langle \rho(t) \rangle\!\rangle = \begin{pmatrix} \rho_{00}(0) & e^{-gt}\rho_{01}(0) \\ e^{-gt}\rho_{10}(0) & \rho_{11}(0) \end{pmatrix}. \quad (64)$$

The exponential decay of the off-diagonal elements can also be obtained from Eq. (60) as

$$\langle\!\langle d\rho_{01} \rangle\!\rangle = \rho_{01}(-gdt). \quad (65)$$

This result is again the solution of the Lindblad master equation for the same system, with the single decoherence operator $L_\mu = \sqrt{\gamma}(P_0 - P_1)$, $\gamma = g/2$. The nonlinear evolution of quantum trajectories, after averaging over their distribution, produces a linear completely positive trace-preserving evolution for the density matrix. It can be expressed in the Kraus

decomposed form the same way as in the case of quantum diffusion.

C. Salient features

The evolution constraint yielding the Born rule $\langle\!\langle dN \rangle\!\rangle = 2g\rho_{11}dt$ relates the strength of the geodesic evolution g to the frequency of the noise dN. It can again be expressed as a fluctuation-dissipation relation.

For the shot-noise measurement, the geodesic parameter is ρ_{00} and $(dN)^2 = dN$. Dropping the subleading $o(dt)$ terms, Eq. (59) gives the size of the fluctuations as

$$\langle\!\langle (d\rho_{00})^2 \rangle\!\rangle = \rho_{00}^2 \langle\!\langle dN \rangle\!\rangle. \quad (66)$$

The geodesic evolution term, also from Eq. (59), is

$$(d\rho_{00})_{\text{geo}} = 2g\rho_{00}\rho_{11}dt. \quad (67)$$

Hence, $\langle\!\langle dN \rangle\!\rangle = 2g\rho_{11}dt$ amounts to the coupling-free relation

$$\langle\!\langle (d\rho_{00})^2 \rangle\!\rangle = \rho_{00}^2 \frac{(d\rho_{00})_{\text{geo}}}{\rho_{00}}. \quad (68)$$

Once more, the noise is proportional to the damping term. Although the proportionality factor differs from that in Eq. (57), because of a different nonlinear evolution, it still becomes independent of (gdt) when the Born rule is satisfied.

In addition, our analysis has brought out the following features of the binary quantum jump model: (1) In the presence of shot noise, the quantum trajectories are monotonic, and smooth except for infrequent discontinuous jumps. They still maintain purity, and mixed states arise when multiple quantum trajectories with different noise histories are combined. (2) The trajectory weights can be interpreted as probabilities since the shot noise has a direct probabilistic interpretation as a Poisson process. (3) Evolution of the diagonal ρ_{ii} is independent of the off-diagonal $\rho_{i \neq j}$, and so is unaffected by decoherence. So, as in case of quantum diffusion, measurement and decoherence can be looked upon as independent and complementary processes. Also, free reparametrization of the "measurement time" is allowed, when the Born rule is satisfied. (4) The measurement dynamics is local between the system and the apparatus. The quantum trajectory distribution, given by Eq. (62), is fully determined in terms of the evolution parameter $\int_0^t g(t)dt$. Weak measurement experiments in quantum optics should be able to verify this theoretical prediction.

V. DISCUSSION

We have described a quantum trajectory formalism for state collapse during measurement, which replaces the discontinuous projective measurement by a continuous stochastic process and remains consistent with the Born rule. It supplements the Schrödinger evolution by addition of quadratically nonlinear measurement terms:

$$d\rho = i[\rho, H]dt + \sum_i w_i g[\rho P_i + P_i \rho - 2\rho \operatorname{Tr}(\rho P_i)]dt$$
$$+ \text{noise}. \quad (69)$$

Instead of attributing the additional terms to novel interactions beyond the standard quantum theory, we look at them as an

effective description of the system-apparatus measurement interaction that replaces the von Neumann projection axiom. The task is then to figure out what restrictions such an effective description imposes on the underlying unknown measurement dynamics (including the type of noise that may be present), and whether or not the necessary ingredients exist in the physical world. Nonlinear superoperator evolution for the density matrix is avoided in quantum mechanics because it conflicts with the probability interpretation for mixtures of density matrices. Nevertheless, nonlinear quantum evolutions need not be unphysical, and our analysis in Sec. II shows that Eq. (69) obeys the well-known rules of the quantum theory.

