Macroscopic realism of quantum work fluctuations

Ralf Blattmann and Klaus Mølmer
Department of Physics and Astronomy, Aarhus University, DK-8000 Aarhus C, Denmark
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We study the fluctuations of the work performed on a driven quantum system, defined as the difference between subsequent measurements of energy eigenvalues. These work fluctuations are governed by statistical theorems with similar expressions in classical and quantum physics. We show that we can distinguish quantum and classical work fluctuations, as the latter can be described by a macrorealistic theory and hence obey Leggett-Garg inequalities. We show that these inequalities are violated by quantum processes in a driven two-level system and in a harmonic oscillator subject to a squeezing transformation.

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I. INTRODUCTION

The thermodynamics of quantum systems has become a rapidly expanding field of research in recent years [1,2]. One of its main lines of research has been triggered by the discovery of fluctuation relations, especially the classical work fluctuation theorems [3,4] and the derivation of their quantum mechanical counterparts [5]. One prominent example of such a fluctuation theorem is the Jarzynski relation [3],

$$\langle e^{-\beta w}\rangle_w = e^{-\beta \Delta F}, \tag{1}$$

which relates the average work w performed during a nonequilibrium transformation with the free-energy difference ΔF between two thermal states at inverse temperature $\beta = 1/kT$. It has been shown that if the work performed on a quantum system is defined in a suitable way, the Jarzynski inequality (1) holds for classical as well as for quantum systems [6]. While there has been some debate as to what "suitable" means in this context [5,7,8], a widely accepted definition of quantum work is given by the difference in the outcome of projective measurements of the Hamiltonian operator at different times. With this definition, the quantum work becomes, in general, a fluctuating quantity, similar to the fluctuating work in classical nonequilibrium thermodynamics. While the (nonequilibrium) work is characterized by a (classical) probability density in both cases, the origin of the randomness can be quite different.

In classical physics, energies have definite values and work fluctuations stem from (thermal) fluctuations of mixed initial states or, in the case of an open system, the random exchange of energy with the particles in the surrounding heat bath, while in quantum mechanics, they originate partially from quantum uncertainty and the resulting fundamental randomness of the outcome of measurement, i.e., Born's rule and the projection postulate.

In this work, we address the question of "how quantum is quantum work" by investigating to what extent the different origins of quantum and classical work fluctuation have measurable consequences.

Our approach to the problem will be to consider processes where the energy of the system is measured multiple times (see Fig. 1) and where we can hence study temporal correlations in the work fluctuations. If the work behaves classically, it should be describable as a macroscopic, realistic variable, which is measurable in a noninvasive manner, and hence its correlations should obey the Leggett-Garg inequalities [9,10]. Leggett-Garg inequalities have been used to analyze quantum effects in thermodynamics processes and heat engines [11]. While they only apply for correlation functions of dichotomic variables in their original form, entropic Leggett-Garg inequalities have been derived for correlations of more general variables.

This article is structured as follows: In Sec. II, we introduce the definition of quantum work and its probability distribution. In Sec. III, we recall the dichotomic and entropic Leggett-Garg inequalities and we discuss their application to the work done on a quantum system. In Secs. IV and V, we investigate whether the inequalities are obeyed or violated for a driven two-level system and a squeezed harmonic oscillator, respectively. In Sec. VI, we discuss the consequences and possible applications of our results.

II. QUANTUM WORK AND ITS PROBABILITY DISTRIBUTION

Consider an isolated quantum system with a time-dependent Hamiltonian $H_t = H(\lambda_t)$, where λ_t is a varying control parameter. The state of the system obeys the Liouville–von Neumann equation $i\hbar\dot{\rho}_t = [H_t,\rho_t]$, and, in general, work will be performed on it, i.e., energy will be injected into (or removed from) the system. In order to determine the work performed during a given process (quantum as well as classically), one measures the energy of the system before and after the process. While in classical systems such measurements are unproblematic, the measurement on a quantum system will in general have random outcomes and it will change the state of the system. If one wants to measure the work performed on the system between the times t_0 and t_1, one has to probe the system energy at the beginning and the end of the driving. This will yield one of the eigenvalues $E^0_{k_0}$ of H_{t_0} with a probability p_{k_0}, where $p_{k_i} = \mathrm{tr}[\Pi^i_{k_i}\rho_i]$, and, subsequently, an eigenvalue $E^1_{k_1}$ of H_{t_1}, with the joint probability $p_{k_1,k_0} = p_{k_1|k_0}p_{k_0}$ depending on the first measurement at t_0 and the conditional probability $p_{k_i|k_j} = \langle k_i|U_{i,j}|k_j\rangle$. Here, we have introduced the projection operators on the energy eigenstates, $\Pi^\alpha_{k_\alpha} = |k_\alpha\rangle\langle k_\alpha|$ of H_t at $t = t_\alpha$, $\alpha = 0,1$ and the time-evolution operator $U_{i,j} = \mathcal{T}\exp[-i\int_{t_j}^{t_i} H_t dt]$ from time t_j to t_i ($\mathcal{T}$ denotes the time ordering operator). Subsequent time evolution of the quantum state and measurements are described by the same formalism; cf. Fig. 1.

