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Pseudo-Hermitian Landau-Zener-Stückelberg-Majorana model

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We derive the analytical solution of the model of a two-state system interacting with an external coherent field, in which the Hamiltonian is pseudo-Hermitian. We describe in detail the non-Hermitian generalization of the famed Landau-Zener-Stückelberg-Majorana model, but similar generalizations can be derived in a very simple fashion for the other analytically soluble two-state models. The analytical solutions possess a non-Hermitian dynamical invariant, which replaces the probability conservation condition in the Hermitian case. Implementations in waveguide optics and nonlinear frequency conversion are suggested.

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I. INTRODUCTION

In many experiments in quantum and optical physics, a two-state transition is sufficient to describe the essential changes in the state of the system. The coherent two-state dynamics is studied extensively in relation to many areas of physics, including nuclear magnetic resonance [1], coherent atomic excitation [2], atomic collisions [3], quantum information processing [4], and polarization optics [5], to mention just a few. There are several exactly soluble nonresonant two-state models, including the Rabi [6], Landau-Zener-Stückelberg-Majorana (LZSM) [7], Demkov-Kunike [8], Demkov [9], Nikitin [10], and Carroll-Hioe [11] models. Their significance derives from the ability to design simple recipes for control of the transition probability and, more generally, of the entire propagator. Moreover, they allow one to obtain physical insight into the dynamics and the explicit dependence on the interaction parameters. In particular, one can quantify such important features as the behavior of the oscillation frequency and the amplitude of the Rabi oscillations, the degree of power broadening, the presence or absence of excitation sidebands, etc.

In recent years, there has been growing interest in the use of non-Hermitian (NH) Hamiltonians [12], especially in the context of $\mathcal{PT}$-symmetric systems [13]. It was shown, for instance, that such systems can produce a faster-than-Hermitian evolution in a two-state quantum system, while keeping the eigenenergy difference fixed [14]. NH extensions have been presented on the LZSM model [15], and some schemes for the realization of $\mathcal{PT}$ symmetry have been proposed [16]. Recently, an approximation of the adiabatic condition for NH systems was also derived [17]. Finally, NH Hamiltonians have been used as shortcuts to adiabatic processes [18].

A very interesting (from a practical point of view) subclass of the NH Hamiltonians is the so-called pseudo-Hermitian Hamiltonians. An operator $\mathbf{H}$ is called pseudo-Hermitian if there exists a Hermitian operator η, such that

$$\eta \mathbf{H} \eta^{-1} = \mathbf{H}^{\dagger}. \tag{1}$$

Very recently, the dynamical invariants of a pseudo-Hermitian Hamiltonian have been explicitly derived in a closed form [19]. Examples of practical applications of pseudo-Hermiticity include a description of spinor fields in gravitational Kerr fields [20], optical microspiral cavities [21], microcavities perturbed by particles [22], modeling a possible discrepancy between experiment and the standard model value of the muon's anomalous g-factor [23], describing Maxwell's equations in pseudo-Hermitian form [24], describing a weak backscattering between counterpropagating traveling waves in a general open quantum system [25], modeling the propagation of light in a perturbed medium [26,27], etc. For many other applications, which include quantum cosmology, magnetohydrodynamics, and quantum chaos, see Ref. [28].

To be specific, we focus on a couple of experimental implementations wherein NH Hamiltonians emerge: quantum and classical optics. The application of NH Hamiltonians in quantum physics apparently violates the laws of quantum mechanics, which require that the Hamiltonian must be Hermitian in order to obtain a real energy spectrum, probability conservation, and unitary evolution. However, a NH Hamiltonian usually describes an open subsystem of a larger system, and the latter is described by a Hermitian Hamiltonian. The algebraic separation of the subsystem from the full system, e.g., by adiabatic elimination, may give rise in a NH Hamiltonian. The probability nonconservation in the subsystem naturally reflects its interaction with the larger system and the ensuing outflow or inflow of population. Another example of NH Hamiltonians is the phenomenological inclusion of decay rates in the Schrödinger or Bloch equation.

In classical optics, NH Hamiltonians emerge naturally, e.g., in nonlinear frequency conversion. There the physical reason for the emergence of NH behavior is the undepleted pump approximation used, in which the pump field is considered as an infinite source of energy. Another example of NH behavior is guided-wave optics, in the case when light travels in opposite directions in two coupled waveguides.

