Artificial gauge field for a light ray

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An optical ray equation is derived for a trajectory in a medium with directional birefringence. The birefringence is described as an artificial gauge field for light that gives an effective magnetic field and a Lorentz force on light. The gauge invariance is also confirmed. This paper opens a door to realize an artificial gauge field for a light ray in a nonreciprocal medium.

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I. INTRODUCTION

Gauge invariance is one of the fundamental principles in physics. Intriguing phenomena have been investigated in an electronic system such as quantum Hall effects [1] and Aharonov-Bohm effect [2,3]. By analogy between electrons and light, the gauge theory has given an impact to optics [4,5]. The geometrical optics is generalized by introducing an optical wave packet consisting of Bloch functions [6,7]. The Bloch function is regarded as a modulation of a plane wave. The modified ray equation has some correction terms, which represent geometrical aspects of the Bloch function in parameter spaces such as momentum and phase spaces. A correction term in momentum space predicted the spin Hall effect of light [8], which was experimentally verified via weak measurements [9]. Another correction term in phase space gives rise to the enhanced shift of a wave packet in deformed crystals [10–13]. All these studies are based on the concept of Berry's phase and relate to the gauge theory.

Following the development of these theories, several optical systems have received considerable attention as a platform to study artificial gauge fields [14–18]. One example is a multiferroic material [19,20], in which a light ray bends in its domain wall as if it was subject to the Lorentz force. However, the gauge invariance of the effective Lorentz force has not been shown up to now.

In this paper, we derive a gauge-invariant ray equation for directionally birefringent media. The derived equation is analogous to the Newtonian equation of motion for a point charge under an electromagnetic field. Such gauge fields are realized in magnetochiral media.

II. GAUGE FIELD FOR A LIGHT RAY

Let us consider a ray trajectory in a directionally birefringent medium. We assume that the refractive index depends on position $\vec{r}(s)$ and direction $d\vec{r}/ds \equiv \dot{\vec{r}}$ of the ray: $n(\vec{r},\dot{\vec{r}})$, where s is the arc length of the ray. An effective Lagrangian $\mathcal{L}^{\text{eff}}(\vec{r},\dot{\vec{r}})$ describing the ray is given by

$$\mathcal{L}^{\text{eff}}(\vec{r},\dot{\vec{r}}) = n(\vec{r},\dot{\vec{r}})|\dot{\vec{r}}| + \lambda(|\dot{\vec{r}}| - 1). \tag{1}$$

Here, λ is the Lagrange multiplier calculated in the Appendix. The Euler-Lagrange equation gives the generalized ray equation to be

$$\frac{d}{ds}\left\{n\frac{d\vec{r}}{ds} + \left(1 - \frac{d\vec{r}}{ds}\frac{d\vec{r}}{ds}\cdot\right)\frac{\partial n}{\partial \dot{\vec{r}}}\right\} = \vec{\nabla} n. \tag{2}$$

The second term on the left-hand side of Eq. (2) represents the momentum variation due to the directional dependence of the refractive index.

We introduce a directionally birefringent refractive index described as

$$n(\vec{r},\dot{\vec{r}}) = n_0(\vec{r}) + \vec{T}(\vec{r})\cdot\dot{\vec{r}}, \tag{3}$$

where $n_0(\vec{r})$ is the component independent of propagating directions. The vector $\vec{T}(\vec{r})$ characterizes the directional birefringence. Substituting Eq. (3) into Eq. (2), we obtain the following equation:

$$\frac{d}{ds}\left(n_0(\vec{r})\frac{d\vec{r}}{ds}\right) = \vec{\nabla} n_0(\vec{r}) + \frac{d\vec{r}}{ds}\times(\vec{\nabla}\times\vec{T}(\vec{r})). \tag{4}$$

This equation has almost the same form as the Newtonian equation of motion for a particle under an electromagnetic field. The left-hand side corresponds to the acceleration term, while the first and second terms on the right-hand side, respectively, correspond to the electric and the magnetic Lorentz forces. Compared with the particle dynamics, the vector $\vec{T}(\vec{r})$ looks like vector potential, i.e., a gauge field. This comparison leads to introducing the gauge transformation

$$\vec{T}(\vec{r}) \to \vec{T}(\vec{r}) + \vec{\nabla}\Lambda(\vec{r}), \tag{5}$$

where $\Lambda(\vec{r})$ is an arbitrary function. The effective Lagrangian is transformed into

$$\mathcal{L}^{\text{eff}}(\vec{r},\dot{\vec{r}}) \to \mathcal{L}^{\text{eff}}(\vec{r},\dot{\vec{r}}) + \frac{d\Lambda}{ds}. \tag{6}$$