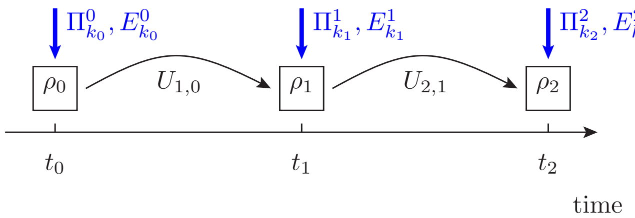

FIG. 1. Illustration of our work-measurement process. The system is prepared in an initial thermal state and a measurement of its energy projects it into an energy eigenstate $|k_0\rangle$ of the Hamiltonian H_{t_0} at $t = t_0$. The Hamiltonian leads to a time evolution of the quantum state between t_0 and the later time t_1, which can be expressed as a unitary operator $U_{1,0}$, which governs the expansion of the state on the energy eigenstates of H_{t_1} and causes the stochastic nature of the energy measurements at t_1. The work is defined as the difference $E^1_{k_1} - E^0_{k_0}$ between the system eigenenergies measured, and, by evolving the system further and measuring the energy at the later time t_2, we can study correlations between the work done during the two evolution processes.

One obtains the work by merely subtracting the measured energies,

$$W_{k_1,k_0} = E^1_{k_1} - E^0_{k_0}, \tag{2}$$

and since it depends on the random outcome of projective energy measurements, it is an inherently fluctuating quantity. Its probability distribution is given by [5]

$$p(w) = \sum_{k_0,k_1} \delta(w - W_{k_1,k_0}) p_{k_1,k_0}, \tag{3}$$

where, according to the above arguments,

$$p_{k_i,k_j} = \mathrm{tr}\big[\Pi^i_{k_i} U_{i,j} \Pi^j_{k_j} \rho_j \Pi^j_{k_j} U^\dagger_{i,j}\big] \tag{4}$$

is the joint probability distribution for measuring $E^j_{k_j}$ at t_j and $E^i_{k_i}$ at t_i.

Assuming that the system is initially in a thermal state $\rho_0 = \exp(-\beta H_0)/Z_0$ with $Z_0 = \mathrm{tr}[-\beta H_0]$, it is rather straightforward to use the characteristic function [6] for the work probability distribution given by Eq. (3) and verify that the quantum Jarzynski equality (1) holds, where $\langle \ldots \rangle_w = \int \ldots p(w) dw$ and $\Delta F = F_1 - F_0$ with the free energy $F_t = -(1/\beta)\ln(Z_t)$.

Instead of measuring the energy only in the beginning and the end of a process, one might perform several energy measurements at different times; see Fig. 1. Probing the system energy at three instants of time, t_0, t_1, t_2, then leads to the joint probability distribution,

$$p_{k_2,k_1,k_0} = \mathrm{tr}\big[\Pi^2_{k_2} U_{2,1} \Pi^1_{k_1} U_{1,0} \Pi^0_{k_0} \rho_0 \Pi^0_{k_0} U^\dagger_{1,0} \Pi^1_{k_1} U^\dagger_{2,1}\big]. \tag{5}$$

By summing over indices, one obtains, e.g., p_{k_1,k_0} and p_{k_2,k_1} from (5). Note that since in general $[\Pi^j_{k_j}, U_{i,j}] \neq 0$, the distribution arising from summing over the middle index k_1 is not equivalent to the distribution p_{k_2,k_0} without measurement at t_1.