In this paper, we demonstrate how a certain type of pseudo-Hermitian two-state problem can be solved by using the Hermitian solution after a certain symmetrization procedure. We apply this method to the special case of the LZSM model, but the same approach is suitable for any two-state model.

The paper is organized as follows. In the next section, we introduce the pseudo-Hermitian generalization of the LZSM model and we derive the exact analytical solution for the propagator. In Sec. III, we generalize our method to other two-state models, and we show that it can be applied to generalize any Hermitian two-state problem. We discuss possible physical implementations in Sec. IV. Finally, the conclusions are summarized in Sec. V.

II. LANDAU-ZENER-STÜCKELBERG-MAJORANA MODEL

A. The model

The model that we consider is for a coupled two-state quantum system, described by the ordinary differential equations for the probability amplitudes c_1 and c_2 of the two states,

$$i\frac{d}{dt}c_1(t) = -\frac{1}{2}\Delta(t)c_1(t) + \frac{1}{2}\Omega(t)c_2(t), \tag{2a}$$

$$i\frac{d}{dt}c_2(t) = \frac{1}{2}k\Omega(t)c_1(t) + \frac{1}{2}\Delta(t)c_2(t), \tag{2b}$$

where $\Omega(t)$ is the coupling (assumed real) between the two states, $\Delta(t)$ is the frequency detuning, and the parameter k brings asymmetry in the system and makes the Hamiltonian non-Hermitian. If k is real, the Hamiltonian is pseudo-Hermitian, while if k is complex, it has a more general NH nature. In this paper, for simplicity and due to implementation feasibility, we will only focus on real values of k, but most of the results are also valid for complex k.

In the LZSM model [7], we have a constant coupling and a linear detuning,

$$\Omega(t) = \Omega_0, \quad \Delta(t) = \beta^2 t, \tag{3}$$

where Ω_0 and β are real constants. We consider a finite time duration, which means that the coupling lasts from some initial moment t_i until some final moment t_f, and we will express the solution in terms of the evolution matrix $\mathbf{U}(t_f,t_i)$, which connects the initial and final amplitudes,

$$\mathbf{c}(t_f) = \mathbf{U}(t_f,t_i)\mathbf{c}(t_i), \tag{4}$$

where $\mathbf{c}(t) = [c_1(t),c_2(t)]^T$. For the NH model, which we consider, the evolution matrix is *not* unitary.

B. Exact solution

To derive the solution, it is convenient to introduce the parameters

$$\tau = \frac{\beta t}{\sqrt{2}}, \quad \alpha = \frac{\sqrt{k}\,\Omega_0}{\sqrt{2}\,\beta}. \tag{5}$$

Next, we decouple Eqs. (2) by repeated differentiation and obtain the following second-order equation for $c_1(\tau)$:

$$\frac{d^2}{d\tau^2}c_1(\tau) + (\alpha^2 + \tau^2 - i)c_1(\tau) = 0. \tag{6}$$

The solution of this equation is expressed in terms of the parabolic cylinder (Weber) function $D_\nu(z)$ [29] as

$$c_1(\tau) = AD_\nu(z) + BD_\nu(-z), \tag{7}$$

where A and B are integration constants and

$$\nu = \tfrac{1}{2}i\alpha^2, \quad z = \beta t e^{-i\pi/4}. \tag{8}$$

The solution for $c_2(\tau)$ can be obtained from here and Eq. (2), and is

$$c_2(\tau) = \frac{\sqrt{k}\,\alpha}{\sqrt{2}}e^{-i\pi/4}[-AD_{\nu-1}(z) + BD_{\nu-1}(-z)]. \tag{9}$$

The constants A and B are to be found from the initial values $c_1(\tau_i)$ and $c_2(\tau_i)$ and read

$$A = \frac{\Gamma(1-\nu)}{\sqrt{2\pi}}\left[D_{\nu-1}(-z_i)c_1(\tau_i) - \frac{\sqrt{2}}{\sqrt{k}\,\alpha}e^{i\pi/4}D_\nu(-z_i)c_2(\tau_i)\right], \tag{10a}$$

$$B = \frac{\Gamma(1-\nu)}{\sqrt{2\pi}}\left[D_{\nu-1}(z_i)c_1(\tau_i) + \frac{\sqrt{2}}{\sqrt{k}\,\alpha}e^{i\pi/4}D_\nu(z_i)c_2(\tau_i)\right]. \tag{10b}$$