Let us consider the physical meaning of the gauge degree of freedom. Following the transformation, the refractive index changes to be

$$n(\vec{r},\dot{\vec{r}}) \to n(\vec{r},\dot{\vec{r}}) + \frac{d\Lambda}{ds}. \tag{7}$$

The transformed refractive index gives the same trajectory as that of the original refractive index, because the term $d\Lambda/ds$ does not affect the trajectory. This indicates that refractive indices related by Eq. (7) give the same trajectory. The

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TABLE I. Analogy between geometrical optics and classical dynamics. m and q are mass and electric charge of an electron, respectively. ϕ and $\vec{A}$ are vector and scaler potentials for an electromagnetic field, respectively.

	Geometrical optics	Classical mechanics
Dynamical parameter	Arc length s	Time t
Equation of motion	$d(n_0\dot{\vec{r}})/ds = \vec{\nabla}n_0 + \dot{\vec{r}} \times (\vec{\nabla} \times \vec{T})$	$d(m\dot{\vec{r}})/dt = -q\vec{\nabla}\phi + q\dot{\vec{r}} \times (\vec{\nabla} \times \vec{A})$
Electric Lorentz force	$\vec{\nabla}n_0$	$-q\vec{\nabla}\phi$
Magnetic Lorentz force	$\dot{\vec{r}} \times (\vec{\nabla} \times \vec{T})$	$q\dot{\vec{r}} \times (\vec{\nabla} \times \vec{A})$
Gauge transformation	$\vec{T} \to \vec{T} + \vec{\nabla}\Lambda$	$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$

analogy between geometrical optics and classical mechanics is summarized in Table I.

The theoretical calculation so far has shown an advantage of our theory that it can explicitly deal with the directional birefringence ($d\vec{r}/ds$) of a refractive index. As far as we know, this feature is not found in other theories dealing with an effective magnetic field for light. Because of the advantage, the derived ray equation is directly applicable for ray tracing in a nonreciprocal medium. The direct ray calculation is a useful tool for designing optical devices with novel functionality such as nonreciprocal invisibility cloaking [21,22].

III. REALIZATION OF THE GAUGE FIELD

In this section, we consider the directional birefringence realized in the optical magnetoelectric [23,24] and the magnetochiral (MCh) [25,26] media.

We assume that a microwave is incident on a medium with homogeneous magnetization and isotropic chirality as shown in Fig. 1. The constitutive relations are given by

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E}, \tag{8}$$

$$\vec{H} = \mu_0^{-1}\vec{B} - \vec{M}, \tag{9}$$

where $\vec{D}$, $\vec{E}$, $\vec{H}$, $\vec{B}$, and $\vec{M}$ are the electric displacement field, the electric field, the magnetic-field strength, the magnetic field, and the magnetization, respectively. The permittivity and the permeability of the vacuum are ε_0 and μ_0, respectively. The permittivity of the magnetic medium is ε. The magnetization originates from two contributions: $\vec{M} = \vec{M}_0 + \vec{M}_{\text{Ch}}$. The magnetization $\vec{M}_0$ is from the spin precession and the other one $\vec{M}_{\text{Ch}}$ is from the electromagnetic induction. Let us first consider $\vec{M}_0$, which is described as a sum of a linear magnetization and a magneto-optical (MO) effect:

$$\vec{M}_0 = \chi_m \vec{H} - i\kappa \vec{H} \times \vec{B}_0. \tag{10}$$

Here, χ_m is the linear magnetic susceptibility, κ is a proportional coefficient characterizing the MO effect, and $\vec{B}_0$ is an external magnetic field. We next consider the magnetization by the electromagnetic induction due to the circular current flowing inside the helix of the chiral structure. This response is proportional to the magnetization current density and described by $\vec{M}_{\text{Ch}} = \eta\vec{\nabla} \times \vec{M}_0$, where η is a proportional coefficient characterizing the chirality of the helix. Assuming that the microwave with the wave vector $\vec{k}$ is plane wave and transverse, $\vec{k} \cdot \vec{H} = 0$, we obtain $\vec{M}_{\text{Ch}}$ to be

$$\vec{M}_{\text{Ch}} = -2i\xi\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{E} + 2\xi\frac{\kappa}{\chi_m}\frac{1}{\varepsilon k_0}(\vec{k} \cdot \vec{B}_0)\vec{H}. \tag{11}$$

Here, $k_0 = \omega/c$, where ω is the angular frequency of the microwave and c is the speed of light in a vacuum. The chiral parameter $\xi = k_0\eta\chi_m\varepsilon/2$ is proportional to η characterizing the chirality. In Eq. (11), the first term gives rise to the natural optical activity by the chiral response. The second term is dependent linearly on the propagation direction and independent on the polarization states. Namely, this term gives the nonreciprocal response for unpolarized waves, representing the MCh effect.