From Eq. (5), we define the joint probability,

$$p(w_1,w_2) = \sum_{k_0,k_1,k_2} \delta\big(w_1 - W_{k_0,k_1}\big)\delta\big(w_2 - W_{k_1,k_2}\big) \times p_{k_2,k_1,k_0}, \tag{6}$$

to perform the work w_1 between t_0 and t_1, and the work w_2 between t_1 and t_2. Note that its marginal distribution $p(w_1) = \int dw_2 p(w_1,w_2)$ is equivalent to Eq. (3) and hence fulfills the fluctuation relations, while the marginal $p(w_2) = \int dw_1 p(w_1,w_2)$ will in general not fulfill such relations because the system is not in an equilibrium state at t_1.

Using Eq. (6), the probability for the total work $w_{\mathrm{tot}} = w_1 + w_2$ yields

$$p(w_{\mathrm{tot}}) = \int dw_1 dw_2 \delta(w_{\mathrm{tot}} - [w_1 + w_2]) p(w_1,w_2) = \sum_{k_0,k_1,k_2} \delta\big(w_{\mathrm{tot}} - W_{k_2,k_0}\big) p_{k_2,k_1,k_0}, \tag{7}$$

i.e., it depends, as one would expect, only on the difference between the first and the last energy measurements. While the intermediate measurements will, in general, influence the final energy measurement [12], one can prove that the Jarzynski relation (1) still holds for the total work [12,13].

The measurement backaction and, in particular, the destruction of quantum mechanical coherence by the middle measurement (the first measurement acts on a thermal state with already vanishing coherences) presents a fundamental difference between the definition of work in quantum and classical contexts. While a joint pseudoprobability distribution for the values of the noncommuting energy observables at different times can be used to theoretically define work with the proper mean value and fluctuation properties [14], it has been shown that no protocol to *experimentally* measure work exists that yields a mean value equal to the average energy difference and follows the usual statistics in the case of incoherent mixtures [15].

In this article, we shall instead retain the generally accepted definition of work and address the consequences of its invasiveness in a more quantitative manner. To this end, we shall appeal to the Leggett-Garg inequalities [9,10], which precisely concern correlations between measurements performed at different times on a quantum system. We note that Leggett-Garg inequalities have also been applied to characterize the quantumness of a quantum heat engine through the correlation between the working system observables at different times [11].

III. LEGGET-GARG INEQUALITIES FOR WORK MEASUREMENTS

Assuming macroscopic realism and noninvasive measurability of a dichotomic variable that is measured at different times, t_i, with output values $Q_i = \pm 1$, Leggett and Garg derived the inequality [9,10]

$$C_{21} + C_{32} - C_{31} \leqslant 1 \tag{8}$$

for the two-time correlation functions $C_{ij} = \langle Q_i Q_j \rangle$.

Here, macroscopic realism means that a (macroscopic) object with several distinct states is at any time definitely in one of these states, while noninvasive measurability means that it is, at least in principle, possible to determine in which state the system is without disturbing it. If both conditions were fulfilled, it would be possible to describe the process in Fig. 1

as a classical process with a probability distribution for the system energy existing independent from the measurements.

While Eq. (8) is obeyed for classical dynamics, measurements on a quantum system may violate the Leggett-Garg inequality [16,17]. This violation is readily understood as a consequence of the measurement backaction, which is absent in the classical case. In the present work, we shall use the Leggett-Garg inequality to assess how the definition of quantum work as the result of projective energy measurements necessarily implies a quantitative difference between the fluctuations of quantum and classical work. Note that Eq. (8) is not associated with the absolute magnitude of energy and work measurements, but only the statistical correlations of the variables $Q_i = \pm 1$, which we can associate with the projective measurements on two different eigenstates.

For systems with more eigenstates, the measurement outcome Q_i at time t_i may attain more than two values $\{q_i\}$, and an alternative, entropic Leggett-Garg inequality has been derived for the correlations between such multivalued measurement outcomes [18],

$$H(Q_2|Q_0) \leqslant H(Q_2|Q_1) + H(Q_1|Q_0). \tag{9}$$

Here, $H(Q_j|Q_i) = -\sum_{q_j,q_i} p(q_i)p(q_j|q_i)\log_2 p(q_j|q_i)$ is the (classical) conditional entropy, where $p(q_j|q_i)$ is the conditional probability for outcome q_j given the earlier outcome q_i [occurring with probability $p(q_i)$].

We shall now apply the Leggett-Garg and entropic Leggett-Garg inequalities to the correlations given by Eq. (4) of energy measurements. These correlations reflect how the definition of work is affected by measurement backaction effects and, hence, to what extent the underlying thermodynamic transformation can be modeled as a classical process.

