Multiparty quantum mutual information: An alternative definition

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Mutual information is the reciprocal information that is common to or shared by two or more parties. Quantum mutual information for bipartite quantum systems is non-negative, and bears the interpretation of total correlation between the two subsystems. This may, however, no longer be true for three or more party quantum systems. In this paper, we propose an alternative definition of multipartite information, taking into account the shared information between two and more parties. It is non-negative, observes monotonicity under partial trace as well as completely positive maps, and equals the multipartite information measure in literature for pure states. We then define multiparty quantum discord, and give some examples. Interestingly, we observe that quantum discord increases when a measurement is performed on a large number of subsystems. Consequently, the symmetric quantum discord, which involves a measurement on all parties, reveals the maximal quantumness. This raises a question on the interpretation of measured mutual information as a classical correlation.

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I. INTRODUCTION

Quantum correlations [1,2] are essential ingredients in quantum information theory [3]. Various quantum correlations, different in nature and types, find huge applications in quantum information processing tasks. Consequently, their characterization and quantification is inevitable. Several nonclassical correlation measures have been proposed for bipartite quantum systems, and some of them have been extended to multipartite settings. Nonetheless, quantifying multipartite quantum correlations in quantum physical systems remains a challenging problem. Recently, however, significant developments have been made towards this end in the form of multipartite global (symmetric) quantum discord (GKD) [4], conditional entanglement of multipartite information (CEMI), [5], and quantum correlation relativity (QCR) [6]. In Ref. [7], an operational interpretation of GQD was given in terms of the partial state distribution protocol. It was also shown that GQD nearly vanishes for a multiparty quantum state that is approximately locally recoverable after performing measurements on each of the subsystems. An important aspect to notice is that all these developments count on some multipartite information measure (see below) [8]. Multipartite (mutual) information, the reciprocal information that is common to or shared by two or more parties, has an authoritative stand in the arena. Quantum mutual information (QMI), whose definition is motivated by that of classical mutual information (CMI), is well defined for bipartite quantum systems. QMI of a bipartite quantum state ρ_{AB} is defined as

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = S(\rho_{AB} \parallel \rho_A \otimes \rho_B) \geqslant 0, \tag{1}$$

where $S(\rho) = -\text{tr}(\rho \log_2 \rho)$ is the von Neumann entropy and $S(\rho \parallel \sigma) = \text{tr}(\rho \log_2 \rho - \rho \log_2 \sigma)$ is the quantum relative entropy. It is non-negative, and bears the interpretation of total correlation between the two subsystems [9]. It is defined as the amount of work (noise) that is required to erase (destroy) the correlations completely. These properties (non-negativity, interpretation of total correlation) may, however, no longer be true for three or more party quantum systems. The existing quantum version of multipartite information in literature, due to Watanabe [8], is a straightforward generalization of bipartite QMI,

$$\begin{aligned} I_x(A_1 : A_2 : \cdots : A_n) &= \sum_{k=1}^{n} S(\rho_{A_K}) - S(\rho_{A_1A_2\cdots A_n}) \\ &= S(\rho_{A_1A_2\cdots A_n} \parallel \rho_{A_1} \otimes \cdots \otimes \rho_{A_n}) \\ &\geqslant 0. \end{aligned} \tag{2}$$

We refer to it as *conventional* quantum mutual information (CQMI). It is the sum of the individual von Neumann entropies less the joint von Neumann entropy of a multipartite quantum system, $\rho_{A_1A_2\cdots A_n}$. It is non-negative, and monotone nonincreasing under the local discarding of information [i.e., $I_x(A_1X_1 : A_2X_2 : \cdots : A_nX_n) \geqslant I_x(A_1 : A_2 : \cdots : A_n)$ for a multiparty quantum state $\rho_{A_1X_1A_2X_2\cdots A_nX_n}$]. However, unlike two-party quantum mutual information $I_x(A_1 : A_2)$, it does not have any operational interpretation.

