Computational complexity of exterior products and multiparticle amplitudes of noninteracting fermions in entangled states

Dmitri A. Ivanov
Institute for Theoretical Physics, ETH Zürich, 8093 Zürich, Switzerland
and Department of Physics, University of Zürich, 8057 Zürich, Switzerland
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Noninteracting bosons were proposed to be used for a demonstration of quantum-computing supremacy in a boson-sampling setup. A similar demonstration with fermions would require that the fermions are initially prepared in an entangled state. I suggest that pairwise entanglement of fermions would be sufficient for this purpose. Namely, it is shown that computing multiparticle scattering amplitudes for fermions entangled pairwise in groups of four single-particle states is #P-hard. In linear algebra, such amplitudes are expressed as exterior products of two-forms of rank 2. In particular, a permanent of a $N \times N$ matrix may be expressed as an exterior product of N^2 two forms of rank 2 in dimension $2N^2$, which establishes the #P-hardness of the latter.

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I. INTRODUCTION

Quantum devices are believed to have the potential to outperform classical computers [1]. One of the challenges in the field of quantum computing is characterizing the scope of computational tasks where quantum computers would be useful (the most famous example of such a task is Shor's factorization algorithm [2]). On the hardware side, there exist numerous proposals of quantum-computing devices, but a suitable scalable hardware still needs to be invented [3].

One approach which targets the two above goals simultaneously is the so-called "quantum supremacy" demonstration: finding a task (even possibly useless for practical purposes) that can be performed efficiently by a quantum device but not by a classical computer (see, e.g., Ref. [4] and references therein). An example of such a quantum supremacy task is the Boson-Sampling proposal [5]: Noninteracting bosons are sent to a subset of input channels of a specially designed scattering matrix [Fig. 1(a)]. After scattering, the bosons are distributed among the output channels with the probabilities determined by the amplitudes of the corresponding multiparticle scattering processes. The authors of the proposal argue that modeling this sampling process on a classical computer would most likely require an exponentially large computational effort (assuming the P $\neq$ NP conjecture).

The key reason for the quantum supremacy of Boson Sampling is the computational complexity of the corresponding multiparticle amplitudes. For noninteracting bosons, these amplitudes are given by the matrix permanent,

$$\mathrm{Per}(A) = \sum_{\sigma} \prod_{i=1}^{N} a_{i\sigma(i)}. \tag{1}$$

Here A is a square $N \times N$ matrix with entries a_{ij}, and the sum is performed over all permutations σ of N elements (in application to the Boson-Sampling setup, A is the submatrix of the full scattering matrix spanned by the input and output channels). The (exact) computation of a permanent is, in turn, known to be a #P-complete problem [6], which therefore is believed (assuming the P $\neq$ NP conjecture) to be not solvable on classical computers in polynomial time. Note however that since Boson Sampling is not equivalent to computing a permanent, the actual argument in favor of the quantum supremacy of Boson Sampling is more involved: See Ref. [5] for details.

At the same time, the straightforward counterpart of the Boson-Sampling proposal with fermions does not work: The corresponding amplitudes for fermions are given by determinants, which are computable in polynomial time [7]. The resolution of this apparent "supersymmetry breaking" is that it is the non-Gaussian property of the initial state that is crucial for the complexity of the quantum computation [8,9]. For bosons, the state of one boson per channel is non-Gaussian and therefore provides a complexity resource. For fermions, on the other hand, the single-particle state is Gaussian, and therefore manipulations with such states do not raise complexity beyond the single-particle level. This difference was illustrated in Ref. [10] where it was shown that Boson Sampling can, in fact, be simulated with fermions, provided that the fermions initially are prepared in a specially entangled state.

The construction of Ref. [10] involves fermions with a large number of internal quantum degrees of freedom (equal to the number of particles). One can try to optimize this construction by using simpler non-Gaussian states of fermions. One of the simplest non-Gaussian states is the entangled state of two fermions in four single-particle states,

$$\Psi_4 = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle), \tag{2}$$

where 1100 and 0011 refer to the fermionic occupation numbers in the four states [Fig. 1(b)].