The mean conditional entropy for energy measurements at two instants of time is given by

$$H(E^j|E^i) = -\sum_{k_i,k_j} p_{k_j,k_i}\log_2 p_{k_j|k_i}, \tag{10}$$

where the conditional probability $p_{k_j|k_i}$ is the quantum mechanical transition probability $|k_i\rangle \to |k_j\rangle$ between the eigenstates of the Hamiltonian at times t_i and t_j, as governed by the unitary time-evolution operator $U_{j,i}$, defined above.

For a slowly varying Hamiltonian, the system will adiabatically follow the time-dependent eigenstate in which it is prepared by the first measurement, and hence $p_{k_1|k_0} = \delta_{k_1,k_0}$ (note the eigenenergies may generally differ and a definite, nonvanishing amount of work is hence done on the system). Also, during the subsequent evolution and measurement, the system follows the same (kth) eigenstate, and due to the definite outcomes, $H(E^j|E^i) = 0$ and the entropic Leggett-Garg inequality (9) for energy measurements is trivially fulfilled. In the following sections, we shall hence consider systems that do not evolve adiabatically.

Noting that the probability distribution for the work done on the system is governed by the joint probability of the two pertaining energy measurements, $w_{k_j,k_i} = E^j_{k_j} - E^i_{k_i}$, $p(w_{k_j,k_i}) = p(k_j,k_i) = p(k_j|k_i)p(k_i)$, we shall introduce the corresponding entropy $H(w_{ji}) = H(E^j,E^i) = -\sum_{k_i,k_j} p_{k_j,k_i}\log_2 p_{k_j,k_i}$. Using the identity between conditional and joint entropies [19], $H(E^j|E^i) = H(E^j,E^i) - H(E^i)$, where $H(E^i_{k_i}) = -\sum_{k_i} p_{k_i}\log_2 p_{k_i}$ is the entropy of the distribution p_{k_i}, we can rewrite the entropic Leggett-Garg inequality (9) as a relation for the work distributions,

$$\mathcal{H}(w_{20}) \leqslant \mathcal{H}(w_{21}) + \mathcal{H}(w_{10}) - H(E^1). \tag{11}$$

Here, $\mathcal{H}(w_{ij}) = -\sum_{p(w_{ij})\neq 0} p(w_{ij})\log_2 p(w_{ij})$ is the entropy of the work distribution, which is discrete if the corresponding energy spectrum is discrete. Note that Eq. (11) does not only depend on the entropy of work distributions but also on the entropy of the middle energy measurement $H(E^1)$. Disregarding this term leads to an inequality that is more easily fulfilled and which reflects that in a classical system, the entropy of the work distribution, without the middle measurement at t_1, is always smaller than with this measurement taking place because classical measurements do not decrease the information [18]. Note that it is easier to observe a violation of the Leggett-Garg inequality if the entropy of the middle energy measurement $H(E_1)$ is retained in Eq. (11).

As a side remark, we note that to go from Eq. (10) to Eq. (11), we assume that the work distribution has no "degeneracies", i.e., there are no k_j,k_i and k'_j,k'_i with $W_{k_j,k_i} = W_{k'_j,k'_i}$. While such degeneracies may be easy to avoid, they can also be accounted for by using the grouping formula for the Shannon entropy [19] where joint probabilities with $W_{k_j,k_i} = W_{k'_j,k'_i}$ are grouped together. If $\mathbf{p} = \{p_1,\ldots,p_n\}$ is a probability distribution and $\mathbf{q} = \{q_1,\ldots,q_m\}$ with $q_j = \sum_{i\in I_j} p_i$ is another distribution formed by "grouping" the probabilities p_i which correspond to a subset of events with indices I_j, the Shannon entropy yields

$$\begin{aligned} H(\mathbf{q}) &= H(\mathbf{p}) - \sum_j q_j H\left(\left\{\frac{p_i}{q_j}|i\in I_j\right\}\right) \\ &= H(\mathbf{p}) - \bar{H}(\mathbf{q}). \end{aligned} \tag{12}$$

Hence, the grouping reduces the Shannon entropy by an amount given by weighted entropies of subset probabilities.

