Spectral properties of trapped two-level ions interacting with quantized fields

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In this paper we study the full quantum approach to the interaction of N trapped two-level ions with N quantized fields in such a way that each of the trapped ions separately interacts with one of the quantized fields. A semiclassical model of this type of interaction has been already studied, using classical laser fields instead of quantized fields, by Cirac and Zoller [Phys. Rev. Lett. **74**, 4091 (1995)]. During each interaction, the excitation of a trapped ion occurs due to the occurrence of resonance between the trapped ion and one of the modes of multimode quantized field. In this way, we are able to find the quantized radiation field, which propagates from each excited trapped ion after its deexcitation. It is demonstrated that such a propagated field is a function of the position of ion within the trap, ion number, trapping and ionic transition frequencies, spontaneous emission rate, and time evolution of trapped ionic operators. In the continuation, to achieve the explicit form of such a field we propose a method to evaluate the temporal evolution of operators of the system by utilizing the Heisenberg operator approach. Via the propagated field of each trapped ion, the spectral properties such as intensities as well as power spectra of their radiations are evaluated. Furthermore, the quantum statistics of field and in particular, its photon antibunching and sub-Poissonian statistical properties are investigated. Finally, resonance spectroscopy of the trapped ions is obtained by which one can allocate a characteristic spectrum to each of them.

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I. INTRODUCTION

Nowadays, the ensembles of identical two-level systems that interact with the quantized fields are an interesting subject that has attracted a lot of attention [1–7]. In this regard, Scully *et al.* investigated the spontaneous emission, which is directly emitted from an ensemble of N atoms (which are fixed in a crystal at a low temperature) and interacts with an incident quantized field [8]. In the continuation of their work, Sete *et al.* considered the collective spontaneous emission from an ensemble of identical N two-level atoms, which are prepared by absorption of a single photon Dicke superradiance (see Ref. [9] and references therein). The authors discussed the dynamical properties of superradiance for small and large atomic ensembles and also evaluated the effects of virtual processes on the collective decay rate and Lamb shift. Svidzinsky *et al.* studied the evolution of a collection of N atomic states in single photon superradiance and presented the analytical solution for such evolution [10]. In addition, the cooperative spontaneous emission has been investigated in Ref. [11], in which the authors studied the single photon emission from a spherically symmetric cloud of atoms and presented exact analytical expressions for the relevant eigenvalues and eigenstates. They also found that some atomic states can decay much faster than the single-atom state, while other states experience very slow decay.

In direct relation to the above-mentioned models, in the present work we consider an ensemble of N trapped two-level ions (instead of the atoms in Refs. [8–11]) in a linear Paul trap with the condition that each ion interacts separately with a quantized field in the Lamb-Dicke as well as the first blue (vibrational) sideband regimes [12]. In this model, we intend to analyze the emitted spectrum (propagated photons) from each trapped ion after its deexcitation in the trap. By this study, the effects of the trapping process on the temporal behavior of the ionic emitted spectra is quantitatively investigated. The main advantage of this model is to allocate a characteristic spatial as well as temporal behavior to the spectrum of each trapped ion in the first blue sideband for detecting them.

Trapped ions are frequently used in quantum information science and quantum optics [13–18]. These systems possess internal and external degrees of freedom. Their internal (external) degrees are related to their two internal ionic levels [are related to their center-of-masses (c.m.) vibrational motion] [12,19]. Quantum computing using trapped ions originates from the proposal of Cirac and Zoller [20]. They proved that the trapped ions in the interaction with intense classical (laser) fields can construct some C-NOT gates for storing and transporting the information. They accompanied the internal ionic levels (as the qubits) for storing the data and the external vibrational modes of the trapped ions for transferring the data to design a quantum computer architecture [20]. The linear Paul trap has been frequently used for trapping the ions in the literature [21–23]. This kind of trap possesses a particular arrangement of oscillating electric potential at the radio frequency (rf) [24–26]. In such a trap, the ions can be confined to oscillate in normal modes around their c.m. equilibrium positions on the axis of the trap [21]. After trapping the ions via a linear Paul trap, one can apply the electromagnetic field to the ions to perform the interaction between the produced trapped ions and the applied fields. There are two different approaches to study such an interaction: first, a semiclassical approach, which has been proposed by Blockly *et al.* [27]. In this model the classical intense electromagnetic traveling field interacts with the trapped ions. Second, a full quantum approach may be considered in which the field is also quantized; this has

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been suggested by Bužek *et al.* [28]. The authors implied that the interaction between a trapped two-level ion and a quantized field can be performed in the Lamb-Dicke regime as well as the first vibrational sideband. This kind of interaction is very similar to the well-known Jaynes-Cummings model (JCM) [29], which is for the atom-field interaction. It is well known that the JCM can illustrate quantum features such as collapse and revival phenomena, bipartite entanglement, etc., while the latter effects cannot be observed in the semiclassical approach [30]. So, as is explained in the above descriptions, the main difference between the two mentioned approaches (Blockly *et al.* and Bužek *et al.*) is the nature of the applied field. Meanwhile, both approaches have usually been solved via the probability amplitude method for the evaluation of their system states by considering appropriate initial conditions of the ion-field state vectors.

In this paper, to provide a more complete investigation about the full quantum interaction between the trapped ions and the radiation fields we focus on the operator approach. In detail, we try to utilize the Heisenberg operator method [31] to obtain the analytical form of the time evolution of the operators of subsystems (ions and fields) involving in the interaction. As an advantage of working with the Heisenberg picture, we may refer to the fact that one can obtain the evolution of multitime correlation functions straightforwardly [32]. A correlation function in quantum mechanics is a statistical correlation between the operators of the system at two different points in space (spatial) or time (temporal) [32–34]. Also, by using the operator techniques, the propagated field from each of the trapped ions after their interactions with the field may also be obtained, by which one can investigate the spectral properties of the ions and also allocate a characteristic spectrum to each of them. These applications led us to use the Heisenberg operator approach in our present study. Briefly, in this work we first consider N multimode quantized fields interacting separately with N trapped two-level ions and then specialize this model to the first vibrational sideband. It is supposed that there exists one mode (among a lot of modes) of a field, which is in resonance with a trapped ion (resonance fluorescence phenomenon). Performing such an interaction may lead to the emission of a quantized radiation by each ion distinctly. We calculate the analytical form of such a quantized radiated field explicitly. Our main purposes for applying the quantized field in this investigation may be listed as below. This model helps us in obtaining (i) the emitted quantized field of each trapped ion, (ii) the lifetime of the upper trapped ionic level, (iii) spectral features of the radiations of trapped ions, (iv) statistical properties of the external quantized field.

This paper is organized as follows. In the next section we study the full quantum interaction between a string of N trapped ions and N quantized fields. The explicit form of the propagated (electric) field from jth trapped ion is obtained in Sec. III and in Sec. IV the probability of such an emission is manipulated. In Sec. V the time evolutions of operators of the subsystems are obtained. Section VI deals with the first-order multitime correlation function of the trapped ion and Sec. VII with quantum statistics of the field. Next, in Sec. VIII the intensities of the emitted fields from the trapped ions and their power spectra are studied. Our concluding remarks will be presented in the final section.

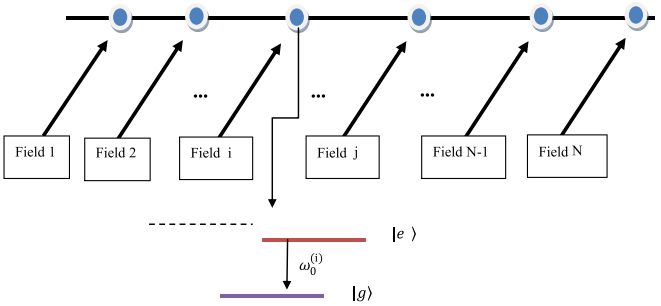

FIG. 1. N trapped ions interacting with N quantized fields, distinctly.

II. IONS-FIELDS MODEL HAMILTONIAN

We consider a system in which a string of N two-level ions (qubits) are injected from an ion source into a linear Paul trap [21]. This trap is surrounded by a cavity quantum electrodynamics (QED) and supported by an oscillating rf electric potential. The ions are confined to oscillate in c.m. mode around their equilibrium positions on the trapping $\hat{z}$ axis. In particular, for $N = 1$, $\bar{z}_1 = 0$, and for $N = 2$, $\bar{z}_1 = \kappa(\frac{1}{4})^{\frac{1}{3}}$ and $\bar{z}_2 = -\kappa(\frac{1}{4})^{\frac{1}{3}}$ in which $\kappa = (q^2/4\pi\epsilon_0 M\nu^2)^{\frac{1}{3}}$ [12]; the parameter $\bar{z}_j$ denotes the equilibrium position of jth trapped ion and q, M, and ν are, respectively, the charge, mass, and external vibrational frequency in c.m. mode and ϵ_0 is the vacuum permeability. For more trapped ions ($N > 2$), the equilibrium positions have been determined in Ref. [19]. In the c.m. mode, (by considering the quantized vibrational motion) the position of the jth trapped ion on the $\hat{z}$ axis is defined as $z_j = \bar{z}_j + \sqrt{\frac{\hbar}{2M\nu N}}(\hat{b}^\dagger + \hat{b})$ where $\hat{b}$ ($\hat{b}^\dagger$) is the annihilation (creation) operator of the quantized collective vibrational motion in the c.m. mode [12]. In this situation, we assume that N quantized fields are separately applied to N trapped ions and interact with them (see Fig. 1).