We calculate the refractive index of the MCh medium with magnetization and chirality. Polarization-independent parts of the linear magnetization and the MCh effects are, respectively, represented by the first and second terms on the right-hand side of Eqs. (10) and (11). Therefore, we obtain the constitutive equation representing the polarization independent response to be

$$\vec{B} = \mu_0(\vec{H} + \vec{M})_{\text{unpol}} \tag{12}$$

$$= \mu_0\left\{\vec{H} + \chi_m\vec{H} + 2\xi\frac{\kappa}{\chi_m}\frac{1}{\varepsilon k_0}(\vec{k} \cdot \vec{B}_0)\vec{H}\right\}, \tag{13}$$

where the subscript "unpol" denotes the averaged response by polarizations. Substituting Eqs. (8) and (13) into Maxwell's equations and using the dispersion relation, $\omega = c|\vec{k}|/n$, we

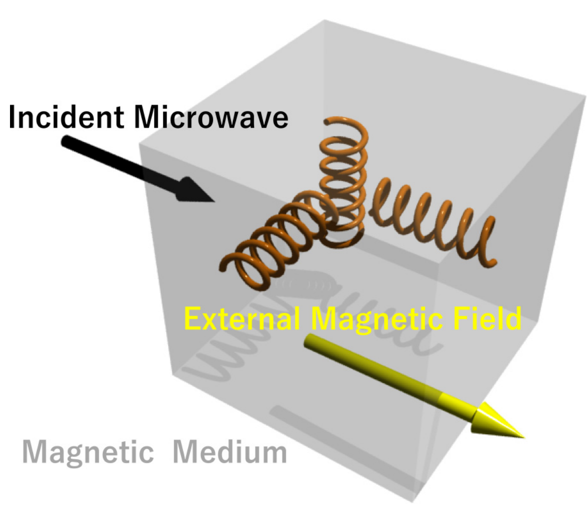

FIG. 1. Schematic of the magnetochiral medium. The helices indicate not physical components but the isotropic chirality of the system. The black and yellow arrows indicate the propagation direction of the incident microwave and an external magnetic field, respectively.

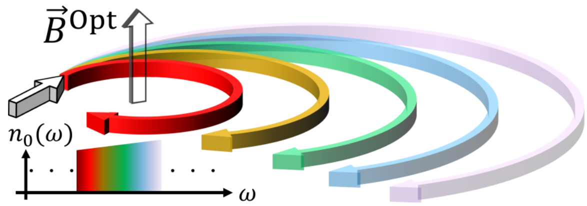

FIG. 2. Schematic of light dispersion by the optical cyclotron motion. The incident white light is spatially dispersed depending on the frequency dispersion of n_0, which is analogous to the variation of the cyclotron radius depending on mass of a particle. The inset shows the frequency dispersion.

obtain the refractive index to be

$$n = \sqrt{\varepsilon\mu} + \xi \frac{\kappa}{\chi_m} \vec{B}_0 \cdot \frac{d\vec{r}}{ds}, \tag{14}$$

where $\mu = 1 + \chi_m$. In Eq. (14), we took the first order of ξ, and replaced $\hat{\vec{k}} = \vec{k}/|\vec{k}|$ by $d\vec{r}/ds$. From Eqs. (3) and (14), we find $\vec{T} = \xi(\kappa/\chi_m)\vec{B}_0$. Thus the MO effect $(\kappa \vec{B}_0/\chi_m)$ coupled with the chirality (ξ) gives the MCh effects, and plays the role of an artificial gauge field for a light ray. As we have shown in Sec. II, the rotation of $\vec{T}$ is regarded as an effective magnetic field on a light ray. The effective magnetic field is calculated to be

$$\vec{B}^{\mathrm{Opt}} = \frac{\kappa}{\chi_m} \vec{\nabla}\xi(\vec{r}) \times \vec{B}_0. \tag{15}$$

This becomes nonzero when the chiral parameter is inhomogeneous and the chiral axis is perpendicular to the external magnetic field. To realize this, some artificial structures are useful such as the gradient index metasurface with a phase gradient along its surface [27]. The gradient of the chiral parameter is also realized in a chiral metasurface.

