Extreme-value statistics of intensities in a cw-pumped random fiber laser

Bismarck C. Lima,[1] Pablo I. R. Pincheira,[1] Ernesto P. Raposo,[2,*] Leonardo de S. Menezes,[1] Cid B. de Araújo,[1] Anderson S. L. Gomes,[1] and Raman Kashyap[3]

[1]*Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife-PE, Brazil*

[2]*Laboratório de Física Teórica e Computacional, Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife-PE, Brazil*

[3]*Fabulas Laboratory, Department of Engineering Physics and Department of Electrical Engineering, Polytechnique Montreal, Montreal H3C 3A7, Canada*

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We report on the extreme-value statistics of output intensities in a one-dimensional cw-pumped erbium-doped random fiber laser, with a strongly scattering disordered medium consisting of randomly spaced Bragg gratings. The experimental findings from the analysis of a large number of emission spectra are well described by the Gumbel distribution below and above the laser threshold, whereas the Fréchet distribution, typical of strongly fluctuating extreme events with heavy power-law probability tails, provides a nice support to the data near the threshold. We establish a close connection, relying on theoretical arguments, between the reported extreme-value statistics and the shifts in the statistics of intensity fluctuations, from the Gaussian to the Lévy distribution at the threshold and back to the Gaussian well above threshold.

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I. INTRODUCTION

Extreme events in optics and photonics, arising from rare intensity fluctuations, have been investigated intensively in the last decade in diverse linear and nonlinear optical phenomena and devices [1–18]. In fact, the evidence of statistically uncommon optical events in rogue waves, freak waves, instabilities, and breathers have been reviewed in [19,20]. These studies followed earlier works that reported on analog extreme events that take place in various natural environments, such as sea waves [21], DNA [22], multifractal systems [23], solar wind [24], and astrophysical bodies [25].

Among the optical systems, lasers and fiber lasers have been some of the most explored devices due both to their experimental access and control, as well as to their reported complex behavior [3,7,10,12,17,18]. In contrast to conventional lasers, in which cavities bounded by mirrors provide the optical feedback that sustains the oscillation in the gain medium, random lasers [26–28] and random fiber lasers [29,30] constitute a special class of open (cavity-less) complex systems whereby the feedback is related to the presence of a gain medium in a disordered scattering environment. These ingredients can actually be present both in the same particles, such as in ZnO, or rare-earth-doped three-dimensional (3D) crystal powders [31,32].

In the case of one-dimensional (1D) random fiber lasers, we remark that the system investigated in [33–35] differs markedly from the one used by the authors of Ref. [13] for both the scattering and gain media. Indeed, in [13] the optical gain arises from stimulated Raman scattering, and the Rayleigh scattering due to refractive index fluctuations acts as random distributed feedback reflectors. On the other hand, the system employed in [33–35], as well as in the present work, is formed by an erbium-based gain medium, with the scattering originating from a specially designed fiber Bragg grating [36].

The above features led random lasers and random fiber lasers to be explored both experimentally and theoretically in the past few years as platforms to study several complex phenomena, such as Bose-Einstein condensation [37], astrophysical lasers [38], spin-glass analogy [33–35,39–47], Lévy statistics [34,35,46–56], and turbulence [57–60]. Recently, the occurrence of rare events was also reported in random lasers and random fiber lasers, leading to extreme-value statistics (EVS) of optical observables [8,9,13–15]. Indeed, on the one hand extreme-value analysis has been applied [8,9] to sets of coupled fiber lasers to study the distribution of phase locking levels [8], with the Gaussian regime related to the Gumbel density function, as well as to describe the statistics of the combined output power, which was shown [9] to agree with the Tracy-Widom, Majumdar-Vergassola, and Vivo-Majumdar-Bohigas distributions of the largest eigenvalue of Wishart random matrices.

On the other hand, in 2015 the pioneer works by Gorbunov *et al.* [13] and Uppu and Mujumdar [14,15] investigated extreme events related to the emission intensity in single random laser and random fiber laser systems. In [13], the fast-intensity dynamics of random distributed feedback fiber lasers was shown to exhibit pronounced fluctuations that deviate from the Gaussian behavior depending on the excitation power. We mention, however, that, although probability functions with power-law behavior in the large-intensity regime have been demonstrated in [13], no attempt was made to relate the experimental results to the theory of generalized extreme-value distributions, such as Gumbel and Fréchet. Moreover, the statistics of extreme intensity events has also been studied [14,15] in a 3D random laser with a double pulsed pump source. The characterization of the statistical regimes of maximum intensities in terms of the Gumbel or Fréchet distribution has been particularly performed in [15], even though the relatively small number (5000) of emission spectra

*ernesto@df.ufpe.br

analyzed has possibly hindered the extreme-value analysis of intensity measurements, as we argue below.

In this work, we report on the observation of extreme intensity events in a 1D cw-pumped random fiber laser. The gain medium is formed by erbium ions, so that the process of single-photon induced nonlinear absorption, with microscopic origin in the erbium electronic levels, constitutes the mechanism underlying the optical nonlinearity. The strong disorder in this erbium-based random fiber laser (Er-RFL) is provided by a large number of customized random Bragg grating scatterers inscribed with a random spacing along the doped fiber [36]. A very large number (150 000) of emission spectra was collected for each excitation power in the regimes below, near, and above the random laser threshold. The system is the same studied in Ref. [60]. However, while in [60] photonic turbulence was investigated relying on a hierarchical stochastic model, here, in contrast, extreme intensity events are analyzed with basis on the EVS theory. Indeed, the present study complies nicely with the theoretical predictions related to the generalized extreme-value distributions. More specifically, depending on the excitation power regime, we find that the EVS of maximum intensities is well described by the Gumbel and Fréchet distributions, respectively, away and near the threshold. In particular, the Fréchet regime is typical of strongly fluctuating extreme events with heavy power-law tail in the probability function. We also obtain a good agreement, supported by the theoretical analysis, between the distinct regimes of intensity fluctuations (Gaussian or Lévy) and the statistics of extreme intensity events for all excitation powers considered.