Each of the evolved trapped ions possesses two degrees of freedom. The first relates to the internal ionic levels and the second relates to its external vibrational motion (external degree of freedom). These degrees can both interact and couple with the applied fields. The general Hamiltonian for such a system reads as

$$\hat{H}(t) = \hat{H}_0(t) + \hat{H}_{\rm int}(t), \tag{1}$$

where

$$\hat{H}_0(t) = \hbar\nu\hat{b}^\dagger(t)\hat{b}(t) + \sum_{j=1}^{N}\frac{\hbar\omega_0}{2}\hat{\sigma}_z^{(j)}(t) + \sum_{\vec{k}_j,\lambda,j=1}^{N}\hbar\omega_{\vec{k}_j}^{(j)}\,\hat{c}_{\vec{k}_j,\lambda}^{(j)\dagger}(t)\hat{c}_{\vec{k}_j,\lambda}^{(j)}(t), \tag{2}$$

and $\hat{H}_{\rm int}(t) = \sum_{j=1}^{N}\hat{H}_{\rm int}^{(j)}(t)$ with

$$\hat{H}_{\rm int}^{(j)}(t) = -\left(\vec{Q}_{eg}^{(j)}\hat{\sigma}_+^{(j)}(t) + \vec{Q}_{ge}^{(j)}\hat{\sigma}_-^{(j)}(t)\right) \times\left(\sum_{\vec{k}_j,\lambda}\varepsilon_{\vec{k}_j}^{(j)}\,\hat{\epsilon}_{\vec{k}_j}^{\lambda,(j)}\,\hat{c}_{\vec{k}_j,\lambda}^{(j)}(t)e^{i\vec{k}_j z_j\cos\theta_j} + \text{H.c.}\right). \tag{3}$$

In the above equations, ω_0 is the frequency of internal ionic transition and $\omega_{\vec{k}_j}^{(j)} = c|\vec{k}_j|$ is the frequency of kth mode of the jth field with c as the velocity of light. In addition, $\hat{c}_{\vec{k}_j,\lambda}^{(j)}$ ($\hat{c}_{\vec{k}_j,\lambda}^{(j)\dagger}$) represents the annihilation (creation) operator of kth mode of the jth quantized field and $\lambda = 1,2$ denotes the two polarization directions. Furthermore, $\varepsilon_{\vec{k}_j}^{(j)} = \sqrt{\frac{\hbar\omega_{\vec{k}_j}^{(j)}}{2\epsilon_0 V}}$ is the amplitude of field, V is the effective volume of the cavity and $\hat{\epsilon}_{\vec{k}_j}^{\lambda,(j)}$ is the polarization direction of kth mode. The dipole moment vector of the jth trapped ion represents as $\vec{Q}_{eg}^{(j)}(=\vec{Q}_{ge}^{(j)*})$, and $\hat{\sigma}_{\pm}^{(j)}$ and $\hat{\sigma}_z^{(j)}$ are the pseudoionic operators of it. Also, θ_j is the incident angle of jth applied field with respect to the positive direction of the trapping axis (the polar angle between the vector $\vec{k}_j$ and positive $\hat{z}$ axis). Using the rotating-wave approximation (RWA) [35], one can arrive at the interaction Hamiltonian

$$\hat{H}_{\rm int}^{(j)}(t) = \hbar\sum_{\vec{k}_j,\lambda}\mu_{\vec{k}_j,\lambda}^{(j)}\hat{c}_{\vec{k}_j,\lambda}^{(j)}(t)\hat{\sigma}_+^{(j)}(t)e^{i\eta_j(\hat{b}^\dagger(t)+\hat{b}(t))\cos\theta_j}e^{i\phi_j} + \text{H.c.}, \tag{4}$$

where $\mu_{\vec{k}_j,\lambda}^{(j)} = -\frac{\vec{Q}_{eg}^{(j)}.\hat{\epsilon}_{\vec{k}_j}^{\lambda,(j)}}{\hbar}\varepsilon_{\vec{k}_j}^{(j)}$ is the jth trapped ion-field coupling and $\phi_j = k_j\bar{z}_j\cos\theta_j$. Also, we have defined $\eta_j = k_j\sqrt{\frac{\hbar}{2M\nu N}}$, in which by considering $E_r^{(j)} = \frac{\hbar^2k_j^2}{2M}$ as the recoil energy of jth trapped ion from jth field, η_j can be rewritten as $\eta_j = \sqrt{\frac{E_r^{(j)}}{N\hbar\nu}}$. We consider the Lamb-Dicke regime, i.e., $\eta_j^2[\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle + \frac{1}{2}] \ll 1$ [12], therefore the effective interaction Hamiltonian can be rewritten as

$$\hat{H}_{\rm eff}^{(j)}(t) = i\hbar\eta_j\cos\theta_j\sum_{\vec{k}_j,\lambda}\mu_{\vec{k}_j,\lambda}^{(j)}\hat{c}_{\vec{k}_j,\lambda}^{(j)}(t)\hat{\sigma}_+^{(j)}(t) \times[\hat{b}^\dagger(t)+\hat{b}(t)]e^{i\phi_j} + \text{H.c.} \tag{5}$$

As is observed, the absorption of one photon can excite a trapped ion to its internal upper level and simultaneously increase its external vibrational quanta by one. Inversely, the emission of a photon from a trapped ion accompanies with the deexcitation of it to its internal lower level and decreases its external vibrational quanta by one. The terms $\hat{c}_{\vec{k}_j,\lambda}^{(j)}(t)\hat{\sigma}_+^{(j)}(t)\hat{b}(t)$ and $\hat{c}_{\vec{k}_j,\lambda}^{(j)\dagger}(t)\hat{\sigma}_-^{(j)}(t)\hat{b}^\dagger(t)$ can be ignored in Eq. (5) using RWA. We can again use the RWA and simplify the effective interaction Hamiltonian (5) to the following form

$$\hat{H}_{\rm eff}^{(j)}(t) = i\hbar\sum_{\vec{k}_j,\lambda}g_{\vec{k}_j,\lambda}^{(j)}(\theta_j)\big(\hat{c}_{\vec{k}_j,\lambda}^{(j)}(t)\hat{\Sigma}_+^{(j)}(t)e^{i\vec{k}_j.\vec{r}_j} - \text{H.c.}\big), \tag{6}$$

where $g_{\vec{k}_j,\lambda}^{(j)}(\theta_j) = \eta_j\mu_{\vec{k}_j,\lambda}^{(j)}\cos\theta_j$ and $\hat{\Sigma}_+^{(j)}(t) = \hat{b}^\dagger(t)\hat{\sigma}_+^{(j)}(t)$ and $\hat{\Sigma}_-^{(j)}(t) = \hat{b}(t)\hat{\sigma}_-^{(j)}(t)$ are, by definition, the total raising and lowering operators of jth trapped ion, respectively [36]. Also, $\vec{r}_j = \bar{z}_j\hat{z}$, $\vec{k}_j.\vec{r}_j = \phi_j$ and $\hat{\Sigma}_z^{(j)}(t) = [\hat{\Sigma}_+^{(j)}(t), \hat{\Sigma}_-^{(j)}(t)]$ is defined as the total population inversion operator of jth trapped ion [36].

In the resonance condition between a trapped ion and one of the modes of an electromagnetic applied field, i.e., $\omega = \omega_0 + \nu$, the effective interaction Hamiltonian reads as

$$\hat{H}_{\rm eff}(t) = i\hbar g(\theta)(\hat{c}(t)\hat{\Sigma}_+(t) - \hat{c}^\dagger(t)\hat{\Sigma}_-(t)), \tag{7}$$

where $g(\theta) = \eta\mu\cos\theta$. This equation is valid for each trapped ion that interacts with a quantized field in the Lamb-Dicke regime as well as in the first vibrational sideband. The resonance condition can be regarded as the first vibrational (blue) sideband of the interaction [28]. As is clear, the interaction between a trapped ion and a single-mode applied field in this condition takes the form similar to that of the well-known Jaynes-Cummings model [29]. Using the presented model, we aim to propose a method to calculate the propagated field from each trapped ion.

III. QUANTIZED RADIATED FIELD FROM A TRAPPED ION INTERACTING WITH A QUANTIZED FIELD

We suppose that, when a multimode electromagnetic field interacts with a single trapped ion, among its several modes, there exists only one mode that can be in resonance with the trapped ion. This situation is called resonance fluorescence [37]. In this case, the considered trapped ion receives the radiation (energy) of this mode and excites to its upper level. Then, it tends to propagate the same radiation, which is received via deexcitation to its lower level. In this section, we try to extract the operator form of such a propagated field and in the next section, the probability of this emission is evaluated.

Using the Heisenberg operator approach [31] and the obtained model Hamiltonian [see (2) and (6)] for description of the interaction between jth trapped ion and jth applied electromagnetic field, we deduce:

$$\begin{aligned}
\dot{\hat{\tilde{c}}}_{\vec{k}_j,\lambda}^{(j)}(t) &= -g_{\vec{k}_j,\lambda}^{(j)}(\theta_j)\hat{\tilde{\Sigma}}_-^{(j)}(t)e^{-i\vec{k}_j.\vec{r}_j}\,e^{i(\omega_k^{(j)}-\omega_0-\nu)t},\\
\dot{\hat{\tilde{\Sigma}}}_-^{(j)}(t) &= -\sum_{\vec{k}_j,\lambda}g_{\vec{k}_j,\lambda}^{(j)}(\theta_j)\hat{\Sigma}_z^{(j)}(t)\hat{\tilde{c}}_{\vec{k}_j,\lambda}^{(j)}(t)e^{i\vec{k}_j.\vec{r}_j}\\
&\quad\times e^{-i(\omega_k^{(j)}-\omega_0-\nu)t},
\end{aligned} \tag{8}$$

where we have set

$$\begin{aligned}
&\hat{\tilde{c}}_{\vec{k}_j,\lambda}^{(j)}(t) = \hat{c}_{\vec{k}_j,\lambda}^{(j)}(t)e^{i\omega_k^{(j)}t}, \quad \hat{\tilde{\Sigma}}_-^{(j)}(t) = \hat{\Sigma}_-^{(j)}(t)e^{i(\omega_0+\nu)t},\\
&\hat{\Sigma}_z^{(j)}(t) = \hat{\tilde{\Sigma}}_z^{(j)}(t).
\end{aligned} \tag{9}$$