In another approach, three-variable CMI [10] is defined as

$$K(A : B : C) = K(A : B) - K(A,B|C), \tag{3}$$

where $K(A : B) = H(A) - H(A|B) = H(A) + H(B) - H(A,B) = H(B) - H(B|A)$ is two-variable CMI, $K(A,B|C) = H(A|C) + H(B|C) - H(A,B|C)$ is three-variable conditional mutual information, and $H(\cdot)$ is the Shannon entropy. Though both $K(A : B)$ and $K(A,B|C)$ are non-negative, the three-variable CMI can be negative. Using the chain rule $H(X,Y) = H(X) + H(Y|X)$, the following expressions of CMI are equivalent:

$$\begin{aligned} K_1(A : B : C) = {} & [H(A) + H(B) + H(C)] \\ & - [H(A,B) + H(A,C) + H(B,C)] \\ & + H(A,B,C), \end{aligned} \tag{4}$$

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$$K_2(A:B:C) = H(A,B) - H(B|A) - H(A|B) - H(A|C) - H(B|C) + H(A,B|C), \quad (5)$$

$$K_3(A:B:C) = [H(A) + H(B) + H(C)] - [H(A,B) + H(A,C)] + H(A|B,C). \quad (6)$$

The above definitions of CMI can be extended to the quantum domain. They are obtained by replacing the random variables by density matrices and Shannon entropies by von Neumann entropies, with appropriate measurements in the quantum conditional entropies. Hence,

$$I(A:B:C) = [S(A) + S(B) + S(C)] - [S(A,B) + S(A,C) + S(B,C)] + S(A,B,C), \quad (7)$$

$$J_1(A:B:C) = S(A,B) - S_{\mathcal{M}}(B|A) - S_{\mathcal{M}}(A|B) - S_{\mathcal{M}}(A|C) - S_{\mathcal{M}}(B|C) + S_{\mathcal{M}}(A,B|C), \quad (8)$$

$$J_2(A:B:C) = [S(A) + S(B) + S(C)] - [S(A,B) + S(A,C)] + S_{\mathcal{M}}(A|B,C), \quad (9)$$

where $S(X) \equiv S(\rho_X)$, and $S_{\mathcal{M}}(X|Y) = S(\rho_{X|\mathcal{M}^Y})$ is the quantum conditional entropy obtained after some generalized measurement $\mathcal{M}$ has been performed on subsystem Y. It is asserted that the above quantum expressions are not equivalent as measurement assumes its role in the quantum conditional entropies. These QMIs have certain drawbacks. First, surprisingly enough, $I(A:B:C)$ is identically zero for arbitrary three-party pure quantum states [10], implying that mutual information among the subsystems of three-party pure quantum systems is zero. This is not true in the case of bipartite QMI. Second, $I(A:B:C)$ and other versions of QMI can be negative [10,11]. How is this negative correlation useful for quantum information tasks? Though the existing definition of three-party QMI is argued to reveal the true nature of quantum correlations [10], the fact that QMI, being a measure of correlation, can assume a negative value is challenging. This perplexing stance, handicapped with any operational interpretation of multipartite information, motivated us to propose an alternative definition of multiparty QMI.

Our multipartite quantum mutual information for quantum state $\rho_{A_1A_2\cdots A_n}$ assumes the following form,

$$I(A_1:A_2:\cdots:A_n) = \sum S\big(X_{k_1}X_{k_2}\cdots X_{k_{n-1}}\big) - (n-1)S(A_1A_2\cdots A_n), \quad (10)$$

where $X_{k_i} \in \{A_1, A_2, \ldots, A_n\}$. It takes into account the shared information among m-parties, $2 \leqslant m \leqslant n$, and not only the common information among all parties. This is quite reasonable as information can be distributed or stored among m-parties. We show that it is non-negative, and argue that it, by its very construction, manifests total correlation. Also, it equals CQMI for pure states. Moreover, we obtain its lower and upper bounds in terms of CQMI.

The rest of this paper is organized as follows: In the following section, we provide an alternative definition of multiparty quantum mutual information, compute its value for some typical states, and prove its non-negativity and monotonicity. Then, in the next section, we discuss multiparty quantum discord and present a few illustrations. Surprisingly, we observe that the symmetric quantum discord reveals the maximal quantumness. Finally, we conclude.

II. QUANTUM MUTUAL INFORMATION

We propose an alternative definition of multiparty quantum mutual information, via the Venn diagram approach, for an n-party quantum state $\rho_{A_1A_2\cdots A_n}$. As information can be distributed or stored among m-parties, $2 \leqslant m \leqslant n$, our definition takes into account the shared information among m-parties, and not only the common information among all parties. This can be understood readily using a Venn diagram.