After some simple algebra, the evolution matrix elements acquire the form

$$U_{11} = \frac{\Gamma(1-\nu)}{\sqrt{2\pi}}[D_{\nu-1}(-z_i)D_\nu(z_f) + D_{\nu-1}(z_i)D_\nu(-z_f)], \tag{11a}$$

$$U_{12} = \frac{\Gamma(1-\nu)}{\alpha\sqrt{k\pi}}e^{i\pi/4}[D_\nu(z_i)D_\nu(-z_f) - D_\nu(-z_i)D_\nu(z_f)], \tag{11b}$$

$$U_{21} = \frac{\alpha\sqrt{k}\,\Gamma(1-\nu)}{2\sqrt{\pi}}e^{-i\pi/4}[D_{\nu-1}(z_i)D_{\nu-1}(-z_f) - D_{\nu-1}(-z_i)D_{\nu-1}(z_f)], \tag{11c}$$

$$U_{22} = \frac{\Gamma(1-\nu)}{\sqrt{2\pi}}[D_\nu(-z_i)D_{\nu-1}(z_f) + D_\nu(z_i)D_{\nu-1}(-z_f)]. \tag{11d}$$

If we compare these results with the propagator of the Hermitian LZSM model [30], we notice that the elements U_{11} and U_{22} in our solution are the same as for the Hermitian case with the substitution $\Omega \to \Omega\sqrt{k}$. However, the U_{12} and U_{21} elements possess also an additional nontrivial loss and gain factor of $\sqrt{k}$.

The probabilities $P_{j\to l}$, starting from state j to end up in state l, are

$$P_{j\to l} = |U_{lj}|^2. \tag{12}$$

In Fig. 1 we plot the transition probability time evolution for different values of the parameter k. From the left frames, we notice the following:

(i) For the Hermitian case, $k = 1$, the probability sum is conserved,

$$P_{1\to 1} + P_{1\to 2} = 1. \tag{13}$$

(ii) For $k = -1$, it is their difference that is conserved,

$$P_{1\to 1} - P_{1\to 2} = 1. \tag{14}$$

From the right frames, we notice the following:

(i) For $k < 1$, a probability loss occurs from the system,

$$P_{1\to 1} + P_{1\to 2} < 1. \tag{15}$$

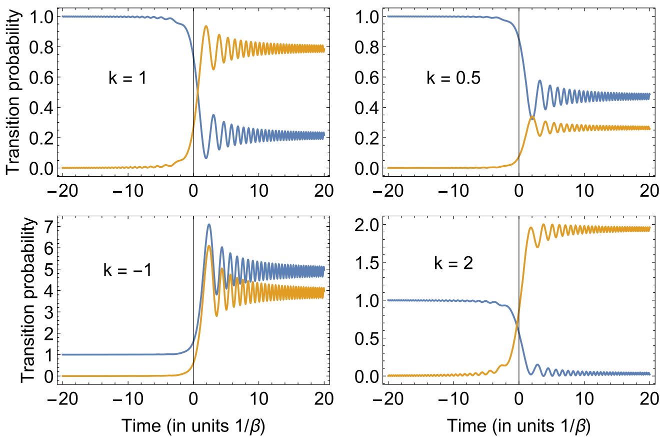

FIG. 1. Transition probabilities $P_{1\to1}$ and $P_{1\to2}$ as a function of scaled time for different values of the parameter k and $\Omega_0 = \beta$.