This article is organized as follows. In Sec. II, we detail the experimental procedure to collect the emission spectra of the Er-RFL system in order to generate the long-time series of maximum intensity values. The theory of EVS is reviewed in Sec. III and particularly applied to the study of the maximum intensity events in Er-RFL. Experimental results are presented and their statistical analysis based on the theory of Sec. III are discussed in Sec. IV. Finally, Sec. V displays the concluding remarks.

II. EXPERIMENTAL PROCEDURE

The fabrication of the Er-RFL system, including the fiber Bragg grating inscription procedure, is detailed in [36]. It suffices to say here that we employed a polarization maintaining erbium-doped fiber with 30 cm length, fabricated by CorActive (peak absorption 28 dB/m at 1530 nm, numerical aperture NA = 0.25, mode field diameter of 5.7 μm), in which a grating was written with randomly distributed phase errors, instead of a random array of gratings as in [61]. This procedure allowed a large number ($\gg 10^3$) of scatterers. Consequently, the fiber used in this work presents a high degree of randomness due to the presence of the strongly scattering disordered gain medium.

Figure 1 illustrates the experimental setup. A home-assembled semiconductor laser operating in the continuous-wave (cw) regime at 1480 nm was used as the pump source. The maximum output power at the fiber pigtail end was 150 mW. The output fiber was connected to the Er-RFL system using a fiber connector. The Er-RFL output was split, through a

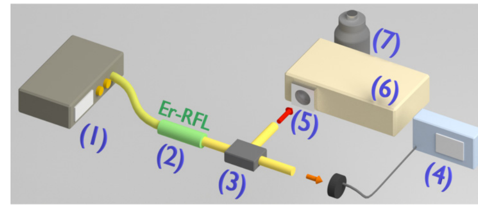

FIG. 1. Experimental setup of the Er-RFL system. (1) Fiber pigtailed semiconductor pump laser operating in the cw regime, (2) Er-RFL, (3) 1480 nm/1550 nm WDM, (4) power meter to measure the output power at 1480 nm, (5) RFL emission out to the spectrometer, (6) spectrometer, and (7) liquid-N_2 cooled InGaAs CCD camera.

1480 nm/1550 nm wavelength-division multiplexer (WDM), to a power meter and a spectrometer with a liquid-N_2 CCD camera of 0.1 nm resolution at 1540 nm [34].

A rather long sequence of 150 000 emission spectra was recorded at each input excitation power, in the regimes below, near, and above the random laser threshold. The spectra were collected with an integration time of 50 ms. We notice that qualitatively similar results for the intensity measurements on the Er-RFL system were obtained in Refs. [33,34] using an integration time of 100 ms (we remark, however, that Refs. [33,34] did not study the extreme intensity events). The measured threshold from the FWHM analysis of the spectra was $P_{\rm th} = (16.30 \pm 0.05)$ mW [34].

We stress that the intensity fluctuations of the pump source of less than 5% were not correlated with the Er-RFL fluctuations, as similarly found in the random laser devices considered in [13,39] and also specifically inferred [34] in the present experimental setup through the measurement of the normalized standard deviation of both the pump laser and the Er-RFL system. Indeed, while that quantity remained constant in the pump source, it varied substantially in Er-RFL [34]. Moreover, we also remark that the number of longitudinal modes in Er-RFL, measured using a speckle contrast technique, was ~204 [33], demonstrating the multimode character of this system.

To proceed with the analysis of extreme intensity events in the Er-RFL system, we denote as I_j the intensity value at the wavelength of maximum output intensity of the spectrum j. Consequently, the discrete sequence $\{I_j\}$ of subsequent spectra values gave rise to a long-time series, with $j = 1,2,\ldots,N$ (=150 000), at each excitation power. Each spectrum was obtained in a 50-ms time window, and therefore the 150 000 spectra at a given power were acquired during a total time measurement of 125 min. The stochastic nature of such intensity values results from the intrinsic disorder related to the strongly scattering medium with randomly spaced Bragg gratings. Indeed, the intensity dynamics is described [46,48] by a set of coupled Langevin equations with nonlinear disorder terms, from which a probability density function (PDF) of intensity values $P(I)$ can be determined for each power.

We next subdivided the sequence $\{I_j\}$ into M blocks of N/M intensity values each. For statistical purposes, we chose the number of blocks such that $M \gg 1$ and $N/M \gg 1$ (see discussion below). Therefore, a new long-time series $\{I_{\max,n}\}$ was generated at each excitation power, with $n = 1,2,\ldots,M$, where $I_{\max,n}$ denotes the maximum intensity among the values belonging to the nth block. The distribution of $I_{\max,n}$ values at a given power defines the PDF $f(I_{\max})$, as well as its corresponding cumulative density function (CDF) $F(I_{\max}) = \int_0^{I_{\max}} f(I'_{\max}) dI'_{\max}$. In the next sections, the statistics of extreme intensity events in the Er-RFL system is investigated with basis on the analysis of the sequence $\{I_{\max,n}\}$ and its associated CDF $F(I_{\max})$ for each excitation power.