Through the first relation in (8), we obtain $\hat{\tilde{c}}_{\vec{k}_j,\lambda}^{(j)}(t) = \hat{\tilde{c}}_{\vec{k}_j,\lambda}^{(j)}(0) + \hat{\tilde{a}}_{\vec{k}_j,\lambda}^{(j)}(t)$ where

$$\begin{aligned}
\hat{\tilde{a}}_{\vec{k}_j,\lambda}^{(j)}(t) &= -g_{\vec{k}_j,\lambda}^{(j)}(\theta_j)e^{-i\vec{k}_j.\vec{r}_j}e^{i(\omega_k^{(j)}-\omega_0-\nu)t}\\
&\quad\times\int_0^t dt'\hat{\tilde{\Sigma}}_-^{(j)}(t')e^{-i(\omega_k^{(j)}-\omega_0-\nu)(t-t')}.
\end{aligned} \tag{10}$$

The quantity $\hat{\tilde{c}}_{\vec{k}_j,\lambda}^{(j)}(0)$ physically refers to the annihilation operator of jth applied field before starting the interaction with jth trapped ion at $t = 0$ with t as the interaction time. Similarly $\hat{\tilde{a}}_{\vec{k}_j,\lambda}^{(j)}(t)$ denotes the annihilation operator of field during its interaction with this trapped ion. After receiving the radiation from the field, the trapped ion emits the same

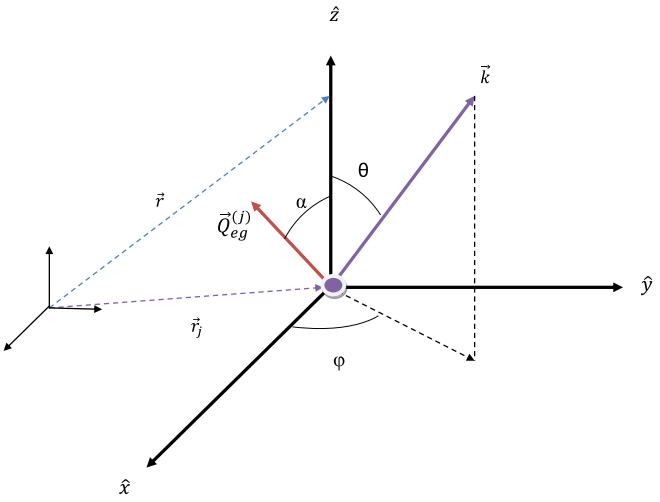

FIG. 2. Coordinate system for studying the interaction between the jth trapped ion with the jth quantized field.

radiation, which is received via an interlevels transition. We consider this emitted quantized field as

$$\hat{E}_j(\vec{r},t) = \hat{E}_j^{+}(\vec{r},t) + \hat{E}_j^{-}(\vec{r},t), \tag{11}$$

in which

$$\hat{E}_j^{+}(\vec{r},t) = \sum_{\vec{k}_j,\lambda} \varepsilon_{\vec{k}_j}^{(j)}\, \hat{\epsilon}_{\vec{k}_j}^{\lambda,(j)}\, \hat{a}_{\vec{k}_j,\lambda}^{(j)}(t)\, e^{i\vec{k}_j.\vec{r}}, \tag{12}$$

where $\hat{a}_{\vec{k}_j,\lambda}^{(j)}(t) = \hat{\tilde{a}}_{\vec{k}_j,\lambda}^{(j)}(t)e^{-i\omega_k^{(j)}t}$ and $\hat{E}_j^{-}(\vec{r},t) = [\hat{E}_j^{+}(\vec{r},t)]^{\dagger}$. By exchanging $\sum_{\vec{k}_j}$ with $\frac{V}{(2\pi)^3}\int d^3\vec{k}_j$ and utilizing the Weisskopf-Wigner theory [38], we finally arrive at:

$$\begin{aligned}\hat{E}_j^{+}(\vec{r},t) &= E_j^{+}(\vec{r})\, \hat{\Sigma}_{-}^{(j)}\left(t - \frac{|\vec{r}-\vec{r}_j|}{c}\right)\hat{x} \\ &= E_j^{+}(\vec{r})\, \hat{\tilde{\Sigma}}_{-}^{(j)}\left(t - \frac{|\vec{r}-\vec{r}_j|}{c}\right)e^{-i(\omega_0+\nu)(t-\frac{|\vec{r}-\vec{r}_j|}{c})}\hat{x},\end{aligned} \tag{13}$$

where $E_j^{+}(\vec{r}) = \frac{-i\eta_j(\omega_0+\nu)^2|\vec{Q}_{eg}^{(j)}|\sin\alpha_j}{4\pi\epsilon_0 c^2|\vec{r}-\vec{r}_j|}$. In deriving the above relation, we used the vector definitions as shown in Fig. 2 ($\vec{Q}_{eg}^{(j)}$ is in xz plane). Equation (13) has been obtained via ignoring the incoming wave contribution and in the far-field region. To apply the Weisskopf-Wigner theory, the resonance fluorescence between the kth mode of jth applied field and the jth trapped ion is supposed to have occurred, i.e., $\omega_k^{(j)} = \omega_0 + \nu$. It is obviously found from (13) that the propagated electric field from a trapped ion depends on its internal ionic transition frequency, its external vibrational frequency, its position in the ion string and the Lamb-Dicke parameter, which itself depends on the ion number N. Now, to complete our investigation about the radiation of trapped ion, we should encounter the probability of occurrence of such an emission; this is the subject of the next section.

IV. PROBABILITY OF EMISSION OF A QUANTIZED FIELD VIA A TRAPPED ION

In this section we intend to obtain the probability of emission of a quantized field (photon) via a trapped two-level ion in a special polarization direction. For this purpose, our model Hamiltonian is comprised of Eqs. (2) and (6) with substituting $N = 1$ and $\lambda = 1$. We suppose that, at $t = 0$ the jth trapped ion lives in its ground state $|g,m-1\rangle$ and via absorbing a photon from the jth applied field at time t_0, excites to its upper level $|e,m\rangle$. The state $|e\rangle$ ($|g\rangle$) refers to the internal upper (lower) level of trapped ion and $|m\rangle$ refers to the number state of its external vibrational mode. The emitted field is initially considered in vacuum state $|0_{\vec{k}_j}\rangle$ at t_0. So, the initial state vector reads as $|\psi(t_0)\rangle = \sum_{m=0}^{\infty}|e,m,0_{\vec{k}_j}\rangle$. The suggested state vector after occurring the emission from the trapped ion denotes as

$$\begin{aligned}|\psi(t)\rangle = &\sum_{m=0}^{\infty} C_{e,m,0_{\vec{k}_j}}(t)\big|e,m,0_{\vec{k}_j}\big\rangle \\ &+ \sum_{m=0}^{\infty}\sum_{\vec{k}_j} C_{g,m-1,1_{\vec{k}_j}}(t)\big|g,m-1,1_{\vec{k}_j}\big\rangle,\end{aligned} \tag{14}$$

with the initial conditions $C_{e,m,0_{\vec{k}_j}}(t_0) = 1$ and $C_{g,m-1,1_{\vec{k}_j}}(t_0) = 0$. The interaction Hamiltonian in the interaction picture may be obtained as follows:

$$\begin{aligned}\hat{\vartheta}(t) = i\hbar\sum_{\vec{k}_j}\Big(&g_{\vec{k}_j}^{(j)}(\theta_j,\vec{r}_j)\hat{c}_{\vec{k}_j}^{(j)}(t)\hat{\Sigma}_{+}^{(j)}(t)e^{i\delta_j t} \\ &- g_{\vec{k}_j}^{*(j)}(\theta_j,\vec{r}_j)\hat{c}_{\vec{k}_j}^{(j)\dagger}(t)\hat{\Sigma}_{-}^{(j)}(t)e^{-i\delta_j t}\Big),\end{aligned} \tag{15}$$

where $g_{\vec{k}_j}^{(j)}(\theta_j,\vec{r}_j) = g_{\vec{k}_j}^{(j)}(\theta_j)e^{i\vec{k}_j.\vec{r}_j}$ and $\delta_j = \omega_0 + \nu - \omega_k^{(j)}$. Now, by utilizing the time-dependent Schrödinger equation, i.e., $i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{\vartheta}(t)|\psi(t)\rangle$, we arrive at the following coupled differential equations:

$$\dot{C}_{e,m,0_{\vec{k}_j}}(t) = \sum_{\vec{k}_j} g_{\vec{k}_j}^{(j)}(\theta_j,\vec{r}_j)\,\sqrt{m}\,C_{g,m-1,1_{\vec{k}_j}}(t)\,e^{i\delta_j t}, \tag{16}$$

$$\dot{C}_{g,m-1,1_{\vec{k}_j}}(t) = -g_{\vec{k}_j}^{*(j)}(\theta_j,\vec{r}_j)\,\sqrt{m}\,C_{e,m,0_{\vec{k}_j}}(t)\,e^{-i\delta_j t}. \tag{17}$$

From Eq. (17), one readily obtains:

$$C_{g,m-1,1_{\vec{k}_j}}(t) = -g_{\vec{k}_j}^{*(j)}(\theta_j,\vec{r}_j)\sqrt{m}\int_{t_0}^{t} dt'\,C_{e,m,0_{\vec{k}_j}}(t')e^{-i\delta_j t'}. \tag{18}$$

Substituting Eq. (18) into (16), replacing $\sum_{\vec{k}_j}$ with $\frac{V}{(2\pi)^3}\int d^3\vec{k}_j$ and utilizing the Weisskopf-Wigner theory (after some manipulations exactly similar to the previous section) yields:

$$\dot{C}_{e,m,0_{\vec{k}_j}}(t) = -\frac{\Gamma_j}{2}C_{e,m,0_{\vec{k}_j}}(t), \tag{19}$$

where

$$\Gamma_j = \frac{\eta_j^2(\omega_0+\nu)^3\, m\, |\vec{Q}_{eg}^{(j)}|^2}{10\,\pi\epsilon_0\hbar c^3}\left(\sin^2\alpha_j + \frac{4}{3}\cos^2\alpha_j\right). \tag{20}$$