A. Two-party QMI

From Fig. 1(a), we see that the only way the subsystems A and B interact with each other is via region ab, i.e., $ab = A \cap B = A + B - A \cup B$. In entropy language, this translates as $I(A:B) = S(A) + S(B) - S(AB)$. This is usual bipartite QMI.

B. Three-party QMI

From Fig. 1(b), the possible ways the subsystems A, B, and C interact with each other are via region abc, which is common to all three, and regions ab, ac, bc, which are pairwise common. Taking just the region abc, i.e., $abc = A \cap B \cap C = A + B + C - (A \cup B + A \cup C + B \cup C) + A \cup B \cup C$, this "common information" translates into $I_c(A:B:C) := [abc] = [S(A) + S(B) + S(C)] - [S(AB) + S(AC) + S(BC)] + S(ABC)$. Note that in doing so we have discarded pairwise interactions. However, *a priori*, there is no reason to throw them away. Moreover, they can provide important information

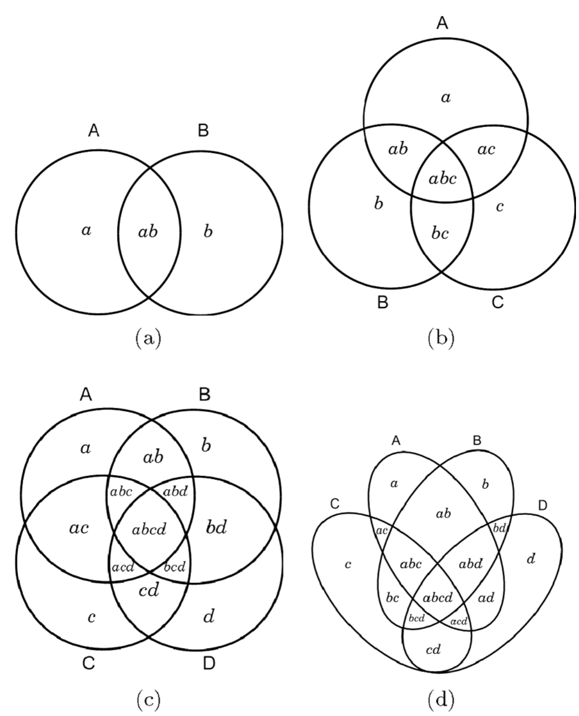

FIG. 1. (a) Two-variable, (b) three-variable, and (c), (d) four-variable Venn diagrams with possible intersecting regions. While (c) does not represent the true Venn diagram of four variables, (d) does.

TABLE I. Values of common information (I_c) and QMI (I) of $|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$ [13], $|D_n^r\rangle = \frac{1}{\sqrt{\binom{n}{r}}}\sum_{\mathcal{P}} \mathcal{P}[|0\rangle^{\otimes n-r}|1\rangle^{\otimes r}]$ [14], three-qutrit totally antisymmetric state $|\psi_{\mathrm{as}}\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle)$ [15], and four-qubit cluster state $|C_4\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$ [16]. While I_c can be negative, I is non-negative. For pure states, $I = I_x$, from Theorem 2.

State	I_c	I
$\|\mathrm{GHZ}_2\rangle$	2	2
$\|\mathrm{GHZ}_3\rangle$	0	3
$\|D_3^1\rangle$	0	2.75489
$\|\psi_{\mathrm{as}}\rangle$	0	4.75489
$\|\mathrm{GHZ}_4\rangle$	2	4
$\|D_4^1\rangle$	0.490225	3.24511
$\|D_4^2\rangle$	0.490225	4
$\|C_4\rangle$	−2	4

when examined together. Then "two-party shared information" reads as $I_{s2}(A:B:C) := [ab + ac + bc]$. Thus, the total three-party QMI is the sum of the common information and the pairwise shared information: $I(A:B:C) = I_c(A:B:C) + I_{s2}(A:B:C) := [A \cup B \cup C - (a+b+c)]$. After simple algebra, $I(A:B:C)$ can be expressed in entropy language as

$$I(A:B:C) = S(AB) + S(AC) + S(BC) - 2S(ABC) = S(\rho_{ABC}^{\otimes 2} \| \rho_{AB} \otimes \rho_{AC} \otimes \rho_{BC}). \quad (11)$$

It guarantees that $I(A:B:C)$ is not identically zero for arbitrary three-party pure quantum systems.