III. EVS THEORY APPLIED TO THE MAXIMUM INTENSITY VALUES IN THE Er-RFL SYSTEM

A remarkable result from the EVS theory [62,63] is that any CDF $F(I_{\max})$ must tend *asymptotically* in the limit $N/M \to \infty$ to a stable generalized extreme-value (GEV) distribution given by

$$F(I_{\max}) = \exp\left\{-\left[1+\xi\frac{(I_{\max}-m)}{\sigma}\right]^{-1/\xi}\right\}, \qquad (1)$$

where $1+\xi(I_{\max}-m)/\sigma > 0$, and $\xi \in (-\infty,+\infty)$, $m \in (-\infty,+\infty)$, and $\sigma > 0$ denote, respectively, the shape, location, and scale parameters. Moreover, the EVS theory also states that this continuous family of GEV distributions can be cast into three classes, depending only on the value of the shape parameter ξ, namely [62,63], (i) Weibull, for $\xi < 0$ and $I_{\max}$ limited to some upper cutoff value; (ii) Gumbel, for $\xi \to 0$, which implies

$$F(I_{\max}) = \exp\{-\exp[-(I_{\max}-m)/\sigma]\}; \qquad (2)$$

and (iii) Fréchet, for $\xi > 0$. The latter two cases are related to density functions of $I_{\max}$ values with no upper bound.

A formal theoretical connection can be established [62,63] between the primary PDF $P(I)$ comprising all intensity values (i.e., not only the maxima of the blocks) and the associated asymptotic GEV distribution $F(I_{\max})$ of unbounded maximum intensity values. Indeed, if the PDF $P(I)$ falls off asymptotically at large I as a power-law heavy tail, $P(I) \sim I^{-\mu}$, with $\mu > 1$, then its corresponding GEV distribution converges [62,63] in the limit $N/M \to \infty$ to the Fréchet density function, Eq. (1), with $\xi = 1/(\mu-1) > 0$. Conversely, if $P(I)$ falls off faster than a power law, the convergence function is the $\xi \to 0$ Gumbel CDF, Eq. (2).

This asymptotic connection of the EVS to one of the GEV stable distributions resembles particularly the attraction of the PDF of the sum of a large number of random variables to one of the possible asymptotic stable distributions, namely, the Gaussian or the Lévy α-stable family [64]. In the present photonic context, if the stochastic values assumed by the intensity I are identically distributed and uncorrelated over the long ($N \gg 1$) sequence of spectra (or even if they present finite-time correlations), and if the second moment of the PDF $P(I)$ is finite, then the central limit theorem (CLT) assures [64] that the intensity fluctuations are driven by the Brownian (Gaussian, normal) dynamics. On the other hand, if the second moment of $P(I)$ diverges, the generalized CLT states [64] that the fluctuations are asymptotically governed by the Lévy statistics.

The continuous family of Lévy α-stable distribution is described [64] by the Fourier transform of the characteristic function defined in k space,

$$\overline{P}(k) = \exp\{-|ck|^{\alpha}[1 - i\beta\,\mathrm{sgn}(k)\Phi] + ik\nu\}. \qquad (3)$$

The Lévy index $\alpha \in (0,2]$ is the most important parameter, since it drives the magnitude of the intensity fluctuations. Indeed, whereas strong fluctuations with relevant deviations from the Gaussian behavior are associated with values in the range $0 < \alpha < 2$, the Gaussian statistics with relatively weak fluctuations and the result of the CLT are recovered for the boundary value $\alpha = 2$. Thus, the parameter α, which can be experimentally determined from the direct analysis of the PDF $P(I)$, effectively works as an indicator of the statistical regime (Gaussian or Lévy) of intensity fluctuations. The other independent parameters describe the asymmetry or skewness of the distribution ($\beta \in [-1,1]$), location [$\nu \in (-\infty,+\infty)$], and scale [$c \in (0,\infty)$], along with the function $\Phi(k) = -(2/k)\ln|k|$ if $\alpha = 1$, whereas $\Phi = \tan(\pi\alpha/2)$ if $\alpha \neq 1$.

Therefore, in addition to determining the type of EVS that drives the maximum intensity values, the asymptotic large-I behavior of $P(I)$ also defines its attraction to the result of the CLT or the generalized CLT. In fact, if the PDF $P(I)$ is power-law tailed at large I, $P(I) \sim I^{-\mu}$, then its statistical behavior is governed [64] by the Lévy PDF with $\alpha = \mu - 1$, if $1 < \mu < 3$ (diverging second moment). On the other hand, if $\mu \geqslant 3$, or alternatively when the large-I decay of $P(I)$ is faster than the power law (for example, in the case of exponential behavior), the $\alpha = 2$ Gaussian statistics takes place (finite second moment).

From the discussion above, we conclude that if the experimental CDF $F(I_{\max})$ best fits the Fréchet ($\xi > 0$) GEV distribution, a large-I power-law dependence can be inferred for the PDF $P(I)$, with exponent $\mu = 1 + 1/\xi > 1$. In this case, since the Lévy (Gaussian) regime is characterized by $1 < \mu < 3$ ($\mu \geqslant 3$), then we can define the parameter $\overline{\alpha}$ as a further experimental indicator of the statistics of intensity fluctuations, which is determined from the EVS analysis of maximum intensities, instead of from the direct analysis of $P(I)$ itself. In this context, we observe that $\overline{\alpha} = \mu - 1 = 1/\xi < 2$ if $\xi > 1/2$ (Lévy regime), and $\overline{\alpha} = 2$ if $0 < \xi \leqslant 1/2$ (Gaussian regime). The Fréchet boundary limit $\xi = 1/2$ thus separates the statistical domains with Lévy and Gaussian behaviors. In contrast, if the best fit of the experimental $F(I_{\max})$ matches the $\xi \to 0$ Gumbel distribution, then the $\overline{\alpha} = 2$ Gaussian regime is implied. Of course, one would actually expect that $\overline{\alpha} = \alpha$ in the asymptotic limit $N/M \to \infty$, thus indicating that the same statistics of intensity values should be obtained in this limit either from the EVS analysis of the CDF $F(I_{\max})$ ($\overline{\alpha}$) or by investigating the PDF $P(I)$ (α). Nevertheless, in real experimental studies this precise equivalence can be hindered, so that one can possibly find $\overline{\alpha} \approx \alpha$ from experimental data due to a number of limitations, as argued below.