As a simple case, let us consider a ray trajectory in a uniform effective magnetic field. We assume that n_0 is spatially homogeneous and has a frequency dispersion: $n_0(\omega)$. The gauge field is given by the "Landau gauge," $\vec{T}(\vec{r}) = (0, B^{\mathrm{Opt}}x, 0)$. The ray trajectory is a circle with an optical cyclotron radius:

$$R_{\mathrm{c}}^{\mathrm{Opt}} = \frac{n_0(\omega)|\dot{\vec{r}}|}{B^{\mathrm{Opt}}} = \frac{n_0(\omega)}{B^{\mathrm{Opt}}}. \tag{16}$$

This radius varies depending on the frequency dispersion. Therefore, when a white light ray is incident into a material subject to the effective magnetic field, it is spatially dispersed depending on the frequency as shown in Fig. 2.

IV. CONCLUSION

In conclusion, we derived the ray equation for a trajectory in a medium with directional birefringence. We have discussed the trajectory in the medium described as $n(\vec{r},\dot{\vec{r}}) = n_0(\vec{r}) + \vec{T}(\vec{r}) \cdot (d\vec{r}/ds)$, and found that the ray equation is reduced to almost the same form as the Newtonian equation of motion for a charged particle under a magnetic field. Using the analogy between electrons and light, we found the vector $\vec{T}(\vec{r})$ plays the role of vector potential for a light ray. The gauge invariance is also confirmed. Such a gauge field is realized in a MCh medium. The present paper paves a way for realizing an artificial gauge field for a light ray.

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APPENDIX: DERIVATION OF THE EFFECTIVE LAGRANGIAN

We derive the effective Lagrangian given by Eq. (1) from the variational principle. The equation of motion for a light ray is derived from Fermat's principle. Hence, we consider the variation of an optical path described as $\int n dl$. In the ray analysis, we denote a ray parametrized by the arc length $s = \int ds$, namely, $\vec{r}(s) = (x(s), y(s), z(s))$, in Cartesian coordinates, where the line element dl is defined as $dl = (d\vec{r} \cdot d\vec{r})^{1/2} = (\dot{\vec{r}} \cdot \dot{\vec{r}})^{1/2} ds$. This relation imposes a constraint condition that $|\dot{\vec{r}}| - 1 = 0$ on the length of the ray direction vector. The variational problem to be solved is described as $\delta \int \mathcal{L}^{\mathrm{eff}}(\vec{r},\dot{\vec{r}}) ds = 0$, where

$$\mathcal{L}^{\mathrm{eff}}(\vec{r},\dot{\vec{r}}) = n(\vec{r},\dot{\vec{r}})|\dot{\vec{r}}| + \lambda(|\dot{\vec{r}}| - 1). \tag{A1}$$

The second term on the left-hand side of Eq. (A1) gives rise to a significant correction to the conventional ray equation when the refractive index has directional birefringence. This Lagrangian yields the Euler-Lagrange equation written as

$$\frac{d}{ds}\left[n(\vec{r},\dot{\vec{r}}) \frac{d\vec{r}}{ds} + \frac{\partial n(\vec{r},\dot{\vec{r}})}{\partial \dot{\vec{r}}} + \lambda \frac{d\vec{r}}{ds} \right] = \vec{\nabla} n(\vec{r},\dot{\vec{r}}). \tag{A2}$$

Let us determine the Lagrange multiplier. Integrating Eq. (A2) with respect to s, we obtain the following equation:

$$n(\vec{r},\dot{\vec{r}}) \frac{d\vec{r}}{ds} + \frac{\partial n(\vec{r},\dot{\vec{r}})}{\partial \dot{\vec{r}}} + \lambda \frac{d\vec{r}}{ds} = \vec{\nabla} \int ds n(\vec{r},\dot{\vec{r}}). \tag{A3}$$

Premultiplying $d\vec{r}/ds$ on both sides of Eq. (A3), the Lagrange multiplier is given by

$$\lambda = -\frac{d\vec{r}}{ds} \cdot \frac{\partial n(\vec{r},\dot{\vec{r}})}{\partial \dot{\vec{r}}}. \tag{A4}$$

In a medium with $n(\vec{r},\dot{\vec{r}}) = n_0(\vec{r}) + \vec{T}(\vec{r}) \cdot \dot{\vec{r}}$, the effective Lagrangian is calculated to be

$$\mathcal{L}^{\mathrm{eff}}(\vec{r},\dot{\vec{r}}) = n_0(\vec{r})|\dot{\vec{r}}| + \vec{T}(\vec{r}) \cdot \dot{\vec{r}}. \tag{A5}$$

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