IV. VIOLATION OF THE CONVENTIONAL AND THE ENTROPIC LEGGETT-GARG INEQUALITIES FOR A TWO-LEVEL SYSTEM

We recall that the work is defined through projective measurements in the eigenstate basis of the time-dependent Hamiltonian. One should hence determine the time-evolution operator in Eq. (4) by solving the Schrödinger equation, and subsequently evaluate its matrix elements between the eigenstates at different times. Since both the time evolution of the state of the system and the time dependence of the eigenstates are governed by unitary operations, the evolution with respect to the time-dependent eigenbasis of the Hamiltonian is also given by a unitary matrix. For the evolution between the first and the middle measurement, this matrix can hence be expressed as

$$U_{1,0} = \begin{pmatrix} e^{\frac{i}{2}(\alpha+\beta)}\cos\frac{\theta}{2} & e^{\frac{i}{2}(\alpha-\beta)}\sin\frac{\theta}{2} \\ -e^{\frac{i}{2}(\alpha-\beta)}\sin\frac{\theta}{2} & e^{\frac{-i}{2}(\alpha+\beta)}\cos\frac{\theta}{2} \end{pmatrix}, \tag{13}$$

with angles α, β, and θ. Since this matrix yields the evolution of the system with respect to the basis of energy eigenstates, the value of θ directly parametrizes the outcome probabilities for the energy measurements and a small value of θ, e.g., represents the case of adiabatic evolution. In that limit, the system follows the energy eigenstates and the measurement outcomes at different times are correlated. With the representation of the unitary evolution operators $U_{1,0}$ and the similarly defined $U_{2,1}$, we readily obtain the joint probabilities in Eq. (4).

For simplicity of analysis, we set $\alpha = \beta = 0$, so that $U_{1,0}$ becomes a real rotation matrix. The angle θ in Eq. (13) controls how much population is transferred between the eigenstates at different times and, for simplicity, we shall assume that $U_{2,1} = U_{1,0}$.

Mutatis mutandis. We can obtain the joint probabilities p_{k_2,k_0} and p_{k_2,k_1}, so that we can study the violation given by Eq. (8), where the dichotomic variable takes the values ± 1 in the ground and excited states, and of Eq. (11).

In order to quantify the violation of the Eqs. (8) and, respectively, Eq. (11), we define the Leggett-Garg parameters,

$$\mathcal{K}_3^{\mathrm{cor}} = \tfrac{1}{4}(1 - C_{01} - C_{02} + C_{12}) \tag{14}$$

and

$$\mathcal{K}_3^{\mathrm{en}} = \tfrac{1}{2}[\mathcal{H}(w_{21}) + \mathcal{H}(w_{10}) - \mathcal{H}(w_{20}) - H(E_1)], \tag{15}$$

where negative values of the parameters are a signature of nonclassical behavior. In Fig. 2, we plot $\mathcal{K}_3^{\mathrm{cor}}$ and $\mathcal{K}_3^{\mathrm{en}}$ as functions of the angle θ. There is some arbitrariness in assigning the values $Q_i = \pm 1$ to the two outcomes of the energy measurements and, in addition, to for $\mathcal{K}_3^{\mathrm{cor}}$, which does not violate the Leggett-Garg inequalities for values of θ around π, we show the function $\mathcal{K}_3^{\mathrm{cor}'}$ which follows from $\mathcal{K}_3^{\mathrm{cor}}$ by exchanging the values of $Q_1 = \pm 1$ [10]. By plotting both $\mathcal{K}_3^{\mathrm{cor}}$ and $\mathcal{K}_3^{\mathrm{cor}'}$, we show that energy measurements, indeed, violate the Leggett-Garg inequality for all values of $\theta \neq n\pi/2$. As the initial state, we chose a thermal state with $\beta = 1/\Delta E$, where ΔE is the energy splitting of the ground and excited states. Actually, for a two-level system, the violation of the Leggett-Garg inequality does not depend on the initial state and temperature, as the matrix $U_{1,0}$ yields the same probability

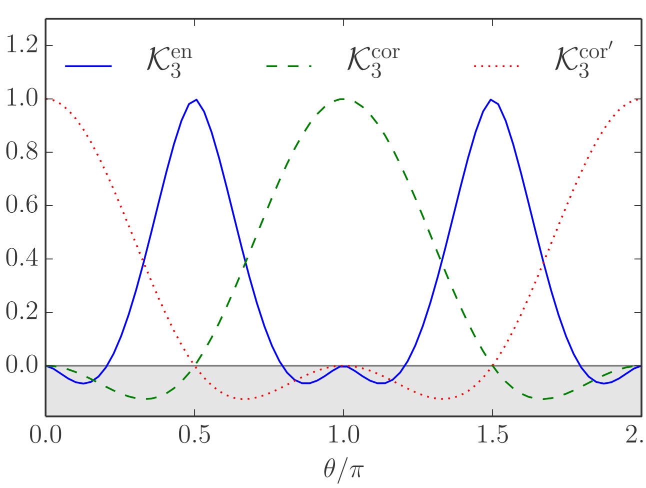

FIG. 2. The Leggett-Garg parameters given by Eqs. (14) and (15) as a function of the angle θ. Negative values (the gray area) imply a violation of the corresponding inequality.