C. Four-party QMI

Figure 1(c) does not represent the true Venn diagram of four variables as pairwise interacting regions ad and bc are missing. The correct four-variable Venn diagram is represented in Fig. 1(d). The total four-party QMI is then defined as $I(A:B:C:D) := [A \cup B \cup C \cup D - (a+b+c+d)]$, which, again, after some simple algebra, can be expressed as [12]

$$I(A:B:C:D) = \sum_{X_1,X_2,X_3} S(X_1X_2X_3) - 3S(ABCD) = S\left(\rho_{ABCD}^{\otimes 3} \,\Big\|\, \bigotimes_{\{X_i\}} \rho_{X_1X_2X_3}\right), \quad (12)$$

where $X_i \in \{A,B,C,D\}$. We list in Table I the values of common information (I_c) and QMI (I) of some typical states (see also Fig. 2).

An n-party QMI can be analogously defined,

$$\begin{aligned} I(A_1:A_2:\cdots:A_n) &:= [A_1 \cup A_2 \cup \cdots \cup A_n - (a_1 + a_2 + \cdots + a_n)] \\ &= \sum_{k=1}^{n} S(\rho_{\overline{A_k}}) - (n-1)S(\rho_{A_1A_2\cdots A_n}) \\ &= \sum_{k=1}^{n} S(\rho_{A_1A_2\cdots A_n} \| \rho_{A_k} \otimes \rho_{\overline{A_k}}) - S\left(\rho_{A_1A_2\cdots A_n} \,\Big\|\, \bigotimes_{k=1}^{n} \rho_{A_k}\right) \\ &= S\left(\rho_{A_1A_2\cdots A_n}^{\otimes n-1} \,\Big\|\, \bigotimes_{k=1}^{n} \rho_{\overline{A_k}}\right), \end{aligned} \quad (13)$$

where n-party common information is evaluated as $I_c(A_1:A_2:\cdots:A_n) = \sum_{k=1}^{n}(-1)^{k+1}\sum_{\{A_{I_k}\}} S_{A_{I_k}}$, with $A_{I_k} \equiv A_{i_1}A_{i_2}\cdots A_{i_k}$, $S_X \equiv S(\rho_X)$, and $S_{\overline{A_i}} \equiv S(\rho_{\overline{A_i}}) = S(\rho_{A_1\cdots A_{i-1}A_{i+1}\cdots A_n})$. Hence, QMI can be rewritten as $I(A_1:A_2:\cdots:A_n) = \sum_{i=1}^{n} S_{\overline{A_i}} - (n-1)S_{A_1A_2\cdots A_n}$. *An n-party quantum mutual information is the sum of $(n-1)$-party von Neumann entropies less $(n-1)$ times the joint von Neumann entropy of an n-party quantum system.* We argue here that multiparty QMI $I(A_1:A_2:\cdots:A_n)$, as