The goal of the present paper is to demonstrate that, if the initial state of fermions is given by a product of the entangled states (2), then the multiparticle amplitudes of a general noninteracting evolution are #P-hard, similar to the Boson-Sampling proposal. Specifically, consider a scattering problem for $2M$ noninteracting fermions distributed over $4M$ input channels divided into M quadruplets, each of those quadruplets being prepared in state Ψ_4 [Fig. 1(c)]. Then the multiparticle amplitude for any noninteracting evolution is given by a sum of 2^M determinants $2M \times 2M$ composed of $4M \times 2M$ elements of the scattering matrix spanned by the

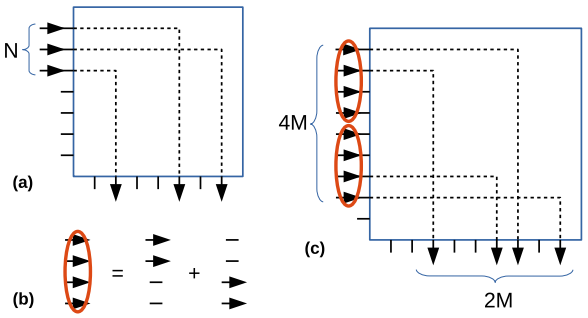

FIG. 1. (a) The Boson-Sampling setup. The square represents the scattering matrix with rows corresponding to the input channels and columns corresponding to the output channels. The dashed lines represent one of the multiparticle processes participating in the interference. (b) The four-channel entanglement of Eq. (2) [the factor $1/\sqrt{2}$ is omitted for simplicity]. (c) The fermion-sampling setup with entangled quadruplets.

input and output channels. For notational convenience, we group these elements into $4M$ rows of $2M$ elements each and denote these rows $v_1,\ldots,v_{4M}$. Then the multiparticle amplitude (multiplied by the factor 2^M for convenience) is

$$D_{2,2}(v_1,\ldots,v_{4M}) = \sum_{\substack{i_k=0,1,\\ k=1,\ldots,M}} \det \begin{bmatrix} v_{2i_1+1} \\ v_{2i_1+2} \\ v_{2i_2+5} \\ v_{2i_2+6} \\ \vdots \\ v_{2i_M+4M-3} \\ v_{2i_M+4M-2} \end{bmatrix}, \qquad (3)$$

where the matrix on the right-hand side is composed of the corresponding rows v_i. For each pair of rows, two pairs of vectors v_i are considered (for the first pair of rows, either v_1 and v_2 or v_3 and v_4, and so on). In linear algebra, the same function may be identified with the exterior product of two forms of rank 2,

$$\begin{aligned} &D_{2,2}(v_1,\ldots,v_{4M}) \\ &\quad = (v_1\wedge v_2 + v_3\wedge v_4)\wedge(v_5\wedge v_6 + v_7\wedge v_8) \\ &\qquad \wedge\cdots\wedge(v_{4M-3}\wedge v_{4M-2} + v_{4M-1}\wedge v_{4M}), \end{aligned} \qquad (4)$$

where $\wedge$ denotes an exterior product (see, e.g., Ref. [11] or other textbooks).

I explicitly show that the above function is computationally #P-hard by a reduction of a N-dimensional permanent to this function at $M = N^2$. This proves that, modulo a polynomial overhead, the considered multiparticle amplitude is at least as computationally difficult as as a permanent, which, in turn, is known to be #P-complete [6]. The details of the proof are presented in Sec. II.

An alternative proof of the #P-hardness of this function was communicated to me by Gurvits [12] based on a relation to mixed discriminants [13–15]. This proof is outlined in Sec. III.

Section IV contains a brief discussion of the result, including some simple generalizations and a possible extension to approximate computations.

Finally, the Appendix contains an explicit form of the construction of the proof of Sec. II for the $N = 3$ case.

II. PROOF OF #P-HARDNESS USING PERMANENTS

The proof can most easily be formulated in terms of graphs. The permanent of a matrix A of dimension N with coefficients a_{ij} is defined by Eq. (1). We can think of A as a weighted adjacency matrix for a graph with N nodes so that a_{ij} is the weight attributed to the edge directed from i to j. In this representation, $\mathrm{Per}(A)$ can be thought of as the sum of products of weights over all cycle covers of this directed graph [7].