Indeed, at this point some words of caution are in order before applying in the next section the above theoretical results to the actual experimental data of the Er-RFL system.

On the one hand, the limitation in the numbers of spectra (N) and spectra blocks (M) should yield some expected variation [24] in the fitting parameters of the GEV distribution of maximum intensity values (notably, in the shape parameter ξ), as larger N and M are considered toward the asymptotic limits $M \to \infty$ and $N/M \to \infty$. In particular, since the total number of emission spectra is fixed in this work ($N = 150\,000$), the suitable choice of M actually becomes a quite subtle issue. Indeed, a large number N/M of spectra per block would imply a rather small number M of blocks, making the calculation of the experimental CDF $F(I_{\max})$ not statistically significant. Conversely, increasing M indefinitely would decrease very much the number of spectra per block, undermining the statistical relevance of the maximum intensity of the blocks. Thus, some proper compromise value of M has to be chosen when a large N is fixed in order to satisfy both $M \gg 1$ and $N/M \gg 1$ requirements in the EVS analysis (see also discussion in the next section).

On the other hand, we observe that a PDF with a diverging second moment actually represents an unphysical possibility, either in the present case of intensity measurements or even in any case of realistic stochastic phenomena. At first sight, this fact would in principle prevent the experimental realization of the asymptotic connections described above regarding the power-law tailed PDFs with $1 < \mu < 3$ (diverging variance), in the context of both the EVS analysis [Fréchet distribution with $\xi = 1/(\mu - 1) > 1/2$] and the generalized CLT (Lévy distribution with $0 < \alpha = \mu - 1 < 2$). Nevertheless, it has been demonstrated [65,66] that a *truncated* power-law PDF, with a large but finite second moment, behaves rather similarly to the Lévy PDF to a considerable extent (i.e., a Lévy-like statistical regime). In this case, the crossover to the Gaussian dynamics predicted by the CLT is attained only in the very long term. Theoretically justified truncation schemes have been suitably implemented, for example, by restricting the values of the random variable to a finite range [65–67], with $P(I) = 0$ for $I > I_{\rm cutoff}$, or by tempering the power law with an exponential attenuation [48,53], $P(I) \sim \exp(-\eta I)/I^{\mu}$. Therefore, the experimental reports of Lévy PDFs of intensities with $0 < \alpha < 2$ or the EVS analysis of maximum intensity values described by a Fréchet GEV distribution with $\xi > 1/2$ should be properly interpreted as representative of this extensive Lévy-like statistical regime of intensity measurements.

IV. RESULTS AND DISCUSSION

Recent experimental results in Er-RFL, as well as in other random laser systems, have indicated that the statistics of intensity values changes as a function of the input excitation power [34,35,46,47,49–53,56]. In fact, the prelasing Gaussian regime observed at low excitation power shifts to the Lévy behavior near the threshold, and back to the Gaussian well above threshold, consistently with the changes in the large-I power-law dependence of the experimental PDF $P(I)$. As a consequence, according to the discussion in the previous section, we also expect that this shifting pattern will impact on the characterization of the CDF $F(I_{\max})$ of extreme intensity values in the Er-RFL system, which is expressed below in terms of the Gumbel or Fréchet distribution, depending on the excitation power regime.

We begin by displaying in Fig. 2 the sequence of values $\{I_j\}$ of the maximum intensity (in arbitrary units) of the spectra $j = 1,2,\ldots,N$ (=150 000) emitted by the Er-RFL system. Data are shown for four values of the excitation power P (normalized by the threshold power $P_{\rm th} = 16.30$ mW): (a) $P/P_{\rm th} = 0.77$, (b) $P/P_{\rm th} = 0.88$, (c) $P/P_{\rm th} = 1.21$, and (d) $P/P_{\rm th} = 1.72$, which correspond, respectively, to the regimes well below, just below, just above, and well above the threshold of the random laser.

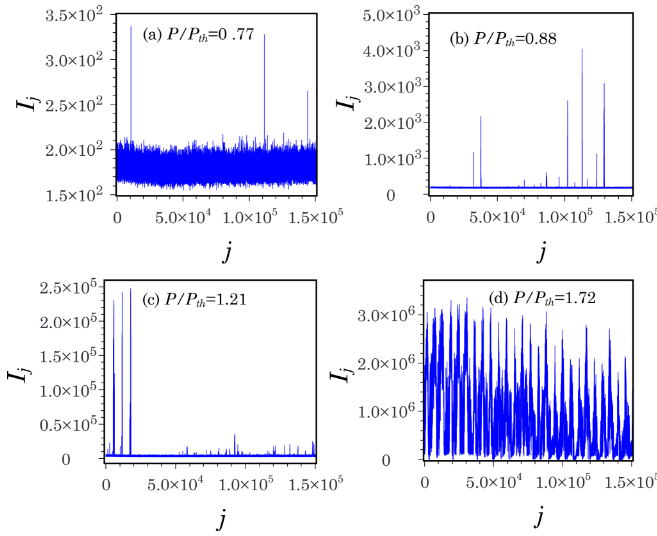

FIG. 2. Maximum intensity value I_j (in arbitrary units) of the spectra $j = 1,2,\ldots,N$ (=150 000) emitted by the Er-RFL system. Data are shown for four values of the excitation power P (normalized by the threshold power $P_{\rm th} = 16.30$ mW): (a) $P/P_{\rm th} = 0.77$, (b) $P/P_{\rm th} = 0.88$, (c) $P/P_{\rm th} = 1.21$, and (d) $P/P_{\rm th} = 1.72$, corresponding, respectively, to the regimes well below, just below, just above, and well above the random laser threshold.