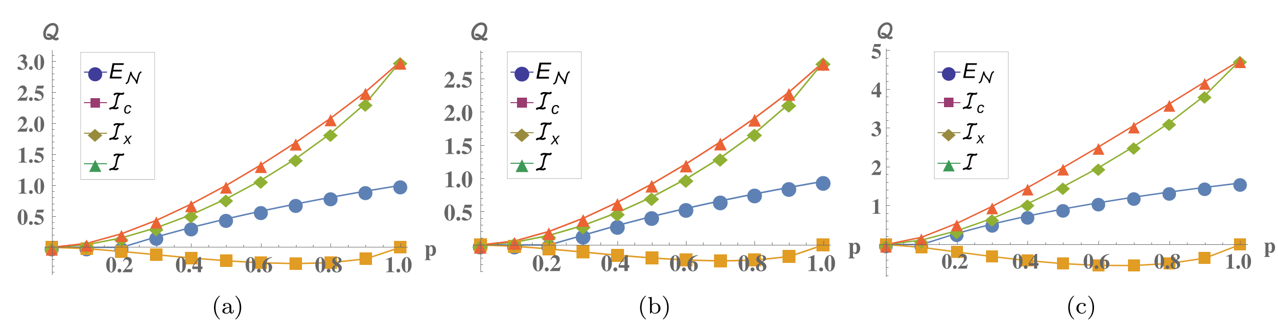

FIG. 2. Plots of logarithmic-negativity $E_{\mathcal{N}}(A:BC)$ [21,22], common information $I_c(A:B:C)$, CQMI $I_x(A:B:C)$, and QMI $I(A:B:C)$ against the white-noise parameter p, of three-party state $\rho_{ABC} = p|\psi\rangle\langle\psi| + (1-p)\frac{I}{8}$ for different $|\psi\rangle$'s: (a) Greenberger-Horne-Zeilinger (GHZ) state, $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ [13], (b) W state $|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ [14], and (c) totally antisymmetric state $|\psi_{\mathrm{as}}\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle)$ [15]. In all three cases, while $I_c(A:B:C)$ vanishes at $p = 0,1$ and is negative at intermediate values, QMI $I(A:B:C)$ vanishes at $p = 0$ only and is positive for other values. QMI is greater than or equal to $I_x(A:B:C)$. We see that logarithmic negativity, an entanglement measure, exceeds common information, indicating that common information cannot be a total correlation.

$I(A_1 : A_2)$, by its very construct, bears the interpretation of total correlation of a multiparty quantum system. Further, we can obtain generalized QMI by replacing the von Neumann entropy, $S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$, with generalized entropies such as the Renyi entropy, $S_q^R(\rho) = \frac{1}{1-q}\log_2[\mathrm{tr}(\rho^q)]$ [17], the Tsallis entropy, $S_q^T(\rho) = \frac{1}{1-q}[\mathrm{tr}(\rho^q) - 1]$ [18], and the (smooth) min-max entropies [19]. Both Renyi and Tsallis entropies reduce to the von Neumann entropy in the limit $q \to 1$.

In subsequent theorems, we prove that $I(A_1 : A_2 : \cdots : A_n)$ is non-negative, and is monotonically nonincreasing under partial trace and completely positive maps. It equals CQMI for pure states, and give its lower and upper bounds in terms of CQMI.

Theorem 1. $I(A_1 : A_2 : \cdots : A_n)$ is non-negative.

Non-negativity of quantum mutual information $I(A_1 : A_2 : \cdots : A_n)$ follows directly from that of quantum relative entropy, $S(\rho \parallel \sigma) \geqslant 0$. Here, we provide an alternative proof. To prove this, we will extensively use a variant of the strong subadditivity relation, $S_{XYZ} + S_Y \leqslant S_{XY} + S_{YZ}$, which states that conditioning reduces entropy, i.e., $S_{X|YZ} \leqslant S_{X|Y}$. The proof for the n-party case follows as [20] $I(A_1 : A_2 : \cdots : A_n) = S_{12\cdots(n-1)} + S_{12\cdots(n-2)n} + \cdots + S_{23\cdots n} - (n-1)S_{12\cdots n} = S_{12\cdots(n-1)} - S_{1|23\cdots(n-1)n} - S_{2|13\cdots(n-1)n} - \cdots - S_{(n-1)|12\cdots(n-2)n} \geqslant S_{12\cdots(n-1)} - S_{1|23\cdots(n-1)} - S_{2|13\cdots(n-1)} - \cdots - S_{(n-1)|12\cdots(n-2)} = \cdots \geqslant S_{123} - S_{1|23} - S_{2|13} - S_{3|12} = S_{12} - S_{1|23} - S_{2|13} \geqslant S_{12} - S_{1|2} - S_{2|1} = S_1 + S_2 - S_{12} \geqslant 0$. Hence, the theorem is proved. ■

Theorem 2. $I(A_1 : A_2 : \cdots : A_n)$ equals CQMI for pure states.

For the pure quantum states, $S_{A_1 A_2 \cdots A_n} = 0$ and $S_{\bar{A}_i} = S_{A_i}$. Hence, $I(A_1 : A_2 : \cdots : A_n) = I_x(A_1 : A_2 : \cdots : A_n)$ using Eqs. (10) and (2). ■

Theorem 3. $I(A_1 : A_2 : \cdots : A_n)$ observes monotonicity under partial trace, and any completely positive map Φ.