(ii) For $k > 1$, a probability gain takes place,

$$P_{1\to1} + P_{1\to2} > 1. \tag{16}$$

C. Symmetric crossing

In a way similar to the Hermitian LZSM model, it is convenient to derive the asymptotic behavior of the transition probabilities for large values of the time parameter. To do this, we use the large-argument asymptotics of the Weber function,

$$D_\nu(z) \sim z^\nu e^{-z^2/4} \times \left[\sum_{n=0}^{N} \frac{\left(-\frac{1}{2}\nu\right)_n\left(\frac{1}{2}-\frac{1}{2}\nu\right)_n}{n!\left(-\frac{1}{2}z^2\right)_n} + O(|z^2|^{-N-1})\right] \left(|\arg(z)| < \frac{3\pi}{4}, \nu \text{ fixed}, |z| \to \infty\right), \tag{17}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is Pochhammer's symbol. In the case of a symmetric time interval ($\tau_i = -\tau$, $\tau_f = \tau$), the asymptotic behavior of the transition probability reads

$$P_{1\to1} = P_{2\to2} \approx e^{-\pi\alpha^2} + \frac{2\alpha}{\tau} e^{-\frac{1}{2}\pi\alpha^2}\sqrt{1-e^{-\pi\alpha^2}}\cos\xi, \tag{18a}$$

$$P_{1\to2} \approx k(1 - P_{1\to1}), \tag{18b}$$

$$P_{2\to1} \approx \frac{1}{k}(1 - P_{1\to1}), \tag{18c}$$

where

$$\xi = \frac{\alpha^2}{2}\ln 2\tau^2 + \tau^2 + \frac{\pi}{4} + \arg\left[\Gamma\left(1 - i\frac{\alpha^2}{2}\right)\right]. \tag{19}$$

For an infinite time duration ($\tau \to \infty$), as in the original LZSM model, we find

$$P_{1\to1} = P_{2\to2} \approx e^{-\pi\alpha^2}, \tag{20a}$$

$$P_{1\to2} \approx k(1 - e^{-\pi\alpha^2}), \tag{20b}$$

$$P_{2\to1} \approx \frac{1}{k}(1 - e^{-\pi\alpha^2}). \tag{20c}$$

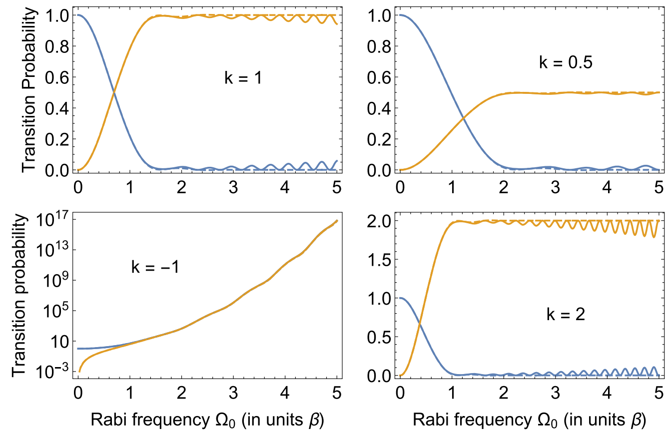

FIG. 2. Transition probabilities $P_{1\to1}$ and $P_{1\to2}$ as a function of Rabi frequency Ω_0 for different values of the parameter k and $t_f = -t_i = 20/\beta$. The dashed lines represent the corresponding asymptotic probabilities.

A few important conclusions follow from here. In the Hermitian limit $k = 1$, we recover the well-known probabilities in the LZSM model. In the NH regime, the probabilities are not confined in the range $[0,1]$. For example, for $k > 1$ it may occur that $P_{1\to2} > 1$, while for $k < 1$ it may occur that $P_{2\to1} > 1$. Moreover, the transition probabilities $P_{1\to2}$ and $P_{2\to1}$ are not equal for $k \neq 1$: their ratio is k^2. In other words, the probability of transition from state 1 to state 2 is different from the probability of transition from state 2 to state 1. However, the no-transition probabilities $P_{1\to1}$ and $P_{2\to2}$ are equal, regardless of the value of k. Moreover, these no-transition probabilities do not depend on k and are the same as in the Hermitian case.

We also note that the probability conservation identities $P_{1\to1} + P_{1\to2} = 1$ and $P_{2\to1} + P_{2\to2} = 1$ are not fulfilled for $k \neq 1$. Instead, the relations

$$kP_{1\to1} + P_{1\to2} = k, \quad kP_{2\to1} + P_{2\to2} = k \tag{21}$$

are satisfied. These are the dynamical invariants of the NH system, which replace the probability conservation relations [19].

In Fig. 2 we plot the transition probability as a function of the Rabi frequency for different values of the parameter k. For $k = -1$, we notice that there is an exponential growth of the probabilities, which is also clearly seen from Eqs. (20), because when k is negative, α^2 is also negative.