It is clear from these plots that a drastic change in the fluctuation patterns of $\{I_j\}$ occurs as the threshold is crossed. For example, in the regime well below threshold at $P/P_{\rm th} = 0.77$, just a few intensity maxima stand out [Fig. 2(a)]. However, their highest values are typically only $\sim$2 times larger than the average intensity. In contrast, the many extreme intensity values observed near the threshold, both at $P/P_{\rm th} = 0.88$ [Fig. 2(b)] and $P/P_{\rm th} = 1.21$ [Fig. 2(c)], can reach up to $\sim$22 and $\sim$66 times above their respective average intensities. Although these extreme events happen not so often, they are statistically relevant, as discussed below. At last, the pattern shown well above threshold at $P/P_{\rm th} = 1.72$ [Fig. 2(d)] does not display rare extreme events, being, instead, more like the prelasing regime of Fig. 2(a), although without the few highest maxima [i.e., it is more similar to the fluctuation pattern of values $I \lesssim 210$ in Fig. 2(a)]. These results can be better understood as follows, with basis on the statistical study of the intensity measurements and EVS analysis as a function of the excitation power.

We next show in Fig. 3 the sequence $\{I_{\max,n}\}$, with $n = 1,2,\ldots,M$, obtained by subdividing the time series $\{I_j\}$ in Fig. 2 into $M = 800$ blocks of 187 intensity values each, with the maximum value of block n denoted as $I_{\max,n}$, according to the prescription in Sec. II. As discussed, large numbers of blocks and spectra per block are necessary for ensuring a significant statistical analysis. We notice in the coarse-grained structure of Fig. 3 that patterns similar to those of Fig. 2 are observed, as expected.

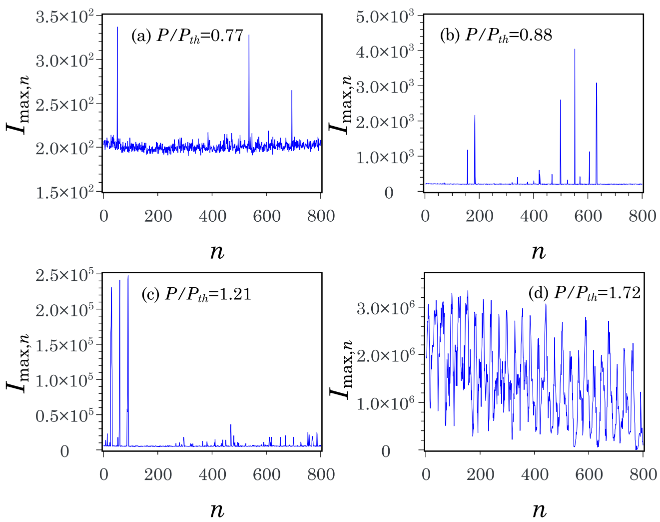

FIG. 3. Maximum intensity value $I_{\max,n}$ (in arbitrary units) of the blocks $n = 1,2,\ldots,M$ (=800) in which the $N = 150\,000$ spectra displayed in Fig. 2 were divided. Data are shown for (a) $P/P_{\rm th} = 0.77$, (b) $P/P_{\rm th} = 0.88$, (c) $P/P_{\rm th} = 1.21$, and (d) $P/P_{\rm th} = 1.72$.

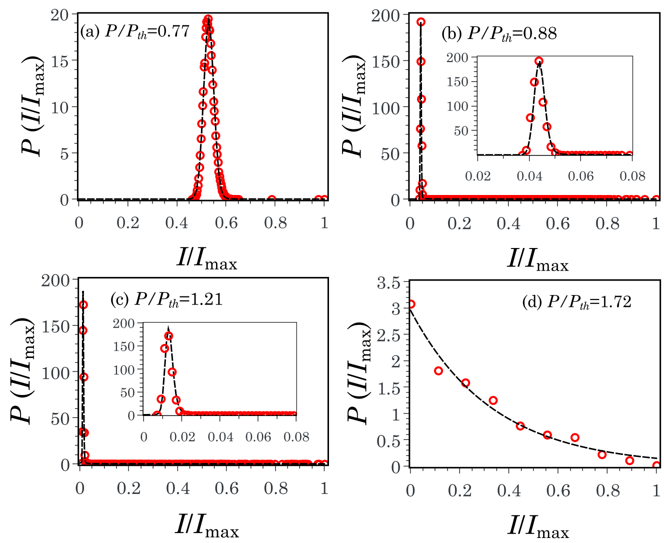

FIG. 4. PDF $P(I)$ of intensity values of the $N = 150\,000$ spectra emitted by the Er-RFL system at (a) $P/P_{\rm th} = 0.77$, (b) $P/P_{\rm th} = 0.88$, (c) $P/P_{\rm th} = 1.21$, and (d) $P/P_{\rm th} = 1.72$. Experimental results from the analysis of the sequences $\{I_j\}$ shown in Fig. 2 are depicted in red circles. Dashed lines in (a)–(c) indicate best-fit curves to the Lévy α-stable distribution, whereas in (d) an exponential best fit emerges. The Lévy statistical behavior with $\alpha = 1.69$ is identified just above the threshold in (c). Gaussian regimes are present both well below the threshold, with the best-fit value $\alpha = 1.98$, and well above, since the exponential PDF is governed by the $\alpha = 2$ Gaussian statistics, according to the CLT. The onset of the transition from the Gaussian to Lévy behavior, characterized by the strong enhancement of intensity fluctuations, is observed as the threshold is approached from below [(b) $\alpha = 1.88$]. The insets show details of the main plots.