These are direct consequences of the monotonicity of quantum relative entropy [23] under partial trace, $S(\rho_A \parallel \sigma_A) \leqslant S(\rho_{AX} \parallel \sigma_{AX})$, and any completely positive map Φ, $S(\Phi(\rho) \parallel \Phi(\sigma)) \leqslant S(\rho \parallel \sigma)$. ■

Theorem 4. $I_x - (n-2)S_{12\cdots n} \leqslant I \leqslant I_x + 2S_{12\cdots n}$.

Using the strong subadditivity entropic relation, $S_X + S_Y \leqslant S_{XZ} + S_{YZ}$, and the Araki-Lieb inequality, $S_X - S_Y \leqslant S_{XY} \Rightarrow S_X - S_{XY} \leqslant S_Y$, we can, respectively, obtain $\sum_{i=1}^n S_{A_i} \leqslant \sum_{i=1}^n S_{\bar{A}_i}$ and $\sum_{i=1}^n S_{\bar{A}_i} - nS_{A_1 A_2 \cdots A_n} \leqslant \sum_{i=1}^n S_{A_i}$. Therefore, $I(A_1 : A_2 : \cdots : A_n) = \sum S_{A_{k_1} A_{k_2} \cdots A_{k_{n-1}}} - (n-1)S_{A_1 A_2 \cdots A_n} \geqslant \sum_{i=1}^n S_{A_i} - (n-1)S_{A_1 A_2 \cdots A_n} = I_x(A_1 : A_2 : \cdots : A_n) - (n-2) S_{A_1 A_2 \cdots A_n}$. Again, $I(A_1 : A_2 : \cdots : A_n) = \sum S_{A_{k_1} A_{k_2} \cdots A_{k_{n-1}}} - (n-1)S_{A_1 A_2 \cdots A_n} = (\sum_{i=1}^n S_{\bar{A}_i} - nS_{A_1 A_2 \cdots A_n}) + S_{A_1 A_2 \cdots A_n} \leqslant \sum_{i=1}^n S_{A_i} + S_{A_1 A_2 \cdots A_n} = I_x(A_1 : A_2 : \cdots : A_n) + 2S_{A_1 A_2 \cdots A_n}$. Hence, the proof. ■

The lower bound being dependent on n is weak. Moreover, we find numerically that for an n-party quantum system $\rho_{A_1 A_2 \cdots A_n} (n = 3,4)$, we have

$$0 \leqslant I^{(n)} - I_x^{(n)} \leqslant I_x^{(n)} - \sum I^{(2)}, \tag{14}$$

where $I^{(k)}$ is the k-party quantum mutual information, and the inequality is saturated for $n = 3$ (this can be shown analytically).

III. MULTIPARTY QUANTUM DISCORD

In this section, we extend the definition of bipartite quantum discord [24,25] to a multipartite setting. Quantum discord for a bipartite quantum state ρ_{AB} is defined as $\mathcal{D}(\rho_{AB}) = I(\rho_{AB}) - \max_{\mathcal{M}} J(\rho_{AB})$, where $I(\rho_{AB}) = I(A : B) = S(A) + S(B) - S(AB)$ and $J(\rho_{AB}) = S(B) - S_{\mathcal{M}}(B|A)$. Here, a measurement is performed on subsystem A with a rank-one projection-valued measurement $\{A_i\}$, producing the states $\rho_{B|i} = \frac{1}{p_i}\mathrm{tr}_A[(A_i \otimes I_B)\rho(A_i \otimes I_B)]$, with probability $p_i = \mathrm{tr}_{AB}[(A_i \otimes I_B)\rho(A_i \otimes I_B)]$. I is the identity operator on the Hilbert space of B. Hence, the conditional entropy of ρ_{AB} is given by $S_{\mathcal{M}}(B|A) = \sum_i p_i S(\rho_{B|i})$.