The spontaneous emission rate Γ_j depends on the ionic characteristics such as ω_0, $|\vec{Q}_{eg}^{(j)}|$ and also on η_j, ν, and m,

which are, respectively, represented to the Lamb-Dicke parameter, vibrational (trapping) frequency, and vibrational mode of the jth trapped ion. So, increasing the trapping (vibrational) energy of the jth trapped ion, $E_m = (m + \frac{1}{2})\hbar\nu$, may lead to an increase in the spontaneous emission rate of its interlevel transition. The lifetime of the upper state of the jth trapped ion denotes as $t_{\rm lt}^{(j)} \approx \frac{1}{\Gamma_j}$ (the subscript "lt" refers to lifetime) and as is discussed above can be varied via changing the trapping energy of ions. Increasing the trapping energy can lead to a decrease in $t_{\rm lt}^{(j)}$. In deducing Eq. (19), $\hat{\epsilon}_{\vec{k}_j}^{(1)}$ has the following form:

$$\hat{\epsilon}_{\vec{k}_j}^{(1)} = (\hat{x}\cos\theta_j\cos\phi_j + \hat{y}\cos\theta_j\sin\phi_j - \hat{z}\sin\theta_j). \quad (21)$$

At last, from (19) we obtain:

$$C_{e,m,0_{\vec{k}_j}}(t) = C_{e,m,0_{\vec{k}_j}}(t_0)e^{-\frac{\Gamma_j}{2}(t-t_0)} = e^{-\frac{\Gamma_j}{2}(t-t_0)}, \quad (22)$$

and using Eqs. (18) and (22) in resonance condition ($\delta_j = 0$) results in:

$$C_{g,m-1,1_{\vec{k}_j}}(t) = \frac{2\, g_{\vec{k}_j}^{*(j)}(\theta_j,\vec{r}_j)\sqrt{m}}{\Gamma_j}(e^{-\frac{\Gamma_j}{2}(t-t_0)} - 1). \quad (23)$$

It can be found from (22) and (23) that the probability of living the trapped ion in its upper or lower states strongly depends on its spontaneous emission rate as well as its trapping energy. The emission of a photon is performed via transition from $|e,m,0_{\vec{k}_j}\rangle$ to $|g,m-1,1_{\vec{k}_j}\rangle$. This occurs with the probability $P_{m,\vec{k}_j}(t) = C_{e,m,0_{\vec{k}_j}}(t)C^{\star}_{g,m-1,1_{\vec{k}_j}}(t)$. We interpret $P_{m,\vec{k}_j}(t)$ as the probability of emission of a quantized field via an excited trapped ion with m vibrational quanta (phonon). Henceforth, this probability (as a correction coefficient) should be multiplied by Eq. (13), so that the final form of emitted field from the jth trapped ion is obtained as

$$\hat{E}_j^{+}(\vec{r},t) = \xi_j^{+}(\vec{r})\,(e^{-\Gamma_j(t-t_0)} - e^{-\frac{\Gamma_j}{2}(t-t_0)}) \times \hat{\tilde{\Sigma}}_{-}^{j}\left(t - \frac{|\vec{r}-\vec{r}_j|}{c}\right)e^{-i(\omega_0+\nu)(t-\frac{|\vec{r}-\vec{r}_j|}{c})}\,\hat{x}, \quad (24)$$

where $\xi_j^{+}(\vec{r}) = \frac{2E_j^{+}(\vec{r})g_{\vec{k}_j}^{(j)}(\theta_j)e^{i\vec{k}_j\cdot\vec{r}_j}\sqrt{m}}{\Gamma_j}$. From Eq. (24), it is obviously seen that to arrive at the explicit form of the electric field operator, determining the quantity $\hat{\tilde{\Sigma}}_{-}^{(j)}(t)$ is necessary. In the next section, we try to do this task.

V. TIME EVOLUTION OF THE OPERATORS OF SYSTEM

To complete the investigation about the emitted quantized field via a trapped ion, we need to determine the explicit analytical form of trapped ionic operators. In this section, we try to find the time evolution of these operators in the Lamb-Dicke regime. The model Hamiltonian [see Eqs. (2) and (7)], which describes the interaction between a trapped ion with a single-mode quantized field in resonance case ($\omega_0 + \nu = \omega$) reads as

$$\hat{H}(t) = \frac{\hbar\omega_0}{2}\hat{\sigma}_z(t) + \hbar\nu\hat{b}^{\dagger}(t)\hat{b}(t) + \hbar\omega\hat{c}^{\dagger}(t)\hat{c}(t) + i\hbar g(\theta)(\hat{c}(t)\hat{\Sigma}_{+}(t) - \hat{c}^{\dagger}(t)\hat{\Sigma}_{-}(t)), \quad (25)$$

where the index "j" from the parameters associated with the jth trapped ion (also jth applied field) is removed for simplicity. The evolution of the operators of system yields:

$$\begin{aligned}\hat{\tilde{c}}(t) &= -g(\theta)\,\hat{\tilde{\Sigma}}_{-}(t),\\ \hat{\tilde{\Sigma}}_{-}(t) &= -g(\theta)\,\hat{\tilde{c}}(t)\,\hat{\tilde{\Sigma}}_{z}(t),\\ \hat{\tilde{\Sigma}}_{+}(t) &= -g(\theta)\,\hat{\tilde{c}}^{\dagger}(t)\,\hat{\tilde{\Sigma}}_{z}(t),\end{aligned} \quad (26)$$

where $\hat{\tilde{\Sigma}}_{z}(t) = \hat{\Sigma}_{z}(t)$. We determine three integrals of motion associated with the Hamiltonian (25) as below:

$$\begin{aligned}\hat{R}_1 &= \hat{b}^{\dagger}(t)\hat{b}(t) + \hat{c}^{\dagger}(t)\hat{c}(t),\\ \hat{R}_2 &= \hat{b}^{\dagger}(t)\hat{b}(t) - \hat{\sigma}_{+}(t)\hat{\sigma}_{-}(t),\\ \hat{R}_3 &= ig(\theta)(\hat{\tilde{c}}(t)\,\hat{\tilde{\Sigma}}_{+}(t) - \hat{\tilde{c}}^{\dagger}(t)\,\hat{\tilde{\Sigma}}_{-}(t)).\end{aligned} \quad (27)$$

So, one can easily find that $\dot{\hat{R}}_i = 0$ due to $[\hat{R}_i, \hat{H}(t)] = 0$ and $[\hat{R}_i, \hat{R}_j] = 0$ (for $i \neq j = 1,2,3$). Utilizing Eqs. (26) and (27) with some straightforward algebraic manipulations (whose details are in Appendix A) results in:

$$\frac{d^2\hat{\tilde{\Sigma}}_{-}(t)}{dt^2} - 2i\,\hat{R}_3\frac{d\hat{\tilde{\Sigma}}_{-}(t)}{dt} + \hat{\chi}\;\hat{\tilde{\Sigma}}_{-}(t) = 0, \quad (28)$$

where $\hat{\chi} = K^2\,(1 + \hat{R}_2)$ and $K = -g(\theta)$. The general solution of (28) is as

$$\hat{\tilde{\Sigma}}_{-}(t) = [\hat{\tilde{\Sigma}}_{+}(t)]^{\dagger} = e^{i\hat{R}_3 t}\left[\left(\cos(\hat{\Omega}t) - \frac{i\hat{R}_3}{\hat{\Omega}}\sin(\hat{\Omega}t)\right)\hat{\tilde{\Sigma}}_{-}(0) + \frac{\hat{\chi}}{k\hat{\Omega}}\sin(\hat{\Omega}t)\hat{\tilde{c}}(0)\right], \quad (29)$$

where $\hat{\Omega} = \sqrt{\hat{\chi} + \hat{R}_3^2}$ (see Appendix A). Using Eqs. (26) and (29), for the operators of field one yields:

$$\hat{\tilde{c}}(t) = [\hat{\tilde{c}}^{\dagger}(t)]^{\dagger} = e^{i\hat{R}_3 t}\left[\left(\cos(\hat{\Omega}t) - \frac{i\hat{R}_3}{\hat{\Omega}}\sin(\hat{\Omega}t)\right)\hat{\tilde{c}}(0) + \frac{k}{\hat{\Omega}}\sin(\hat{\Omega}t)\hat{\tilde{\Sigma}}_{-}(0)\right], \quad (30)$$

Substituting (29) into (24) the explicit form of the emitted field is obtained. With the help of these definitions, we can study the spectral and statistical properties of the evolved interacting subsystems.

VI. FIRST-ORDER MULTITIME CORRELATION FUNCTION OF A TRAPPED ION

In this section we intend to evaluate the first-order multitime correlation function of a trapped ion. This goal originates from the fact that according to Eq. (24), this function can physically be interpreted as the normalized Fourier transform of the power spectrum of emitted field (via the trapped ion) [39]. We define the quantity $g_{TI}^{(1)}(t,\tau)$ as the normalized first-order multitime correlation function of a trapped ion, which may be determined as follows:

$$g_{TI}^{(1)}(t,\tau) = \frac{\langle\hat{\tilde{\Sigma}}_{+}(t)\hat{\tilde{\Sigma}}_{-}(t+\tau)\rangle}{\sqrt{\langle\hat{\tilde{\Sigma}}_{+}(t)\hat{\tilde{\Sigma}}_{-}(t)\rangle}\sqrt{\langle\hat{\tilde{\Sigma}}_{+}(t+\tau)\hat{\tilde{\Sigma}}_{-}(t+\tau)\rangle}}. \quad (31)$$

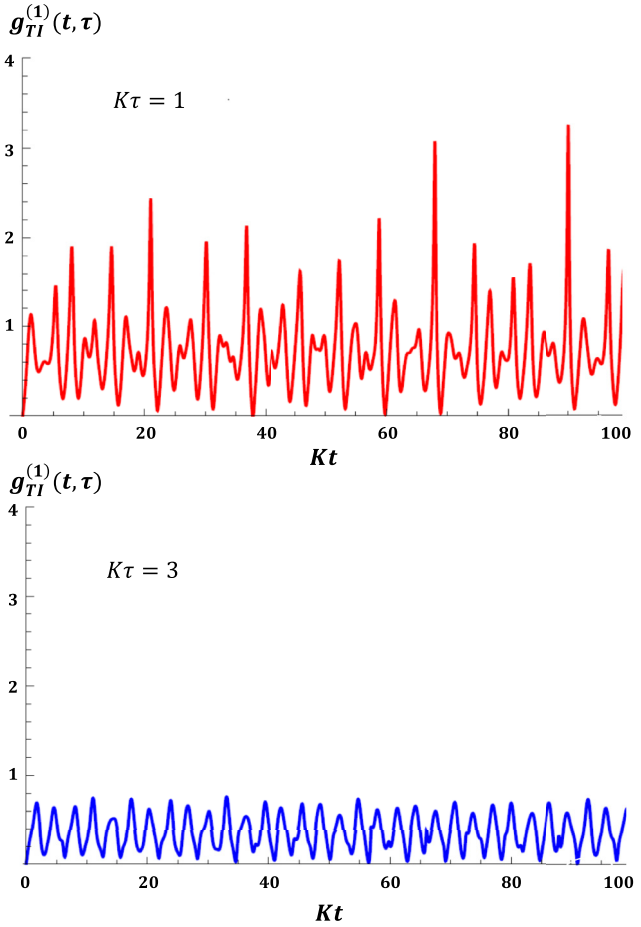

FIG. 3. The variation of $g_{TI}^{(1)}(t,\tau)$ for a trapped ion interacting with a quantized field versus the scaled time Kt for two values of $K\tau = 1$ and $K\tau = 3$. Other parameters are $\beta = 0.5$ and $\gamma = 1$.