The increase in the magnitude of intensity fluctuations observed near the threshold in Figs. 2 and 3 suggests that the PDF of intensity values $P(I)$ can be generally described by the family of Lévy α-stable distributions, including both the Lévy statistical regime if $0 < \alpha < 2$ and the Gaussian limit if $\alpha = 2$, as discussed in Sec. III. Indeed, in agreement with previous reports in Er-RFL [34], by applying the quantile-based method [68] to determine the best-fit parameters of the experimental PDF $P(I)$ to the Fourier transform of Eq. (3), a Gaussian regime was readily identified well below the threshold, with the Gaussian value $\alpha = 1.98$ at $P/P_{\rm th} = 0.77$ [Fig. 4(a)] (see also Table I). On the other hand, the exponential best fit obtained well above threshold at $P/P_{\rm th} = 1.72$ [Fig. 4(d)] also assures that the PDF $P(I)$ is governed by the $\alpha = 2$ Gaussian statistics according to the CLT. We comment, however, that these Gaussian regimes present important differences since, for instance, the former corresponds to the prelasing behavior, whereas the latter has been characterized as a random laser regime with self-averaging of the gain [52].

In contrast to these Gaussian regimes observed far from the threshold, the Lévy statistical behavior is clearly identified just above the threshold, with the best-fit value $\alpha = 1.69$ determined at $P/P_{\rm th} = 1.21$ [Fig. 4(c)]. We mention that even lower values of α can be found when the threshold is approached from above, as reported in [34]. However, for excitation powers very close to the threshold the intensity fluctuates so widely that relatively stable results for the EVS analysis in this regime would require the collection of a much larger number of emission spectra. In this context, since in the present work we focus on the EVS analysis of extreme intensity values, the very proximity of the threshold regime is not explored here.

An interesting scenario also emerges as the threshold is approached from below. On the one hand, as discussed above in Figs. 2 and 3, although the Er-RFL system operates in the prelasing regime in both cases, the fluctuation pattern just below the threshold at $P/P_{\rm th} = 0.88$ is rather distinct from the one in the deep Gaussian phase well below threshold at $P/P_{\rm th} = 0.77$. On the other hand, the best-fit value $\alpha = 1.88$ found at $P/P_{\rm th} = 0.88$ [Fig. 4(b)] can still be considered as characteristic of the Gaussian statistical behavior. In fact, the Lévy regime of intensity fluctuations has been experimentally assigned in random lasers essentially when the value of α falls below 1.8 [47]. In this respect, we actually observe that the issue of determining precisely the Lévy index α from the analysis of the PDF of intensities in random laser systems is in fact a subtle one. This arises in part due to the finite number of spectra considered in the statistical analysis. In this sense, the precise value $\alpha = 2$ that indicates Gaussian statistical behavior of intensity fluctuations is rarely obtained, even when the system is undoubtedly in this regime, such as in the prelasing phase well below the threshold. Some sort of experimental criterion for α (close to, but not exactly, 2) has been thus necessary in order to indicate when the shift to the Lévy regime takes place. If one looks at the dependence of the index α on the input power (see, e.g., Ref. [52]), one generally notices an initial gentle decrease still in the prelasing Gaussian regime, followed by a fast decrease as the system crosses the threshold and enters the Lévy regime. The value $\alpha \approx 1.8$ corresponds approximately to the inflection point of that curve, after which the decrease becomes much more pronounced. From these considerations, we thus conclude that, in spite of

TABLE I. Index α determined from the PDF $P(I)$ of intensities and shape parameter ξ, power-law exponent μ, and index $\overline{\alpha}$ determined from the CDF $F(I_{\max})$ of extreme intensity values in the Er-RFL system. Results were obtained using $N = 150\,000$ emission spectra divided into $M = 800$ blocks. The EVS theory leads to $\overline{\alpha} = 2$ if $0 < \xi \leqslant 1/2$ (Fréchet distribution) or if $\xi \to 0$ (Gumbel distribution), and $\overline{\alpha} = 1/\xi < 2$ if $\xi > 1/2$ (Fréchet distribution), in the asymptotic limit $N/M \to \infty$. The Gaussian statistical regime corresponds to $\alpha = 2$ and $\overline{\alpha} = 2$, whereas values $0 < \alpha < 2$ and $0 < \overline{\alpha} < 2$ are indicative of Lévy behavior. Experimental Gaussian regimes with distinct characteristics are found well below ($P/P_{\text{th}} = 0.77$) and well above ($P/P_{\text{th}} = 1.72$) the laser threshold. In the former, values $\xi \approx 0$ emerge from the nice fits both to the Gumbel ($\xi \to 0$) and Fréchet ($\xi = 0.08$) distributions. The exponent μ is not defined well above the threshold at $P/P_{\text{th}} = 1.72$, in which the $\xi \to 0$ Gumbel CDF is consistent with the PDF $P(I)$ with exponential decay rather than the power law. In contrast, a Lévy statistical regime is found just above the threshold ($P/P_{\text{th}} = 1.21$), with $\alpha = 1.69$ and $\overline{\alpha} = 1.61$. Just below the threshold ($P/P_{\text{th}} = 0.88$), the strong decrease of μ (if compared to the deep Gaussian regime at $P/P_{\text{th}} = 0.77$) toward the $\mu = 3$ Gaussian limit signalizes the onset of the transition to the Lévy behavior. In all cases, we observe that the result $\overline{\alpha} \approx \alpha$ indicates that essentially the same statistics of intensity values is obtained from the EVS analysis of the CDF $F(I_{\max})$ ($\overline{\alpha}$) or by investigating the PDF $P(I)$ (α). The equality $\overline{\alpha} = \alpha$ must hold asymptotically as $N/M \to \infty$.

