Erratum: Exact non-Markovian master equation for the spin-boson and Jaynes-Cummings models [Phys. Rev. A 95, 020101(R) (2017)]

L. Ferialdi

(Received 16 June 2017; published 30 June 2017)

DOI: 10.1103/PhysRevA.95.069908

In our paper, we have derived a master equation for two-level systems interacting with a bosonic bath. Such a master equation was claimed exact but, as we will show in this Erratum, this is not the case. We start by correcting a typo in Eq. (11), that however does not play a role in the following considerations. In particular, different $\hat{\sigma}$'s acting on the same side of $\hat{\rho}$ satisfy the standard anticommutation rules, and Eq. (11) should read

$$\{\hat{\sigma}_L,\hat{\sigma}_L\} = \{\hat{\sigma}_R,\hat{\sigma}_R\} = 2\hat{I}. \tag{11}$$

The main result was derived starting from the most general completely positive, trace preserving, Gaussian, non-Markovian map [1],

$$\mathcal{M}_t = T_D \exp\left\{\int_0^t d\tau \int_0^t ds\, D_{jk}(\tau,s)\big[\hat{A}_L^k(s)\hat{A}_R^j(\tau) - \theta_{\tau s}\hat{A}_L^j(\tau)\hat{A}_L^k(s) - \theta_{s\tau}\hat{A}_R^k(s)\hat{A}_R^j(\tau)\big]\right\}, \tag{E1}$$

that describes the reduced dynamics of a system (with operators $\hat{A}_j$) bilinearly interacting with a Gaussian bosonic bath [with correlation $D_{jk}(\tau,s)$]. We have here added a subscript D to the time ordering operator T in order to stress that this is the Dyson's (time) ordering, defined by

$$T_D[\hat{A}_j(\tau)\hat{A}_k(s)] = \hat{A}_j(\tau)\hat{A}_k(s)\theta_{\tau s} + \hat{A}_j(s)\hat{A}_k(\tau)\theta_{s\tau}, \tag{E2}$$

where $\theta_{\tau s} = 1$ for $\tau > s$, and zero elsewhere. We stress that Dyson's ordering is defined in the same way regardless of whether the system operators $\hat{A}$ are bosonic or fermionic.

The master equation was derived by applying Wick's theorem on the map (E1), according to which one can rewrite time-ordered products in terms of Wick contractions. However, Wick's theorem exploits an operator ordering that discriminates among bosons and fermions. Wick's (time) ordering T_W coincides with Dyson's ordering for bosons: $T_W = T_D$. Indeed, one can exploit Wick's theorem to obtain the exact master equation for a bosonic system [2]. However, for fermionic systems the definition of Wick's ordering is different:

$$T_W[\hat{A}_j(\tau)\hat{A}_k(s)] = \hat{A}_j(\tau)\hat{A}_k(s)\theta_{\tau s} - \hat{A}_j(s)\hat{A}_k(\tau)\theta_{s\tau}. \tag{E3}$$

In our paper, this was accounted for both in the definition of contraction and in the explicit calculations. However, what was overlooked is that the map (E1) is defined with T_D also for fermionic systems: the derivation was performed by implicitly assuming that $\mathcal{M}_t$ was defined with T_W, i.e., T_D was accidentally replaced by T_W in Eq. (E1).

In order to solve this issue, one can conveniently express Dyson's ordering in terms of Wick's one (and vice versa). For the spin-boson model this amounts to

$$T_W[\hat{\sigma}^z(\tau)\hat{\sigma}^z(s)] = T_D[\hat{\sigma}^z(\tau)\hat{\sigma}^z(s)] - 2\hat{\sigma}^z(s)\hat{\sigma}^z(\tau)\theta_{s\tau}. \tag{E4}$$

This clearly shows that the replacement $T_D \to T_W$ implies to neglect systematically some contributions, leading to an approximate master equation in our paper. One might try to correct the previous derivation by taking these new terms into account. However, the relation (E4) between the two orderings becomes very much complicated for higher-order operator products $T[\hat{\sigma}^z(\tau_1)\ldots\hat{\sigma}^z(\tau_n)]$, and one cannot rearrange such terms in order to obtain an overall contribution proportional to $\mathcal{M}_t$ (crucial to obtain a master equation; see Supplemental Material). Another approach is to rewrite explicitly the product of Pauli matrices, by directly exploiting their properties. With the notation of our paper one finds, e.g.,

$$\hat{\sigma}^z(\tau)\hat{\sigma}^z(s) = \mathbf{b}^z(\tau)\cdot\mathbf{b}^z(s)\,\hat{I} + i(\mathbf{b}^z(\tau)\times\mathbf{b}^z(s))\cdot\hat{\boldsymbol{\sigma}}, \tag{E5}$$

where $\cdot$ and $\times$ denote respectively scalar and cross products, and bold symbols denote the vectors with components x,y,z. However, also in this case it is not possible to obtain a compact expression for higher-order products. The failure of these approaches is intimately connected with the algebra of Pauli matrices. In the first case, this is displayed by the fact that it is not possible to reduce a Dyson's ordering for fermions by means of the corresponding Wick's theorem. In the second approach, this is displayed by the terms proportional to $\hat{\boldsymbol{\sigma}}$ in Eq. (E5), that do not allow one to reduce the complexity of higher-order products. Although we have so far mentioned only the spin-boson model, a similar discussion applies to the master equation for the Jaynes-Cumming model given in our paper.

A possible way out is the application of a so-called "generalized Wick's theorem" introduced in Ref. [3]. This method, however, leads to multiply connected diagrams that might be difficult to recast in a master equation. An improvement in the result of this paper might be obtained by exploiting the path-integral formalism with time-non-local Lagrangians [4]. Both these approaches require a detailed analysis and will be subject of further studies.

We have shown that the method of our paper systematically neglects some contributions in the derivation of the master equation, that accordingly cannot be considered exact. We however stress that the weak-coupling limit of our master equation is correct and recovers known results. Indeed, this is obtained by neglecting the ordering of $\hat{\sigma}(s_1)$ with the operators of $\prod_i \diamond_i$ in Eq. (10) of our paper. Accordingly, $\hat{\sigma}(s_1)$ is simply plugged out of the time ordering, bypassing the ordering issues described above. On the other hand, Eq. (E5) suggests that the exact master equation for the spin-boson model should include all combinations over $i,j = x,y,z$ of terms of the type $[\hat{\sigma}_i,[\hat{\sigma}_j,\hat{\rho}]]$ and $[\hat{\sigma}_i,\{\hat{\sigma}_j,\hat{\rho}\}]$. Similarly, the exact master equation for the Jaynes-Cummings model should display terms of the type $\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z$, that were missed by Eq. (30) of our paper, but were obtained in Ref. [5] by means of an approximated technique. One can in principle obtain an exact master equation, although this seems a very hard task and as yet we could not find a closed expression.

The author acknowledges M. Merkli for spotting the typo of Eq. (11), and M. Hall for useful comments. Furthermore, the author is indebted to G. Gasbarri for pointing out the ordering issue, and for collaboration in investigating the problem and drafting this Erratum. This work was supported by the TALENTS^3 Fellowship Programme, CUP code J26D15000050009, FP code 1532453001, managed by AREA Science Park through the European Social Fund.

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