To evaluate the above quantity, we suppose that the trapped ion-field state is initially prepared as

$$|\psi(0)\rangle = |g,\beta,\gamma\rangle. \tag{32}$$

This means that at $t = 0$, the ion is in its ground level ($|g\rangle$), where its vibrational mode and the quantized field are in the coherent state $|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle$ with the complex amplitudes $z = \beta,\gamma$ receptively. Utilizing Eqs. (29) and (32) yields:

$$\langle \hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_-(t+\tau)\rangle = e^{-|\beta|^2}e^{-|\gamma|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{|\beta|^{2m}}{m!}\frac{|\gamma|^{2n}}{n!} \times (m+1)\sin^2(\phi_{n,m}t)\cos(\phi_{n-1,m}\tau)\right\}, \tag{33}$$

where $\phi_{n,m} = K\sqrt{n(m+1)}$. Other expectation values may be defined by the same method. The first-order multitime correlation function is shown in Fig. 3 versus the scaled time Kt for two various values of $K\tau = 1$ and $K\tau = 3$. As is shown in both diagrams, $g_{TI}^{(1)}(t,\tau)$ starts from zero at $t = 0$ and oscillates drastically around some positive value as time goes on. Comparing these diagrams demonstrates that for the case with $K\tau = 3$, the quantity $g_{TI}^{(1)}(t,\tau)$ possesses more regular oscillation behavior, but its maximum value is less than the case with $K\tau = 1$. The Fourier transform of the quantity $\langle \hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_-(t+\tau)\rangle$ is proportional to the power spectrum of the emitted quantized field from a trapped ion. Also, the quantity $\langle \hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_-(t)\rangle$ [according to (24)] is proportional to the intensity of this emitted field at time t. So, an increment in the intensity at either the times t or $t+\tau$ results in the reduction of the quantity $g_{TI}^{(1)}(t,\tau)$ and vice versa. Since the increment (decrement) in the mentioned intensity occurs frequently, the quantity $g_{TI}^{(1)}(t,\tau)$ possesses oscillatory behavior as time passes. Also, according to Eq. (31), it is found that for $\tau \ll t$, one has $g_{TI}^{(1)}(t,\tau) \approx 1$, which possesses nearly a constant behavior.

VII. QUANTUM STATISTICS OF THE FIELD

Now, we concentrate our attention on the quantum statistics of one of the fields and its photon bunching or antibunching properties. For these purposes, we calculate the second-order multitime correlation function of the applied field [30]. This quantity is defined as

$$g_{\rm f}^{(2)}(t,\tau) = \frac{\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\hat{c}(t+\tau)\hat{c}(t)\rangle}{\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle\langle \hat{c}^\dagger(t+\tau)\hat{c}(t+\tau)\rangle}. \tag{34}$$

From physical point of view, the second-order correlation function $g_f^{(2)}(t,\tau)$ determines the probability of detecting a photon of the field at time $t+\tau$ when another photon has been previously detected at time t [30,40]. If $g_{\rm f}^{(2)}(t,0) = 1$, then the field has Poissonian statistics otherwise, the cases $g_{\rm f}^{(2)}(t,0) > 1$ and $g_{\rm f}^{(2)}(t,0) < 1$ show the super-Poissonian and sub-Poissonian statistics, respectively [30]. Also, if $g_{\rm f}^{(2)}(t,\tau) > g_{\rm f}^{(2)}(t,0)$, we have photon antibunching and if $g_{\rm f}^{(2)}(t,\tau) \leqslant g_{\rm f}^{(2)}(t,0)$, photon bunching occurs [30]. Through utilizing (30), (32), and (34), the quantities $g_{\rm f}^{(2)}(t,\tau)$ and $g_{\rm f}^{(2)}(t,0)$ can be evaluated, numerically. These quantities are shown in Fig. 4 versus the scaled time Kt. Figure 4(a) demonstrates the quantity $g_{\rm f}^{(2)}(t,5)$. This quantity starts from 1 at $t = 0$ and oscillates around 1.2 with some typical collapses and revivals. In Fig. 4(b), $g_{\rm f}^{(2)}(t,0)$ is shown, which again starts from 1 at $t = 0$. It is observed that in some periods of time, $g_{\rm f}^{(2)}(t,0)$ is slightly smaller than 1, which proves the sub-Poissonian statistics of the field, which can be interpreted as the nonclassical behavior of the field. Moreover, according to this diagram the super-Poissonian statistics [$g_{\rm f}^{(2)}(t,0) > 1$] is the dominant statistics of field. In Fig. 4(c), the quantities $g_{\rm f}^{(2)}(t,\tau)$ and $g_{\rm f}^{(2)}(t,0)$ are shown together in a single diagram. As is illustrated, the photon bunching or antibunching phenomena occur frequently as time goes on. However, by analyzing this diagram it is generally found that the photons mostly tend to distribute themselves in bunches (photon bunching) rather than at random due to obeying the relation $g_{\rm f}^{(2)}(t,5) \leqslant g_{\rm f}^{(2)}(t,0)$ in more intervals of time.

According to Eq. (34), the temporal behavior of $g_f^{(2)}(t,\tau)$ drastically relates to the time dependence of the operators $\hat{c}(t)$ and $\hat{c}^\dagger(t)$, which have been obtained in Eq. (30). It can be observed from Eq. (30) that the behaviors of $\hat{c}(t)$ and so

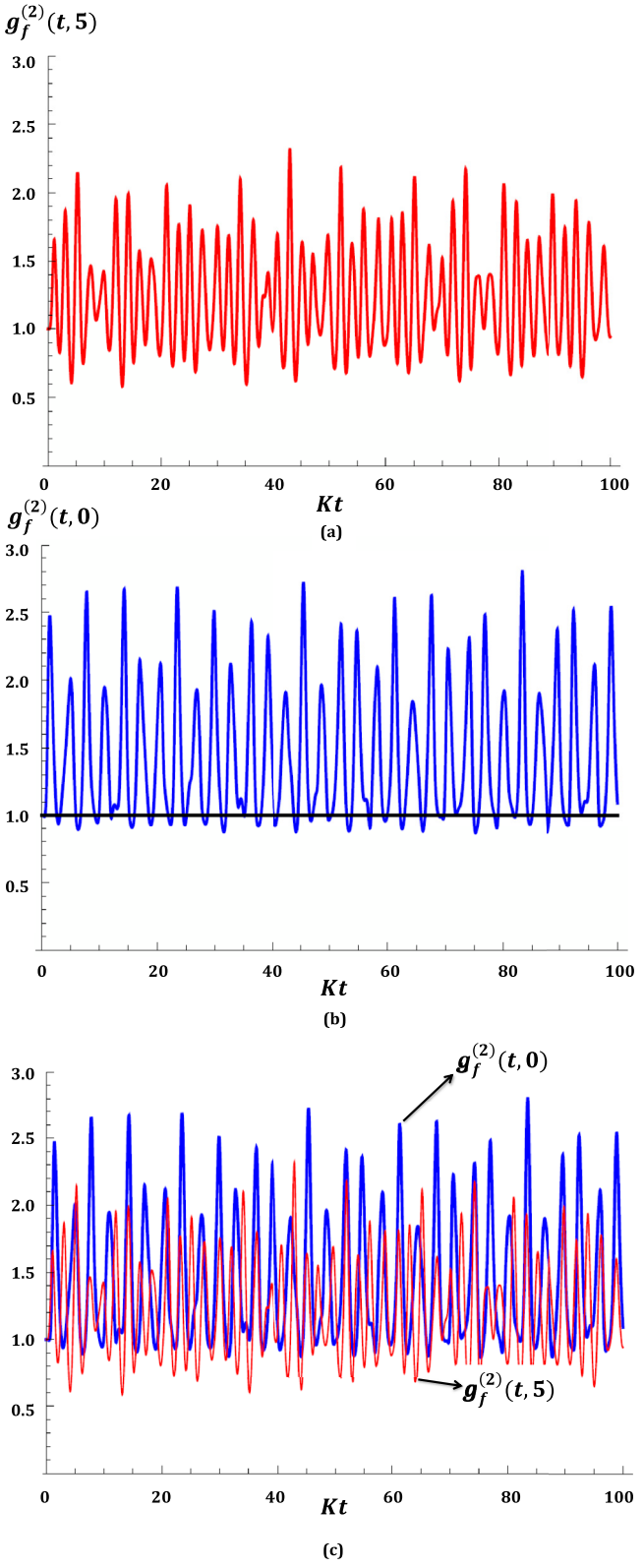

FIG. 4. Second-order multitime correlation function of the applied field versus the scaled time Kt: (a) for $g_{\rm f}^{(2)}(t,5)$, (b) for $g_{\rm f}^{(2)}(t,0)$, (c) for both of them in one diagram. Other parameters are $\beta = 0.5$ and $\gamma = 1$.