to obtain the same and the opposite eigenstates in the middle measurement for both initial outcomes.

Figure 2 shows that for small finite θ, all curves are in the gray area where the Leggett-Garg inequality is violated. They, however, differ considerably. While either $\mathcal{K}_3^{\mathrm{cor}}$ or $\mathcal{K}_3^{\mathrm{cor}'}$ violate the conditions for macroscopic realism for all angles except for multiples of $\pi/2$, nonclassical correlations between measurements are not always revealed by the entropic Leggett-Garg inequality.

V. VIOLATION OF THE ENTROPIC LEGGETT-GARG INEQUALITY FOR A SQUEEZED HARMONIC OSCILLATOR

In this section, we study the entropic Leggett-Garg inequality for a harmonic oscillator. The quantum harmonic oscillator is in many aspects well described by classical physics, e.g., the evolution of the continuous position and momentum operators solve the same coupled linear equations as the classical coordinates. The work done on harmonically trapped particles has been studied quite extensively in both the classical and the quantum case [20–23].

The harmonic oscillator is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \tag{16}$$

with $a = \sqrt{m\omega/2\hbar}[x + (i/m\omega)p]$ and $[a,a^\dagger] = 1$. Driving the system with an arbitrary time-dependent potential strength ω_t will maintain the quadratic form of the Hamiltonian, and hence result in linear coupled equations for the position and momentum operators or, equivalently, for the raising and lowering operators. Their time dependence in the Heisenberg picture can hence be represented by a Bogoliubov (squeezing) transformation [24,25],

$$a(\tau) = U_\tau^\dagger\, a\, U_\tau = \mu(\tau)a + \nu(\tau)a^\dagger, \tag{17}$$

where $\mu(\tau)$ and $\nu(\tau)$ are complex functions depending on details of the driving and U_τ is the corresponding time-evolution operator.

As in the case of the two-level system, we shall express the evolution with respect to the operators defining the energy measurements, $H_t = \hbar\omega_t(a_t^\dagger a_t + \frac{1}{2})$. These are the adiabatically evolved operators and they are also given by a Bogoliubov transformation. Hence, without loss of generality, we can also represent the transformation of the quantum state, expressed in terms of the raising and lowering operators pertaining to the time-dependent Hamiltonian as a Bogoliubov, or squeezing, transform,

$$U_{10}^\dagger\, a_{t_0} U_{10} = \cosh r\, a_{t_1} + \sinh r\, e^{-i\phi} a_{t_1}^\dagger. \tag{18}$$

For simplicity, we omit the phase ϕ and, using Eq. (18) as the propagator between two energy measurements, the joint probability distribution yields

$$p_{k_1,k_0} = \mathrm{tr}\big[\Pi_{k_1}^1 U_{10} \Pi_{k_0}^0 \rho_0 \Pi_{k_0}^0 U_{10}^\dagger\big] \tag{19}$$

$$= G_{k_1,k_0}^2(r)\rho_{k_0,k_0}, \tag{20}$$

with $\rho_{nn} = \langle n|\rho_0|n\rangle$ and $G_{mn}(r) = \langle m|U_r|n\rangle$. The transition matrix elements for squeezed number states $G_{mn}(r)$