A similar representation is possible for the function (3)–(4). Namely, consider a directed graph with $2M$ nodes and edges colored in two colors (dubbed "color 1" and "color 2" below and shown as solid and dashed arrows in the figures). Let vectors v_1 and v_2 contain weights attributed to color-1 edges, originating from nodes 1 and 2, respectively, vectors v_3 and v_4 contain weights attributed to color-2 edges, originating from the same nodes 1 and 2, respectively, and so on. Generally, vectors v_{4i-3} and v_{4i-2} correspond to color-1 edges originated from nodes $2i-1$ and $2i$, respectively, and vectors v_{4i-1} and v_{4i} correspond to their color-2 counterparts. Then $D_{2,2}(v_1,\ldots,v_{4M})$ can be viewed as the sum of products of weights, multiplied by the corresponding signs, over all cycle covers of this directed graph under the constraint that, for each pair of nodes (1,2),(3,4),...,$(2M-1,2M)$, the cycle cover uses the same color for edges originating from the two nodes in the pair. The sign factor is determined as the parity of the total number of cycles.

The idea of the proof is to construct, for each directed graph for a matrix permanent, a two-color directed graph for the function (3)–(4) so that the loop covers are in one-to-one correspondence in the two graphs and produce the same weights. In order to cancel the sign factors, we double the number of nodes: The nodes in the two-color graph will be denoted as "A nodes" and "B nodes," and the edges will only connect nodes of the same type. At the same time, the coloring scheme will be used in such a way as to constrain the cycle cover of A nodes to exactly repeat the cycle cover of B nodes so that the sign factor cancels out.

The construction of such a two-color directed graph is shown in Fig. 2. Without loss of generality, we consider node 1. It has N outgoing edges (to the same node and to the $N-1$ other nodes). This node and the outgoing edges are replaced by $2N$ nodes (N A nodes and N B nodes) and the corresponding outgoing edges as shown in the figure. The A nodes are paired with the corresponding B nodes (in the figure, node 1A forms a pair with node 1B, and node 1A2 forms a pair with node 1B2, etc.).

This construction is repeated for each node of the original directed graph for the matrix permanent. As a result, the two-color directed graph for the function (3)–(4) contains $2N^2$ nodes.

On inspection, the constraint of the cycle covers in the two-color directed graph guarantees that the cycles in the A nodes exactly reproduce the cycles in the corresponding B nodes. This cancels out the sign factor. At the same time, the product of the weights of the edges reproduces the product of the edges in the corresponding cycle cover of the directed

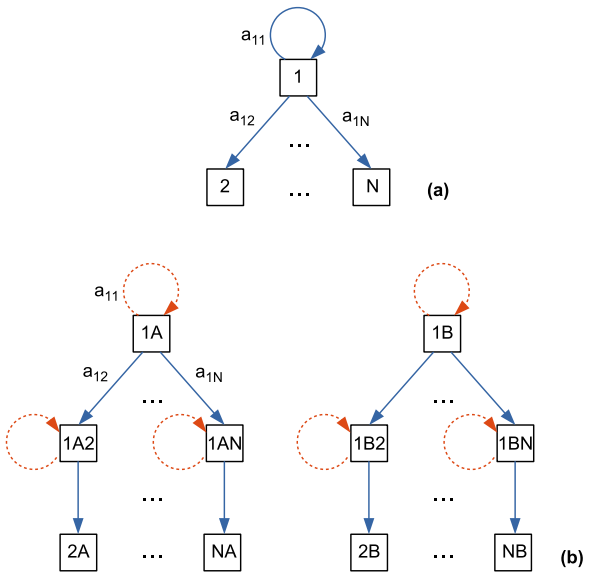

FIG. 2. The construction of the two-color directed graph for the function (3)–(4) [panel (b)] from the graph for a permanent [panel (a)]. The weights of the directed edges are labeled as a_{ij}. The unlabeled edges have weight one. The nodes of the two-color directed graph are grouped in pairs: 1A together with 1B, 1A2 together with 1B2, and so on (every node labeled with letter A is paired with the corresponding node with letter B). Within each pair, the color of the outgoing edges in the cycle cover must be unique.

graph for the permanent. This proves that the function (3)–(4) calculated for the constructed two-color directed graph equal the permanent of the matrix (a_{ij}).

An example of the 3×3 matrix is presented in the Appendix.

Since the computation of the permanent of a matrix with integer elements is #P-complete [6], this proves that the computation of the function $D_{2,2}(v_1, \ldots, v_{4M})$ for integer-valued vectors v_i is #P-hard (a computation of the permanent can be reduced to this function with a polynomial time overhead).