$\hat{c}^{\dagger}(t)$ are oscillatory as time goes on and there is no damping effect, which can stabilize their temporal behavior. So, under this condition the behavior of $g_f^{(2)}(t,\tau)$ is also oscillatory (nonstationary) as time goes on. Such a behavior for the second-order correlation function of fields can also be found in Refs. [41–44]. We would like to emphasize that in our applied fields we did not consider the dissipation effects, which may lead to the steady-state second-order correlation functions.

VIII. TRAPPED ION RESONANCE SPECTROSCOPY

In this section, the emitted spectra (photons) of trapped ions in the c.m. mode of their collective vibrational motion are taken into account via analyzing their emitted fields. The main reason for applying the proposed model in Sec. II is that we intend to obtain the spectral properties (intensity and power spectrum) of all trapped ions (in their collective c.m. vibrational mode) simultaneously in a reference detection point of the cavity (via a photodetector). So, we need to prepare the same interaction condition for each trapped ion in its interaction with a quantized field. By this preparation, the obtained spectrum of each trapped ion only depends on its relative position with respect to the detection point. Moreover, considering only one (instead of N) quantized field in the interaction with a string of N trapped ion leads one to obtain the spectra of trapped ions, which not only depend on their relative positions with respect to the detection point but also on the relative position of applied field with respect to the trapped ions. This situation makes the analysis of trapped ions' spectra more complicated than before and may be impossible. On the other hand, considering a trapped ion in the interaction with a quantized field is a simple case of the proposed general model and can not give us any useful insights about the spectra of the collection of N trapped ions, which have been trapped in the collective c.m. mode. The obtained spectra of trapped ions help us to get more information about the decoherence times due to the transitions between the two-state of considered trapped ions. To achieve this goal, we suppose that a high-precision photodetector has been installed within the trap at the position $\vec{r}$ and is used to detect the radiations, which are emitted via the trapped ions. The width of detector is selected in such a way that it covers all over the cavity during its detection. Using Eq. (24), the intensity of emitted field through the jth trapped ion reads as

$$I^{(j)}(\vec{r},t) = \langle \hat{E}_j^{-}(\vec{r},t)\hat{E}_j^{+}(\vec{r},t)\rangle, \tag{35}$$

where the initial state has been given in Eq. (32). The normalized intensity versus the scaled time $\Gamma_j t$ for $N = 1, 2$, and 3 trapped ions are illustrated in Figs. 5(a), 5(b), and 5(c), respectively. As are shown in these diagrams, this quantity starts from zero at first (in all plots), immediately reaches its maximum possible intensity, and then via some oscillations decays to zero as time goes on. The values of these normalized intensities are sensitive to the relative positions of the trapped ions with respect to the detection point. As is illustrated, by measuring the maximum values of the intensities of emitted fields from the trapped ions, the ions can be separately detected via the photodetector by considering their relative positions from the measuring points.

In fact, the value of the highest peak of the normalized intensity can be regarded as a special characteristic of each ion in the ion string. The normalized intensity of the emitted field

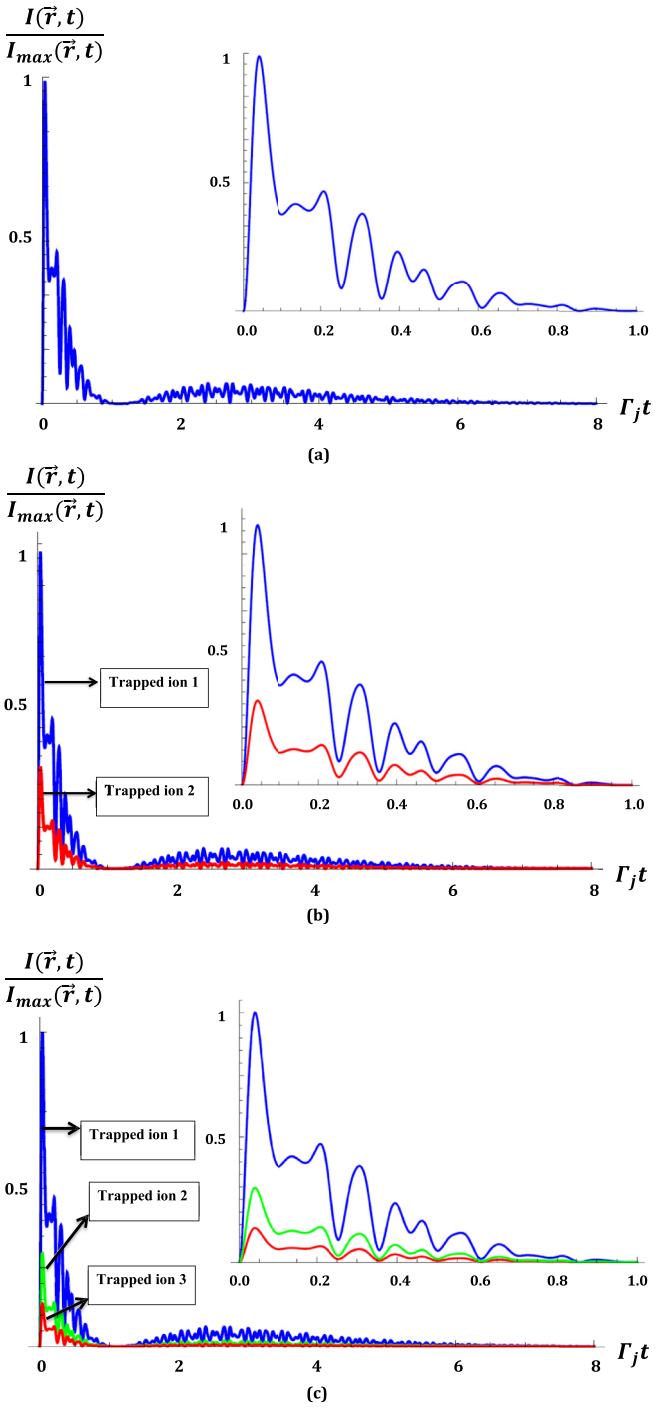

FIG. 5. Normalized intensities of the emission fields from the trapped ions, which are measured via a photodetector (which is installed in the trap at the position $\vec{r}$) versus the scaled time $\Gamma_j t$ for (a) one, (b) two, and (c) three trapped ions interacting with one, two, and three quantized fields, respectively (the same as Fig. 1). Other parameters are $\nu = 600\,\text{kHz}$, $V = 10^{-6}\text{m}^3$, $M = 66.52 \times 10^{-27}\text{kg}$, $\omega_0 + \nu = 5 \times 10^{14}\text{Hz}$, $|\vec{Q}_{eg}^{(j)}| = 2.9 \times 10^{-30}\text{C.m}$, $\alpha = \frac{\pi}{4}$, $\theta = \frac{\pi}{4}$, $\vec{r} = (10,10,50)\mu\text{m}$, $\vec{r}_j$ is selected as in Ref. [12] and the transition is supposed to perform between the levels $|e,1\rangle_j$ and $|g,0\rangle_j$. Other parameters are $\beta = 0.5$ and $\gamma = 1$.

via the nearest trapped ion to the detecting point is larger than the far one. Furthermore, the time in which the emissions via all the trapped ions are approximately stopped in these diagrams is $t_{\text{lt}}^{(j)} \approx \frac{1}{\Gamma_j}$. At this time, the trapped ions decay to their lower levels via emitting radiations. In Fig. 5, a photodetector has been installed in the location $\vec{r} = (10,10,50)\mu\text{m}$. The equilibrium positions of trapped ions are selected as [12]:

$$\begin{aligned} N = 1 &\longrightarrow \bar{z}_1 = 0, \\ N = 2 &\longrightarrow \bar{z}_1 = -\left(\tfrac{1}{4}\right)^{\frac{1}{3}}\kappa \approx -0.63\,\kappa, \\ &\qquad \bar{z}_2 = \left(\tfrac{1}{4}\right)^{\frac{1}{3}}\kappa \approx 0.63\,\kappa, \\ N = 3 &\longrightarrow \bar{z}_1 = -\left(\tfrac{5}{4}\right)^{\frac{1}{3}}\kappa \approx -1.07\,\kappa, \quad \bar{z}_2 = 0, \\ &\qquad \bar{z}_3 = \left(\tfrac{5}{4}\right)^{\frac{1}{3}}\kappa \approx 1.07\,\kappa, \end{aligned} \tag{36}$$

where N is the number of ions. Also, as has been mentioned before, the position of the jth trapped ion on the $\hat{z}$ axis is determined by $z_j = \bar{z}_j + \delta[\hat{b}^\dagger(t) + \hat{b}(t)]$, where $\delta = \sqrt{\frac{\hbar}{2M\nu N}}$. By these considerations, we now turn our attention to the analysis of Fig. 5. Using the plot parameters of Fig. 5 brings us to $\kappa \approx 10^{-6}$ and

$$\begin{aligned} N = 1 &\longrightarrow \delta \approx 0.08\ \mu\text{m}, \\ N = 2 &\longrightarrow \delta \approx 0.06\ \mu\text{m}, \\ N = 3 &\longrightarrow \delta \approx 0.05\ \mu\text{m}. \end{aligned} \tag{37}$$

Also, since we consider the center-of-mass mode for the oscillations of trapped ions, they oscillate simultaneously (and in particular, in the same direction) with the same frequencies [12,19]. So, the quantity $|\vec{r} - \vec{r}_j(t)|$, which denotes the distance between the detector and jth trapped ion represents as [$\vec{r} = (10,10,50)\mu\text{m}$ and $\vec{r}_j(t) = (0,0,z_j)$]:

$$|\vec{r} - \vec{r}_j(t)| = \sqrt{10^2 + 10^2 + (50 - \bar{z}_j - z_j(t))^2}\ (\mu\text{m}), \tag{38}$$

where $z_j(t) = \delta(\hat{b}^\dagger(t) + \hat{b}(t))$ and $\hat{b}(t) = \hat{b}(0)e^{-i\nu t}$ [12]. In other words we can obtain:

$$z_j(t) = \frac{1}{\sqrt{N}}\left(z_j(0)\cos(\nu t) + \frac{p_j(0)}{M\nu}\sin(\nu t)\right), \tag{39}$$

where [12]

$$\begin{aligned} z_j(0) &= \sqrt{\frac{\hbar}{2M\nu}}(\hat{b}^\dagger(0) + \hat{b}(0)), \\ p_j(0) &= i\sqrt{\frac{\hbar M\nu}{2}}(\hat{b}^\dagger(0) - \hat{b}(0)) = \sqrt{2ME_m}, \end{aligned} \tag{40}$$

where $E_m = (m + \frac{1}{2})\hbar\nu$. Consequently, the spectra, which are demonstrated in Fig. 5 have been obtained by using these equations.