Three-party quantum discord can then be defined, when measurement is performed on subsystem A, subsystem AB, and the whole system, as follows,

$$\mathcal{D}_A(\rho_{ABC}) = I(\rho_{ABC}) - \max_{\Phi_A} I(\Phi_A(\rho_{ABC})), \tag{15}$$

$$\mathcal{D}_{AB}(\rho_{ABC}) = I(\rho_{ABC}) - \max_{\Phi_{AB}} I(\Phi_{AB}(\rho_{ABC})), \tag{16}$$

and

$$\mathcal{D}_{ABC}(\rho_{ABC}) = I(\rho_{ABC}) - \max_{\Phi_{ABC}} I(\Phi_{ABC}(\rho_{ABC})), \tag{17}$$

where $I(\sigma_{XYZ}) = I(X : Y : Z)$, $\Phi_A(\rho_{ABC}) = \sum_i \Phi_{A_i}\rho_{ABC}\Phi_{A_i}$, $\Phi_{AB}(\rho_{ABC}) = \sum_{i,j} \Phi_{A_i B_j}\rho_{ABC}\Phi_{A_i B_j}$, and $\Phi_{ABC}(\rho_{ABC}) = \sum_{i,j,k} \Phi_{A_i B_j C_k}\rho_{ABC}\Phi_{A_i B_j C_k}$ with $\Phi_{A_i} = \pi_i \otimes I \otimes I$, $\Phi_{A_i B_j} = \pi_i \otimes \pi_j \otimes I$, and $\Phi_{A_i B_j C_k} = \pi_i \otimes \pi_j \otimes \pi_k$. Equation (17) is the symmetric quantum discord or global quantum

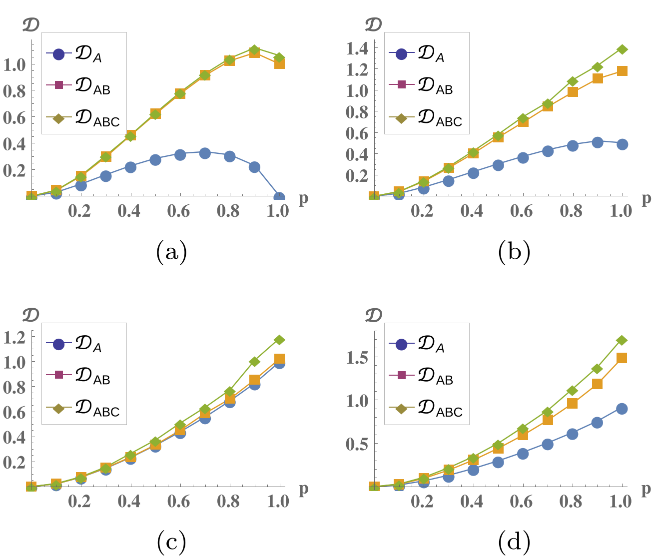

FIG. 3. Plots of quantum discords based on an alternative definition of QMI (top panel) and conventional quantum discords (bottom panel) $\mathcal{D}_A$, $\mathcal{D}_{AB}$, and $\mathcal{D}_{ABC}$ against the white-noise parameter p, of three-party states $\rho_{ABC} = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{I}{8}$ [(a) and (c)], and $\rho_{ABC} = p|\mathrm{W}\rangle\langle\mathrm{W}| + (1-p)\frac{I}{8}$ [(b) and (d)]. Contrary to our intuition, quantum discord increases when measurement is performed on a larger number of subsystems. The higher values of quantum discord suggest that the W state is more robust, as compared to the GHZ state, against measurement.

discord (GQD) [4]. Similarly, multiparty quantum discord can be defined.

Quantum discord, employing the von Neumann entropy, of a three-party GHZ state and W state admixed with white noise is shown in Fig. 3. Quite unexpectedly, we find that $\mathcal{D}_A \leqslant \mathcal{D}_{AB} \leqslant \mathcal{D}_{ABC}$, that is, quantumness increases when a measurement is performed on a large number of subsystems. This observation seems to be independent of the definition of quantum mutual information. The symmetric quantum discord, which requires measurement on all the parties, reveals the maximal quantumness. This contradicts the interpretation of measured mutual information as classical information because measuring more than one subsystem should yield more classical information and hence less quantum discord.

IV. CONCLUSION

To sum up, we have proposed an alternative definition of quantum mutual information for a multipartite setting. It is non-negative, and obeys monotonicity under partial trace and any completely positive map. We argue that it manifests a total correlation of a multiparty quantum system. We then employed this definition of quantum mutual information to define multiparty quantum discord. Surprisingly, we found that more quantumness can be harnessed by performing measurements on a larger number of parties, which is quite counterintuitive. Symmetric quantum discord reveals a maximal quantumness. This suggests that measured mutual information should not be interpreted as a classical correlation. We believe that our work will provide further insights in understanding the nature of nonclassical correlations.

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