Separation of the quantum trajectory evolution into attraction towards the measurement eigenstates and stochastic measurement noise exposes the striking fact that the magnitudes of these two dynamical contributions have to be precisely related for the Born rule to emerge as a constant of evolution. In general stochastic processes, vanishing drift and fluctuation-dissipation relations are quite unrelated properties, involving first and second moments of the distribution, respectively. The fact that both follow from the same constraint ($g S_\xi = 1$ or $\langle\langle dN \rangle\rangle = 2g\rho_{11}dt$ in the cases we have analyzed) is an exceptional feature of quantum trajectory dynamics. It means that the Born rule can be looked upon as a consequence of Eqs. (57) and (68), instead of Eqs. (25) and (61). This change in viewpoint has powerful implications regarding the cause of probabilistic observations in quantum theory. Since the dissipation (convergence to the measurement eigenstates) is produced by the system-apparatus interaction, the precisely related fluctuations (noise giving rise to probabilistic measurement outcomes) too must be produced by the same system-apparatus interaction. The rest of the environment may contribute to decoherence, but it can influence the measurement outcomes only via the apparatus and not directly!

Another feature brought forth by our analysis is the complementary relationship between the processes of decoherence and measurement. An important consequence of experimental interest is to check whether the system relaxation can be suppressed by reducing the apparatus decoherence (or vice versa).

Each quantum trajectory with its noise history can be associated with an individual experimental run, and can be considered one of the many possibilities that make up the ensemble. A model for the measurement apparatus is needed, however, to understand where the noise comes from. During measurement, the observed signal is amplified from the quantum to the classical regime [25]. The interactions involved are usually electromagnetic, and often the dynamics is nonlinear. Coherent states that continuously interpolate between quantum and classical regimes are a convenient choice for the apparatus pointer states. They are the minimum uncertainty (equal to the zero-point fluctuations) states in the Fock space. *The crucial point is that amplification incorporates quantum noise when the extracted information is not allowed to return* (e.g., spontaneous versus stimulated emission with precisely related magnitudes). So, amplifiers can indeed provide attraction towards the measurement eigenstates together with the requisite noise. That is a direction worth investigating further, in order to find the cause of the noise and the irreversible collapse, and hopefully to construct a more complete theory of quantum measurements.

The quantum trajectory framework that we have advocated does not solve the fundamental measurement problem. What it does is to separate the Born rule from the irreversible collapse, by explaining the system-dependent probabilistic measurement outcomes in terms of a system-independent (but apparatus-dependent) stochastic noise. The location of the "Heisenberg cut", defining the crossover between quantum and classical regimes, is thus shifted higher up in the dynamics of the amplifier. This cut is not a universal feature, but depends on the hardware of the measurement apparatus, in terms of the type of the noise and how it originates in the amplification process. The fluctuation-dissipation relation, and the Born rule implied by it as per our analysis, quite likely transcend the specific nature of the noise. It is certainly a challenge to figure out whether the fluctuation-dissipation relation is universal for all amplifiers, or whether it is possible to design amplifiers that would bypass or modify the noise under some unusual conditions.

Finally, the quantum trajectory framework we have analyzed can be vindicated by verifying its predicted trajectory distributions in weak measurement experiments. In these experiments, the coupling g is a characteristic parameter for each system-apparatus pair, and is not a universal constant. Also, g can be tuned by varying the circuit parameters without changing the apparatus size, and it has to be made small enough to observe the intervening stages between the initial state and the final projective outcome. Given the type of the noise, the complete trajectory distribution (not just its first two moments) is determined in terms of a single evolution parameter, as evidenced by Eqs. (38) and (62). The experimental technology has developed enough for observing such trajectory distributions in case of superconducting qubits [18,19], and would generalize to other quantum systems. Work in this direction is in progress.

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