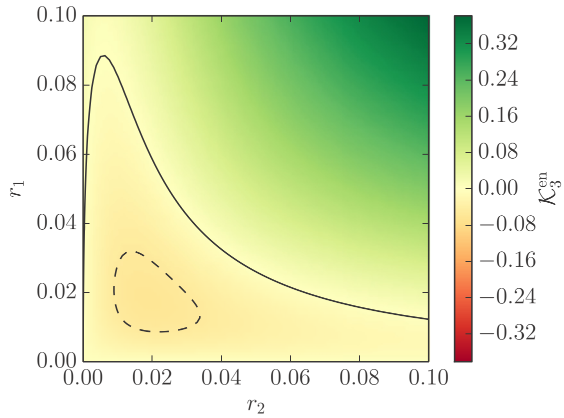

FIG. 3. Violation of Eq. (11) for a squeezed harmonic oscillator as a function of the squeezing parameter z (between t_0 and t_1) and $\bar{z}$ (between t_0 and t_1). The solid contour encloses the area where $\mathcal{K}_3^{\text{en}}$ gets negative, i.e., where the inequality gets violated. The dashed contour denotes $\mathcal{K}_3^{\text{en}} = -0.05$. As the initial state, we assume a thermal state with $\beta = 0.1(\hbar\omega_0)^{-1}$.

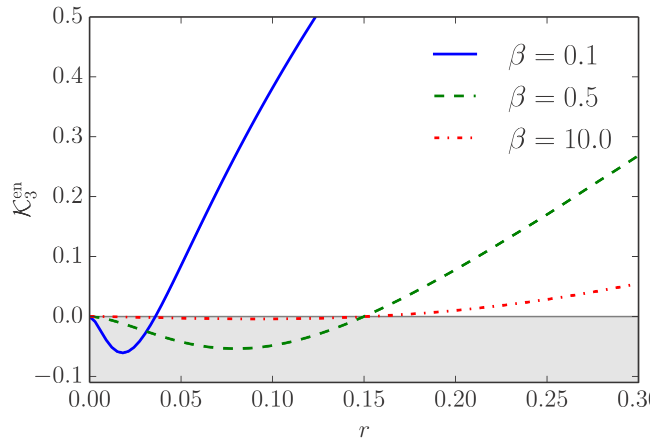

FIG. 4. The entropic Leggett-Garg parameter for a squeezed harmonic oscillator as a function of the squeezing parameter and different thermal initial states with inverse temperatures β expressed in multiples of $1/\hbar\omega_0$.

are provided in the Appendix following analytical results in [26,27]. With the above preparation, we are ready to address the entropic Leggett-Garg inequality (11), where the time evolution between t_0 and t_1 and between t_1 and t_2 are both governed by (18), but possibly with two different arguments r_1 and r_2.

Plotting the entropic Leggett-Garg parameter $\mathcal{K}_3^{\text{en}}$ as a function of r_1 and r_2 for a thermal initial state with $\beta = 0.1(\hbar\omega_0)^{-1}$ in Fig. 3 reveals that also for the harmonic oscillator, Eq. (11) can be violated and the quantum work obeys nonclassical statistics. Interestingly, even a vanishing small amount of squeezing is enough to violate the Leggett-Garg inequality. Increasing the squeezing strength increases the violation until a maximal violation is obtained at $r_1 = r_2 \approx 0.02$. Further increase of the squeezing parameter leads to a less pronounced violation and, finally, $\mathcal{K}_3^{\text{en}}$ turns positive, indicating that the works statistics can no longer be distinguished from that of a classical process. This can be understood as a consequence of the fact that the squeezing of thermal states and number states generally broadens the number distribution, turning sub-Poissonian into super-Poissonian statistics [28]. Note also that for strong squeezing, there is an asymmetry between r_1 (the squeezing between t_0 and t_1) and r_2 (the squeezing between t_1 and t_2). In this regime, too much squeezing in the second interval prevents the violation of the Leggett-Garg inequality.

It is an interesting question how the violation depends on the temperature of the thermal initial state. We recall that for the two-level system, the outcome correlations are independent of the outcome of the first measurement, and hence of the initial state. This is different for the oscillator, since the probability for measuring high-energy outcomes in the first measurement depends on the initial temperature and does affect the subsequent correlations. In Fig. 4, we plot the $\mathcal{K}_3^{\text{en}}$ for different values of inverse temperatures β as a function of the squeezing parameter r, where we set $r_1 = r_2$. Interestingly, the violation for larger β, i.e., lower temperature, is less pronounced than for small β. Hence, the quantum work statistics appears more nonclassical for higher initial temperatures. This can also be seen in the upper plot of Fig. 5, where we plot the minimal value of $\mathcal{K}_3^{\text{en}}$ as a function of β. This can be understood by the fact that for higher temperatures, the system is more likely to start in a higher number state after the first measurement, which is more strongly affected by the squeezing.