D. Half-crossing

In a similar manner, one can derive the half-crossing probabilities ($\tau_i = 0, \tau_f \to \infty$), which are

$$P_{1\to1} = P_{2\to2} \approx \frac{1}{2}\left(1 + e^{-\frac{1}{2}\pi\alpha^2}\right), \tag{22a}$$

$$P_{1\to2} \approx \frac{k}{2}\left(1 - e^{-\frac{1}{2}\pi\alpha^2}\right), \tag{22b}$$

$$P_{2\to1} \approx \frac{1}{2k}\left(1 - e^{-\frac{1}{2}\pi\alpha^2}\right). \tag{22c}$$

In Fig. 3 we plot the half-crossing transition probability as a function of the Rabi frequency for different values of the parameter k. As in Fig. 2, again we notice that the NH

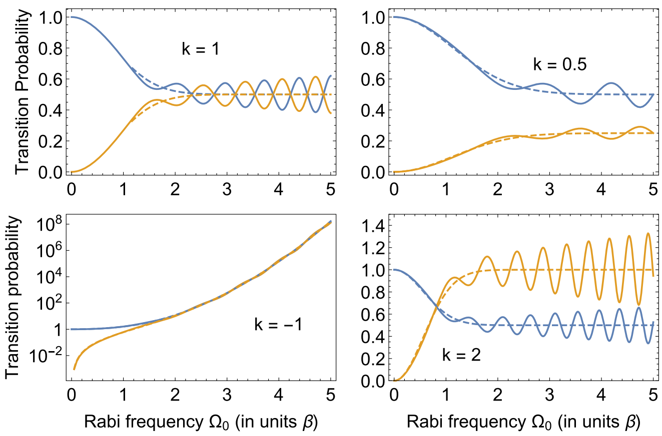

FIG. 3. Transition probabilities $P_{1\to1}$ and $P_{1\to2}$ as a function of Rabi frequency Ω_0 for different values of the parameter k and $t_i = 0$; $t_f = 20/\beta$. The dashed lines represent the corresponding asymptotic probabilities.

conservation laws are fulfilled, and an exponential growth of the probabilities takes place in the case of negative k. We also note that the asymptotic solution fits quite well the exact solution, except for the oscillations in the curves, which can be described by keeping another term in the asymptotic expansion.

III. GENERALIZATION TO OTHER MODELS

The procedure, which we applied to the LZSM model, could be applied to any exactly soluble two-state model. To do this, we write the Hamiltonian of the system as a 2×2 matrix,

$$\mathbf{H}(t) = \frac{\hbar}{2}\begin{bmatrix} -\Delta(t) & \Omega(t) \\ k\Omega(t) & \Delta(t) \end{bmatrix}. \tag{23}$$

To derive the propagator, we make a simple transformation in the amplitude,

$$c_2 \to \sqrt{k}c_2, \tag{24}$$

which leads to a symmetrization of the Hamiltonian, and it becomes

$$\mathbf{H}(t) = \frac{\hbar}{2}\begin{bmatrix} -\Delta(t) & \sqrt{k}\Omega(t) \\ \sqrt{k}\Omega(t) & \Delta(t) \end{bmatrix}. \tag{25}$$

Now if the solution of the Schrödinger equation for this Hamiltonian is given by the propagator

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \tag{26}$$

then the solution for the nonsymmetric Hamiltonian (23) is

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12}/\sqrt{k} \\ U_{21}\sqrt{k} & U_{22} \end{bmatrix}. \tag{27}$$

In this way, one can use the existing solutions for the Hermitian two-state models to derive the non-Hermitian generalizations.

We note here that if $k < 0$, the Hamiltonian in Eq. (25) is still non-Hermitian. Nevertheless, as we shall see, we can still use the results for the Hermitian two-state problems. To do this, we proceed as follows. First, we find the propagator for the two-state Hermitian Hamiltonian,

$$\mathbf{H}(t) = \frac{\hbar}{2}\begin{bmatrix} -\Delta(t) & \Omega(t) \\ \Omega(t) & \Delta(t) \end{bmatrix}, \tag{28}$$

which in many cases is well known and has been derived in the literature. Next, we replace in the solution Ω with $\sqrt{k}\Omega$. Finally, we add the $\sqrt{k}$ factors in the off-diagonal elements of the propagator, as shown in Eq. (27).