III. ALTERNATIVE PROOF USING MIXED DISCRIMINANTS

I am grateful to Gurvits [12] for bringing to my attention the following alternative proof using the theory of mixed discriminants [13–15]. Namely, Theorem 3.4 of Ref. [14] states the #P-hardness of computing the mixed discriminant $D(A_1, \ldots, A_M)$ of rank-2 real symmetric positive semidefinite matrices $A_i = x_{i,0}x_{i,0}^* + x_{i,1}x_{i,1}^*$. If we introduce the $4M$ vectors in the $(M+M)$-dimensional space $v_{4i-3} = x_{i,0} \oplus 0$, $v_{4i-2} = 0 \oplus x_{i,0}$, $v_{4i-1} = x_{i,1} \oplus 0$, $v_{4i} = 0 \oplus x_{i,1}$ (with $i = 1, \ldots, M$), then one can verify that the mixed discriminant can be expressed in terms of the exterior product (4) as

$$D(A_1, \ldots, A_M) = (-1)^{M(M-1)/2} D_{2,2}(v_1, \ldots, v_{4M}) \qquad (5)$$

(the above relation follows, e.g., from Lemma 5.2.1 of Ref. [16]). This proves the #P-hardness of the exterior product (4).

IV. DISCUSSION

The above proof admits several simple generalizations and corollaries.

First, the above relation between the permanent and the function (3)–(4) holds for coefficients in any field or, even more generally, in any commutative ring.

Second, one can generalize the function (4) to an exterior product of k forms of rank $\leqslant r$ in kM-dimensional linear space,

$$D_{k,r}(\omega_1, \ldots, \omega_M) = \omega_1 \wedge \cdots \wedge \omega_k, \qquad (6)$$

where all ω_i's are k-forms of rank $\leqslant r$. The function $D_{2,2}$ is the simplest nontrivial example of this construction. Moreover, for any $k \geqslant 2$ and $r \geqslant 2$, the function $D_{k,r}$ includes $D_{2,2}$ as a particular case. This implies that the more general function $D_{k,r}$ is also #P-hard (for $k \geqslant 2$ and $r \geqslant 2$). On the other hand,

TABLE I. Vectors v_i for the identity (A1). Each of these vectors has 18 components (grouped in pairs for better visualization). The empty spaces denote zeros.

v_1	a_{11}								
v_2		1							
v_3					a_{12}	a_{13}			
v_4						1	1			
v_5			a_{22}							
v_6				1						
v_7							a_{21}	a_{23}		
v_8								1	1		
v_9				a_{33}						
v_{10}					1					
v_{11}									a_{31}	a_{32}
v_{12}										1	1
v_{13}					1					
v_{14}						1				
v_{15}			1							
v_{16}				1						
v_{17}						1				
v_{18}							1			
v_{19}				1						
v_{20}					1					
v_{21}							1			
v_{22}								1		
v_{23}	1									
v_{24}		1								
v_{25}								1		
v_{26}									1	
v_{27}				1						
v_{28}					1					
v_{29}									1	
v_{30}										1	
v_{31}	1									
v_{32}		1								
v_{33}										1
v_{34}											1
v_{35}			1							
v_{36}				1						

for $k = 1$ or $r = 1$, the function $D_{k,r}$ is the determinant and is computable in polynomial time.

The proof above shows that the exact computation of the function (3)–(4) is at least as difficult as the exact computation of the permanent. Yet another interesting question is an approximate computation. Although the permanent of a matrix with positive entries admits an efficient (randomized) approximate calculation in polynomial time [17], even an approximate calculation (up to a multiplicative factor) of a permanent in the general case is believed to be exponentially hard [14,18,19]. If this is indeed the case, the worst-case running time of an approximate calculation of the function (3)–(4) should also be exponential.

Like in the Boson-Sampling case, the #P-hardness of the scattering amplitudes does not automatically imply the quantum supremacy of the proposed fermion-sampling setup. A proof (or a refutation) of such a quantum supremacy would go beyond the scope of this paper and presents a serious challenge, similar to the Boson-Sampling case [5].

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APPENDIX

The simplest nontrivial case that illustrates the construction shown in Fig. 2 is the 3×3 matrix. In that case, we have the identity,

$$\mathrm{Per}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = D_{2,2}(v_1, \ldots, v_{36}), \tag{A1}$$

where the vectors $v_1, \ldots, v_{36}$ are listed in Table I.

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