Figure 4 also reveals that for increasing β, the values for the squeezing parameter $r_1 = r_2$, where the maximal violation occurs, increase. This is studied more generally in Fig. 5, where we plot the value of r leading to maximal Leggett-Garg violation as a function of β. We observe a nonmonotonous dependency with a maximum around $\beta = 1$ and approach towards a constant level for large β.

VI. CONCLUSION

We have shown that quantum work, defined according to Eq. (2), may show statistical correlations that cannot be

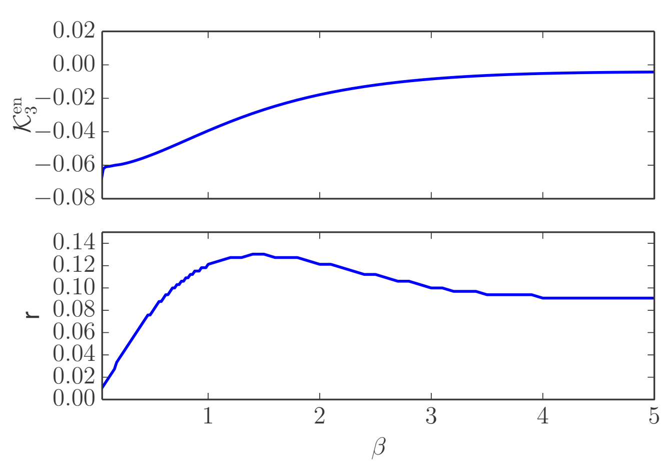

FIG. 5. Maximal Leggett-Garg violation. Top: The smallest value of $\mathcal{K}_3^{\text{en}}$ as a function of β. Bottom: The corresponding values of r, where $\mathcal{K}_3^{\text{en}}$ assumes the minimal value.

described by classical macrorealism. This follows from the violation of Leggett-Garg and entropic Leggett-Garg inequalities for work measurements. In a driven two-level system as well as in a harmonic oscillator subject to squeezing, these inequalities are violated for certain driving parameters and initial temperatures. When both can be evaluated, the entropic and the normal Leggett-Garg inequalities do not necessarily identify the same correlations as nonclassical, as their violation is only a sufficient but not a necessary criterion to abandon macrorealism. This points to the interest in developing tighter bounds to rule out the violation of macroscopic realism over broader parameter ranges.

The study of temporal correlations and their implication for both foundational and practical questions resembles the situation in the 1950s, where a multitude of optical phenomena could be described by stochastically fluctuating classical fields, but where the Hanbury-Brown and Twiss measurements of (classical) intensity correlations spurred [29] discussions about the general validity of classical modeling. This led to the insight that temporal fluctuations in intensity measurements can, indeed, exclude classical descriptions of the light field, and it stimulated the emergence of quantum optics as a research field. Nonclassical properties of light are, e.g., witnessed by temporal noise correlations that violate Cauchy-Schwarz inequalities, antibunching, and higher-order interference effects, which have in several cases turned out to be useful properties, e.g., for precision sensing.

In this spirit, our work is an attempt to quantify temporal quantum correlations involved in thermodynamic processes and might be relevant for the evaluation and design of work extraction protocols [30,31] and (measurement-based) quantum thermal machines [32], where it was shown recently that the efficiency of cyclic processes may nontrivially involve correlations between subsequent cycles [33].

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APPENDIX: ANALYTICAL FORMULA FOR $G_{mn}(z)$

In this appendix, we show the explicit expression for the matrix element $G_{mn}(z)$ derived in [26–28]. It yields

$$G_{mn}(z) = \begin{cases} (-1)^{m/2} \frac{\sqrt{m!n!}}{\cosh r} \sum_{i=1}^{N} \frac{(-4)^i (\sinh z)^{(m+n)/2-2i} (2\cosh z)^{-(n+m)/2}}{2i!(1/2m-i)!(1/2n-i)!} & \text{for } m,n \text{ even} \\ (-1)^{(m-1)/2} \frac{\sqrt{m!n!}}{\cosh r} \sum_{i=1}^{N} \frac{(-4)^i (\sinh z)^{(m+n)/2-2i-1} (2\cosh z)^{-(n+m)/2-1}}{(2i+1)![1/2(m-1)-i]![1/2(n-1)-i]!} & \text{for } m,n \text{ odd} \\ 0 & \text{else,} \end{cases} \quad \text{(A1)}$$

where the summation ends at $N = \min\{m/2,n/2\}$ for n,m even and $N = \min\{(m-1)/2,(n-1)/2\}$ for n,m odd, respectively.

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