It has been shown in Ref. [19] that a pseudo-Hermitian Hamiltonian possesses a set of invariants, which our solutions must keep fixed. For a 2×2 Hamiltonian, there is one independent invariant, which is

$$\mathrm{Tr}(\eta\rho) = kP_{j\to1} + P_{j\to2} = \mathrm{const} \quad (j = 1,2). \tag{29}$$

Since there is a simple connection between the propagator of the pseudo-Hermitian system and a propagator of a Hermitian system, it is trivial to prove that this equation is satisfied by our solution.

IV. PHYSICAL IMPLEMENTATION

Non-Hermitian Hamiltonians can be implemented in a number of physical systems. We will describe two cases.

A. Guided wave optics

One interesting application of the pseudo-Hermitian models is in the area of guided-wave optics [26]. If we consider two electromagnetic modes, traveling through a medium in the opposite directions, the complex amplitudes A and B of the two modes obey the two coupled equations

$$i\frac{dA}{dz} = \kappa e^{i\phi}e^{-i\Delta z}B, \tag{30a}$$

$$i\frac{dB}{dz} = -\kappa e^{-i\phi}e^{i\Delta z}A, \tag{30b}$$

where $\kappa(z)$ is a real coupling function of the propagation direction z, Δ is a real phase mismatch, and $\phi(z)$ is a real phase. These equations are derived within the framework of the coupled-mode theory [26]. After a simple phase transformation of the amplitudes, one can easily obtain the Hamiltonian of Eq. (23) for $k = -1$, if z is considered as the "time" variable.

The dynamical invariant for this system, which replaces the probability conservation for a Hermitian Hamiltonian, reads [cf. Eq. (29) for $k = -1$]

$$P_{1\to2} - P_{1\to1} = \mathrm{const}. \tag{31}$$

The LZSM model, as well as other models, can be implemented by spatially varying the propagation constants of the waveguides by using a photoinduction technique [31].

B. Sum-frequency generation

Another useful application is in the sum-frequency generation (SFG) process. In this nonlinear optical process, we mix a weak signal with frequency ω_1 with a strong signal with frequency ω_2 to convert the ω_1 signal into a signal with frequency $\omega_3 = \omega_1 + \omega_2$. If we denote the amplitudes of the signals with frequencies ω_1 and ω_3 by A_1 and A_3, in the undepleted-pump approximation ($A_2 \approx$ const) they satisfy the following equation [27]:

$$i\frac{d}{dz}\mathbf{A} = \mathbf{HA}, \tag{32}$$

where $\mathbf{A} = [A_1, A_3]^T$ and

$$\mathbf{H} = \begin{bmatrix} -\Delta/2 & K_1(z) \\ K_3(z) & \Delta/2 \end{bmatrix}. \tag{33}$$

Here Δ is the phase mismatch and $K_j \propto \omega_j^2 \chi^{(2)}(z)$, where $\chi^{(2)}$ is the nonlinear susceptibility of the crystal, and $j = 1,3$. We see from this equation that if $\omega_1 \neq \omega_3$, the couplings K_1 and K_3 are different in magnitude, and the pseudo-Hermitian solutions can be applied.

The dynamical invariants are given by Eq. (29). We note that the NH behavior of the SFG equations above stems from the undepleted-pump approximation used, in which the pump field is considered as an infinite source or sink of energy. The LZSM behavior can be implemented by spatially varying the phase mismatch by using an aperiodically poled nonlinear crystal, as in recent experiments [32].

V. DISCUSSION AND CONCLUSIONS

In this work, we derived analytical solutions for a special type of pseudo-Hermitian generalization of the two-state problem. We focused on the LZSM model, but we also showed how to derive the solution for the generalizations of any two-state problem. Several practical applications were considered, namely in waveguide optics and sum-frequency generation. Finally, we note that, because of practical feasibility, we have assumed only real values of the asymmetry parameter k, which lead to a pseudo-Hermitian Hamiltonian. However, most of the derived formulas are also valid for complex values of k, in which case the Hamiltonian has a more general non-Hermitian form.

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