In each of the cases $N = 1$, $N = 2$, and $N = 3$, the relative distance between the detector and all the ions [Eq. (38)] alter with the same frequencies ν. So, although the amplitude of relative position between each of the trapped ions and the detector alternatively changes, moreover, the temporal behavior of this change is approximately the same for all trapped ions. Therefore, in Fig. 5 the amplitudes of normalized intensities change oscillatory and can be different for all trapped ions, however, approximately with the same temporal behavior of intensities as is shown in this figure.

Another quantity that can be utilized for the investigation of the spectrum of emitted field via a trapped ion is its power spectrum [39]. This quantity is defined as [30]

$$S(\vec{r},\Omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty d\tau \langle E^-(\vec{r},t)E^+(\vec{r},t+\tau)\rangle e^{i\Omega\tau}. \quad (41)$$

It is noticeable that $S(\vec{r},\Omega)$ is the Fourier transform of $\langle E^-(\vec{r},t)E^+(\vec{r},t+\tau)\rangle$ as is seen in (41). The quantity $\langle E^-(\vec{r},t)E^+(\vec{r},t+\tau)\rangle$ is a criterion to determine the normalized first-order multitime correlation function of a trapped ion. So, $S(\vec{r},\Omega)$ is indeed a quantity versus Ω, which determines the degree of correlation between $E^-(\vec{r},t)$ and $E^+(\vec{r},t+\tau)$ at the time t. This quantity possesses a maximum, which occurs in the resonance frequency between the trapped ion and the field and refers to the fact that in this frequency the correlation between $E^-(\vec{r},t)$ and $E^+(\vec{r},t+\tau)$ (at the certain time) gets its maximum value. In other words, $S(\vec{r},\Omega)$ describes the power spectrum, which is a criterion to determine the maximum of correlation between the operators $\hat{\tilde{\Sigma}}_+^j(t)$ and $\hat{\tilde{\Sigma}}_-^j(t+\tau)$ of the jth trapped ion. For the emitted field from jth trapped ion $S(\vec{r},\Omega)$ can be calculated with the help of Eqs. (24), (33), and (41) as follows:

$$\begin{aligned} S_j(\vec{r},\Omega) = \frac{\Gamma_j}{2\pi} I_0^{(j)}(\vec{r}) \sum_{m,n=0}^{\infty} I_{m,n}^{(j)}(\vec{r},t) \Bigg[& (e^{-2\Gamma_j(t-t_0)} \\ & - e^{-\frac{3\Gamma_j}{2}(t-t_0)}) \left(\frac{1}{P_{m,n}^2+\Gamma_j^2} + \frac{1}{Q_{m,n}^2+\Gamma_j^2} \right) \\ & + \frac{1}{2}(e^{-\Gamma_j(t-t_0)} - e^{-\frac{3\Gamma_j}{2}(t-t_0)}) \\ & \times \left(\frac{1}{P_{m,n}^2+\frac{\Gamma_j^2}{4}} + \frac{1}{Q_{m,n}^2+\frac{\Gamma_j^2}{4}} \right) \Bigg], \end{aligned} \quad (42)$$

where

$$\begin{aligned} I_0^{(j)}(\vec{r}) &= |\mathbf{E}_j^-(\vec{r})\mathbf{E}_j^+(\vec{r})|, \\ I_{m,n}^{(j)}(\hat{r},t) &= e^{-|\beta|^2} e^{-|\gamma|^2} \frac{|\beta|^{2m}}{m!} \frac{|\gamma|^{2n}}{n!}(m+1) \\ &\quad \times \sin^2\left(\phi_{m,n}\left(t - \frac{|\vec{r}-\vec{r}_j|}{c} \right) \right), \\ P_{m,n} &= (\Omega - (\omega_0 + \nu - \phi_{m,n-1})), \\ Q_{m,n} &= (\Omega - (\omega_0 + \nu + \phi_{m,n-1})). \end{aligned} \quad (43)$$

It can be obviously found that if $P_{m,n} = 0 = Q_{m,n}$, $S_j(\vec{r},\Omega)$ gets its maximum value versus Ω (for a constant value of $\vec{r}$), so:

$$\begin{aligned} P_{m,n} = 0 &\Longrightarrow \Omega_{m,n}^{(1)} = (\omega_0 + \nu - \phi_{m,n-1}), \\ Q_{m,n} = 0 &\Longrightarrow \Omega_{m,n}^{(2)} = (\omega_0 + \nu + \phi_{m,n-1}). \end{aligned} \quad (44)$$

Therefore, the width of peak in $S_j(\vec{r},\Omega)$ for each given m and n reads as $\Delta\Omega_{m,n} = \Omega_{m,n}^{(2)} - \Omega_{m,n}^{(1)} = 2\phi_{m,n-1}$.

The normalized power spectrum for one and two trapped ions interacting, respectively, with one and two quantized fields at $t = \frac{1}{2\Gamma_j}$ have been shown in Figs. 6 and 7, respectively. The frequency of applied field in each plot is $\omega = 5 \times 10^{14}$Hz

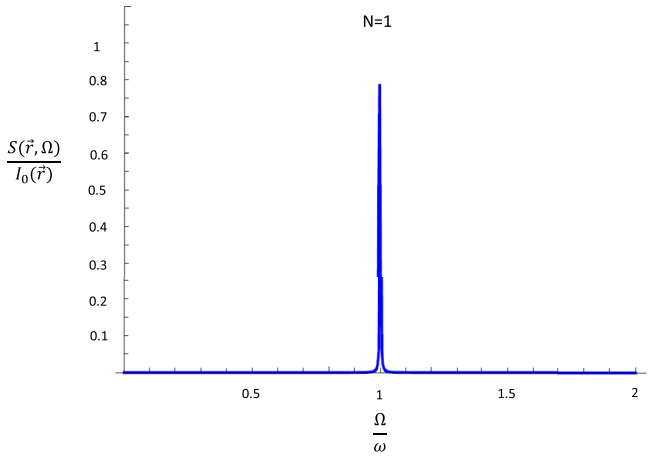

FIG. 6. Normalized power spectrum for one trapped ion interacting with a quantized field versus the normalized frequency. $t(s) = \frac{1}{2\Gamma_j}$ and other parameters are the same as in Fig. 5.

(= $\omega_0 + \nu$ due to the resonance condition). As is demonstrated in these figures, the normalized power spectrum possesses an intense peak exactly in the resonance frequency for each of the trapped ions, but its amplitude for each of them is different due to the differences in their positions in the trap. The maximum correlation between $E^-(\vec{r},t)$ and $E^+(\vec{r},t+\tau)$ for the jth trapped ion occurs at the resonance frequency as is discussed above. So, this quantity can be regarded as another parameter for detecting the trapped ions with respect to their resonance frequencies with the applied field as well as their positions in the trap. To analyze Figs. 6 and 7, it should be

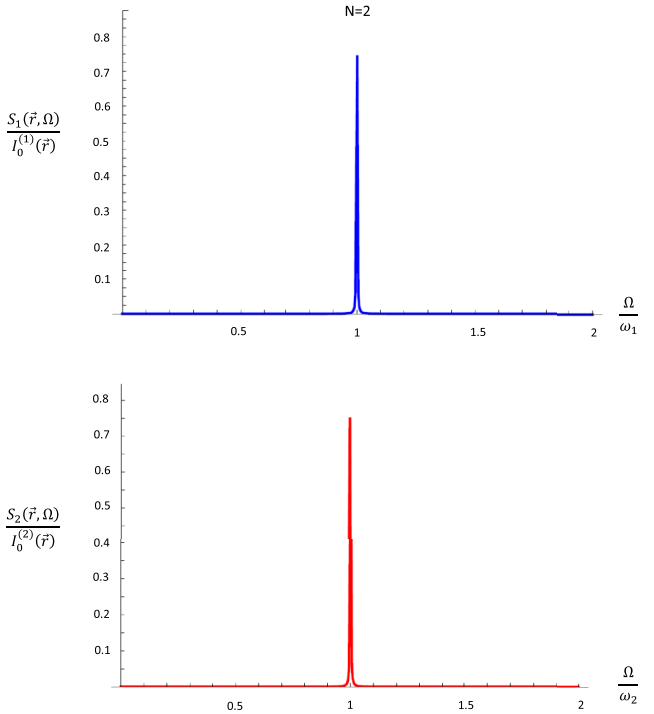

FIG. 7. Normalized power spectrum for two trapped ions in the same conditions as Fig. 6.

noticed that the power spectrum physically determines the power of a signal, which exists at a certain frequency [45]. This quantity, in fact, defines the distribution of spectrum as a function of radiation frequency, so, by calculating this function one can understand about which frequency in a spectrum possesses the maximum probability as well as maximum power for the detection process [45]. According to Fig. 6, which has been plotted for emitted spectrum from a trapped ion in the interaction with a quantized field (with frequency ω), the maximum power (probability) in the emitted spectrum occurs at the frequency $\Omega = \omega$, i.e., the trapped ion propagates the same photon, which has been absorbed from the field with more probability. In Fig. 7, two trapped ions separately interact with two quantized fields at frequencies $\Omega = \omega_1$ and $\Omega = \omega_2$, respectively. As is observed from this figure, the maximum power of the emitted spectrum from each of the ions occurs exactly in the field frequency, which interacts with each ion.