Excitation power	α	ξ	μ	$\overline{\alpha}$
$P/P_{\text{th}} = 0.77$	1.98	≈ 0	13.5	2
$P/P_{\text{th}} = 0.88$	1.88	0.35	3.86	2
$P/P_{\text{th}} = 1.21$	1.69	0.62	2.61	1.61
$P/P_{\text{th}} = 1.72$	2	$\to 0$		2

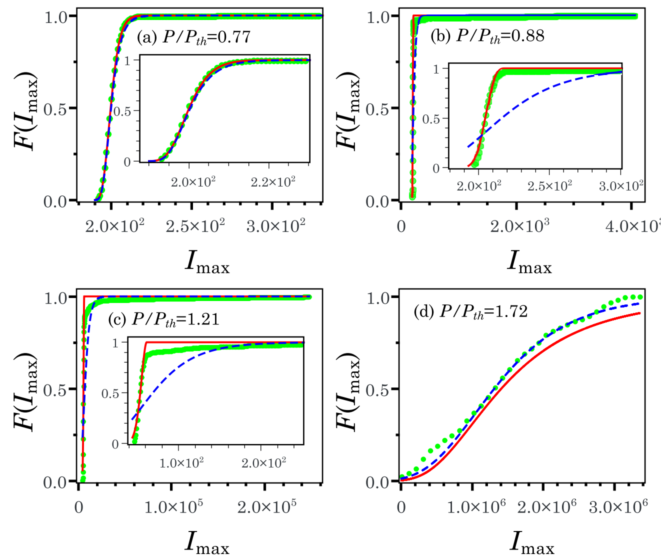

FIG. 5. CDF $F(I_{\max})$ of maximum intensity values of the $M = 800$ blocks of spectra emitted by the Er-RFL system at (a) $P/P_{\text{th}} = 0.77$, (b) $P/P_{\text{th}} = 0.88$, (c) $P/P_{\text{th}} = 1.21$, and (d) $P/P_{\text{th}} = 1.72$. Experimental results from the analysis of the sequences $\{I_{\max,n}\}$ shown in Fig. 3 are depicted in green circles. Best-fit curves to the GEV and Gumbel distributions are indicated by solid red and dashed blue lines, respectively. A Fréchet CDF is identified just above the threshold [(c) $\xi = 0.62 > 1/2$], consistently with the Lévy statistical behavior of intensity fluctuations. Gaussian regimes are found well below and well above the threshold, characterized by distributions with $\xi \approx 0$, either Fréchet [(a) $\xi = 0.08$] or Gumbel [(a) and (d): $\xi \to 0$]. Just below the threshold [(b) $\xi = 0.35$], the strong increase of the shape parameter toward the $\xi = 1/2$ Fréchet limit signalizes the onset of the transition from the Gaussian to Lévy behavior. The insets show details of the main plots.

displaying enhanced intensity fluctuations (if compared to the deep Gaussian behavior well below threshold), which signalize towards the onset of the transition to the Lévy statistical behavior, the system is still in the prelasing Gaussian regime just below the threshold at $P/P_{\text{th}} = 0.88$.

We now turn to the analysis of the extreme intensity events in the output spectra produced by the Er-RFL system.

From the data of the $\{I_{\max,n}\}$ sequence shown in Fig. 3, the experimental CDF $F(I_{\max})$ is determined for each input excitation power, with the results displayed in green circles in Fig. 5. We have tried best fits of $F(I_{\max})$ to both the GEV and Gumbel distributions, Eqs. (1) and (2), which are respectively depicted by solid red and dashed blue lines in Fig. 5.

We start with the analysis of the Gaussian regimes far from the threshold. Well above the threshold at $P/P_{\text{th}} = 1.72$ [Fig. 5(d)], we observe that the fit of the CDF $F(I_{\max})$ to the $\xi \to 0$ Gumbel distribution is considerably better. According to Sec. III, since $\xi \to 0$ then one obtains $\overline{\alpha} = 2$ at $P/P_{\text{th}} = 1.72$. This result also indicates that the PDF $P(I)$ falls off at large I faster than a power law, consistently with the exponential best fit shown in Fig. 4(d).

On the other hand, the Gaussian regime well below the threshold at $P/P_{\text{th}} = 0.77$ presents in Fig. 5(a) good fits of $F(I_{\max})$ to distributions with $\xi \approx 0$, either the $\xi \to 0$ Gumbel or the $\xi = 0.08$ Fréchet CDF. In the former, the Gaussian behavior of intensity fluctuations is identified, with $\overline{\alpha} = 2$ as discussed above. In the latter, the best-fit value of the shape parameter is typical of a large-I power-law PDF $P(I)$ with exponent $\mu = 1 + 1/\xi = 13.5 > 3$. Therefore, this result is equally consistent with the Gaussian statistical regime, and in this case we also infer $\overline{\alpha} = 2$ since $0 < \xi \leqslant 1/2$ (see Sec. III). Once again, this finding agrees with the Gaussian value $\alpha = 1.98$ obtained directly from the PDF of intensities at $P/P_{\text{th}} = 0.77$ [Fig. 4(a)].

In the Lévy statistical regime just above the threshold at $P/P_{\text{th}} = 1.21$, we observe in Fig. 5(c) that the Fréchet distribution provides a much better fit than the Gumbel to the experimental CDF $F(I_{\max})$. The best-fit shape parameter $\xi = 0.62$ implies the values $\mu = 1 + 1/\xi = 2.61 < 3$ for the large-I asymptotic dependence of $P(I)$, and $\overline{\alpha} = 1/\xi = 1.61$ for the statistical behavior of intensities, since $\xi > 1/2$. As displayed in Table I, this finding also compares well with the result $\alpha = 1.69$ obtained from the direct analysis of the PDF $P(I)$ of intensities shown in Fig. 4(c).

Lastly, although we have seen above that the intensity fluctuations enhance largely in the prelasing regime near the threshold, we show in Fig. 5(b) that the experimental CDF $F(I_{\max})$ of extreme intensity events is still well fitted to the Fréchet distribution at $P/P_{\text{th}} = 0.88$. As the best-fit value $\xi = 0.35$ of the shape parameter at $P/P_{\text{th}} = 0.88$ approaches the $\xi = 1/2$ Fréchet limit, the associated power-law exponent

$\mu = 1 + 1/\xi = 3.86$ becomes much smaller than the one (($\mu = 13.5$) found in the deep Gaussian regime at $P/P_{\text{th}} = 0.77$. This result also gets close to the Gaussian boundary value $\mu = 3$, below which the statistics of the intensity fluctuations shifts to the Lévy regime. Therefore, while the value $\xi = 0.35 < 1/2$ still implies the Gaussian index $\overline{\alpha} = 2$ at $P/P_{\text{th}} = 0.88$, we already observe in the strong decrease of ξ the sign of the threshold proximity, in agreement with the analysis of the PDF $P(I)$ of intensities at $P/P_{\text{th}} = 0.88$ [Fig. 4(b)].

We notice in all regimes above that the result $\overline{\alpha} \approx \alpha$ indicates that essentially the same statistics of intensity values is obtained from the EVS analysis of the CDF $F(I_{\max})$ ($\overline{\alpha}$) or by directly investigating the PDF $P(I)$ (α), in agreement with the discussion in Sec. III. In fact, the precise equality $\overline{\alpha} = \alpha$ must hold in the asymptotic limit $N/M \to \infty$. Moreover, as also argued, the finding $\overline{\alpha} \approx \alpha$ is sensitive to the choice of the number of blocks M in which the emission spectra are divided for the extreme-value analysis. For example, in the $\alpha = 1.69$ Lévy regime just above the threshold at $P/P_{\text{th}} = 1.21$, we obtained $1.6 \lesssim \overline{\alpha} \lesssim 1.8$ when the number of blocks was in the range $700 \lesssim M \lesssim 950$. Values of $\overline{\alpha}$ outside this interval were found for either lower or higher choices of M. This variation with M in the best-fit parameters of the GEV distribution is actually expected [24,63] due to both the undermining of the statistical analysis as the value of M gets much reduced or much increased, and the need to fulfill the large-N/M asymptotic requirement (see Sec. III).

We also observe that this sensitiveness to the choice of M in the extreme-value analysis might be the reason why the statistics of maximum intensity values obtained in [15] from the EVS analysis of $F(I_{\max})$ ($\overline{\alpha}$) was generally not equivalent to that determined in [52] from the PDF $P(I)$ (α). In those works, the authors investigated a random laser system consisting of ZnO nanoparticles suspended in a rhodamine 6G-methanol solution, with a double pulsed Nd:YAG laser as the pump source. In particular, we remark that Ref. [15] seems to be, up to the present, the only statistical study of extreme intensity events in photonic systems that relies on the EVS theory of GEV distributions, Eqs. (1) and (2). In [15], $N = 5000$ emission spectra were collected and divided into $M = 500$ blocks of only $N/M = 10$ spectra each, for the $F(I_{\max})$ analysis. In [52], $N = 2000$ spectra were employed in the study of the PDF $P(I)$. For example, in the case of the sample with transport mean free path $\ell^* = 1500\ \mu\text{m}$, it is possible to compare the conclusion associated with the measurement of the shape parameter ξ and index $\overline{\alpha} = 1/\xi$, obtained from the nice fit of the CDF $F(I_{\max})$ to the Fréchet distribution [15], with the one related to the parameter α determined from $P(I)$ [52]. In fact, one notices in the former that the Lévy statistical regime ($\xi > 1/2$, $0 < \overline{\alpha} < 2$) occurs for excitation pump energies in the range $0.8 \lesssim E_p \lesssim 0.9$, whereas in the latter it happens ($0 < \alpha < 2$) for $0.6 \lesssim E_p \lesssim 2.0$. Therefore, it might be possible that a proper choice of M combined with the collection of a larger set of emission spectra could lead to more similar intervals of the pump energy E_p with Lévy behavior in the random laser system investigated in [15,52].

We finally remark that the origin of the changes in the statistics of the output intensity, as the pump power is varied, is intrinsically related to the complex interplay between the gain, nonlinearity, and the feedback mechanism due to the disordered scatterers (in this case, the random fiber gratings inscribed in the erbium-doped system, in a strongly disordered regime), as the laser action takes place. Indeed, Refs. [46,48] describe theoretically the intensity dynamics through a set of coupled Langevin equations with the presence of these ingredients, from which the PDF of intensity values can be determined for each input power. Consequently, these mechanisms have also a deep influence on the extreme-value statistics of intensity measurements, whose connection with the PDF of intensity values has been described above. In particular, we observe that, due to the complexity of the laser build up in this open cavity system, strong intensity fluctuations emerge around the threshold that can give rise, as demonstrated, to extreme events of statistical significance.

V. CONCLUSIONS

In this work, we have reported on the observation of extreme intensity events in a 1D cw-pumped erbium-based random fiber laser, with a strongly scattering disordered medium consisting of randomly spaced Bragg gratings.

By analyzing a large number of emission spectra in the regimes below, near, and above the threshold, we have inferred that the extreme-value statistics of maximum intensity values complies nicely with the theoretical predictions based on the stable generalized extreme-value distributions. Actually, we have found that the extreme-value statistics of maximum intensities is well described by the Gumbel and Fréchet distributions, respectively, away and near the threshold, with the latter indicating the presence of a strongly fluctuating regime described by a probability function with heavy power-law asymptotic tail.

A good agreement, supported by the theoretical analysis, was also obtained between the distinct regimes of intensity fluctuations (Gaussian or Lévy) and the statistics of extreme intensity events for all excitation powers considered.

We hope that the present results might stimulate further experimental and theoretical investigations of extreme events in random laser and random fiber laser photonic systems.

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