IX. CONCLUSIONS AND REMARKS

In summary, we considered the full quantum mechanical interaction between N multimode quantized fields with N trapped two-level ions, such that each ion interacts with one of the quantized fields. In this respect, we supposed that only one of the modes of each field is in resonance with its front trapped ion (see Fig. 1), as a result, resonance florescence in trapped ion-field system can be occurred. Such a model has been previously discussed in the semiclassical framework [20]. We investigated such interactions to get more information about the effects of quantized (instead of classical [20]) field on the physical properties of system. The trapped ions that have been utilized possess external vibrational degrees of freedom in addition to internal atomic degrees. We also worked with the Heisenberg operator approach to discuss on the second-order correlation function as the criteria of bunching and antibunching properties of the fields. The spontaneous emission rate has been obtained for each trapped ion, which denotes the relations between the lifetime of upper state of each trapped ion and the trapping (vibrational) energy. We concluded that increasing the trapping energy leads to an increase (decrease) in the spontaneous emission rate (lifetime) of interlevel transitions. Along these lines, to get information about the fields, we studied their spectral and statistical properties. In this regard, the quantized field operator, which is emitted from each trapped ion (after its interaction with the field) was encountered. Utilizing the obtained emitted field, the normalized intensity and power spectrum of the radiation of each trapped ion was deduced by which we allocated a characteristic spectrum to each of the trapped ions. Furthermore, via studying the normalized power spectrum we observed that the maximum of multitime correlation between the ladder operators associated with each trapped ion is obtained in its resonance frequency with the field. The sub-Poissonian and antibunching statistical properties of the field during the interaction well established the nonclassicality features of the field. It is therefore found that, even though the applied filed is initially prepared in the coherent state (with Poissonian statistics), its statistics during the interaction may changes to both sub-Poissonian (nonclassical) and super-Poissonian (classical). Moreover, the dominant statistics is recognized as super-Poissonian as time goes on.

APPENDIX: TIME EVOLUTION OF THE OPERATORS OF SYSTEM

To obtain Eq. (28), we use the following method. At first, with the help of Eq. (27) and substituting $\hat{\tilde{\Sigma}}_+(t)$ and $\hat{\tilde{\Sigma}}_-(t)$ in $\hat{\tilde{\Sigma}}_z(t)$ one obtains:

$$\hat{\tilde{\Sigma}}_z(t) = 2\hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_-(t) - \hat{R}_2 - 1, \tag{A1}$$

and so:

$$\hat{\tilde{\Sigma}}_z(t) = 2\frac{d}{dt}[\hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_-(t)]. \tag{A2}$$

To deduce $\hat{\tilde{\Sigma}}_-(t)$ in Eq. (26), we first calculate:

$$\begin{aligned}
&\frac{d}{dt}[\hat{\tilde{c}}(t)\hat{\tilde{\Sigma}}_z(t)] \\
&= \frac{d\hat{\tilde{c}}(t)}{dt}\hat{\tilde{\Sigma}}_z(t) + \hat{\tilde{c}}(t)\,\frac{d\hat{\tilde{\Sigma}}_z(t)}{dt} \\
&= \frac{d\hat{\tilde{c}}(t)}{dt}\,(2\hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_-(t) - \hat{R}_2 - 1) \\
&\quad + 2\hat{\tilde{c}}(t)\left(\frac{d\hat{\tilde{\Sigma}}_+(t)}{dt}\hat{\tilde{\Sigma}}_-(t) + \hat{\tilde{\Sigma}}_+(t)\frac{d\hat{\tilde{\Sigma}}_-(t)}{dt}\right) \\
&= K\,(2\hat{\tilde{\Sigma}}_-(t)\hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_-(t) - \hat{\tilde{\Sigma}}_-(t)\hat{R}_2 - \hat{\tilde{\Sigma}}_-(t)) \\
&\quad + 2K\,\hat{\tilde{c}}(t)\,(\hat{\tilde{c}}^\dagger(t)\,\hat{\tilde{\Sigma}}_z(t)\hat{\tilde{\Sigma}}_-(t) + \hat{\tilde{c}}(t)\,\hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_z(t)),
\end{aligned} \tag{A3}$$

where $K = -g(\theta)$. Since $\hat{\sigma}_+\hat{\sigma}_-\hat{\tilde{\Sigma}}_-$ for a two-level trapped ion is equal to zero ($\hat{\sigma}_-\hat{\sigma}_- = \hat{\sigma}_+\hat{\sigma}_+ = 0$), by utilizing Eq. (27) we can simplify the above relation with the help of the following definitions:

$$\begin{aligned}
\hat{b}^\dagger(t)\hat{b}(t)\hat{\tilde{\Sigma}}_-(t) &= \hat{R}_2\hat{\tilde{\Sigma}}_-(t), \\
\hat{\tilde{\Sigma}}_-(t)\hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_-(t) &= \hat{b}(t)\hat{b}^\dagger(t)\hat{\tilde{\Sigma}}_-(t) = (1+\hat{R}_2)\hat{\tilde{\Sigma}}_-(t), \\
\hat{\tilde{\Sigma}}_z(t)\hat{\tilde{\Sigma}}_-(t) &= -(1+\hat{R}_2)\hat{\tilde{\Sigma}}_-(t), \\
\hat{\tilde{\Sigma}}_+(t)\hat{\tilde{\Sigma}}_z(t) &= -\hat{b}^\dagger(t)\hat{b}(t)\hat{\tilde{\Sigma}}_+(t).
\end{aligned} \tag{A4}$$

So, Eq. (A3) can be rewritten as follows:

$$\begin{aligned}
\frac{d}{dt}[\hat{\tilde{c}}(t)\hat{\tilde{\Sigma}}_z(t)] = K\big[&(\hat{R}_2 - 1 - 2\hat{R}_1 - 2\hat{R}_1\hat{R}_2 + 2\hat{R}_2^2)\hat{\tilde{\Sigma}}_-(t) \\
&- 2\hat{\tilde{c}}^2(t)\hat{b}^\dagger(t)\hat{b}(t)\hat{\tilde{\Sigma}}_+(t)\big].
\end{aligned} \tag{A5}$$

To determine the last term in the above equation, i.e., $\hat{\tilde{c}}^2(t)\hat{b}^\dagger(t)\hat{b}(t)\hat{\tilde{\Sigma}}_+(t)$, we notice that using Eq. (27) yields:

$$\hat{\tilde{c}}(t)\hat{\tilde{\Sigma}}_+(t) = \frac{i}{K}\,\hat{R}_3 + \hat{\tilde{c}}^\dagger(t)\hat{\tilde{\Sigma}}_-(t). \tag{A6}$$

Then by multiplying the operator $\hat{\tilde{c}}(t)\hat{\tilde{\Sigma}}_z(t)$ from left onto (A6), we arrive at:

$$\begin{aligned}
&-\hat{\tilde{c}}^2(t)\hat{b}^\dagger(t)\hat{b}(t)\hat{\tilde{\Sigma}}_+(t) \\
&= \frac{i}{K^2}\,\hat{R}_3\,\hat{\tilde{\Sigma}}_-(t) + (\hat{R}_1 - \hat{R}_2 + \hat{R}_1\hat{R}_2 - \hat{R}_2^2)\hat{\tilde{\Sigma}}_-(t).
\end{aligned} \tag{A7}$$

Substituting $\frac{1}{K}\hat{\tilde{\Sigma}}_-(t)$ instead of $\hat{\tilde{c}}(t)\hat{\tilde{\Sigma}}_z(t)$ [see Eq. (26)], using (A7), replacing $\hat{c}^\dagger(t)\hat{c}(t)$ with $\hat{R}_1 - \hat{b}^\dagger(t)\hat{b}(t)$ and $\hat{b}^\dagger(t)\hat{b}(t)$ with $\hat{R}_2 + \hat{\sigma}_+(t)\hat{\sigma}_-(t)$ [see Eq. (27)] and applying $\hat{R}_2\ \hat{\sigma}_-(t)$ instead of $\hat{b}^\dagger(t)\hat{b}(t)\ \hat{\sigma}_-(t)$ in Eq. (A5), one arrives at the differential Eq. (28). In obtaining (28), we also used the following commutators:

$$[\hat{R}_1, \hat{\tilde{\Sigma}}_-(t)] = -\hat{\tilde{\Sigma}}_-(t), \quad [\hat{R}_2, \hat{\tilde{\Sigma}}_\pm(t)] = 0. \tag{A8}$$

The general solution of the differential Eq. (28) may be written as

$$\hat{\tilde{\Sigma}}_-(t) = e^{i(\hat{R}_3 - \hat{\Omega})t}\ \hat{\alpha}_1 + e^{i(\hat{R}_3 + \hat{\Omega})t}\ \hat{\alpha}_2, \tag{A9}$$

where $\hat{\Omega} = \sqrt{\hat{\chi} + \hat{R}_3^2}$ and $[\hat{R}_3, \hat{\Omega}] = 0$. Using Eq. (26) one can obtain:

$$\hat{\tilde{c}}(t) = K\left(e^{i(\hat{R}_3 - \hat{\Omega})t}\ \frac{-i\hat{\alpha}_1}{\hat{R}_3 - \hat{\Omega}} + e^{i(\hat{R}_3 + \hat{\Omega})t}\ \frac{-i\hat{\alpha}_2}{\hat{R}_3 + \hat{\Omega}}\right), \tag{A10}$$

where $\hat{\tilde{\Sigma}}_-(0) = \hat{\alpha}_1 + \hat{\alpha}_2$ and $\hat{\tilde{c}}(0) = -iK(\frac{\hat{\alpha}_1}{\hat{R}_3 - \hat{\Omega}} + \frac{\hat{\alpha}_2}{\hat{R}_3 + \hat{\Omega}})$. Therefore,

$$\begin{aligned}\hat{\alpha}_1 &= \left(\frac{1}{2} - \frac{\hat{R}_3}{2\hat{\Omega}}\right)\hat{\tilde{\Sigma}}_-(0) - \frac{i\hat{\chi}}{2K\hat{\Omega}}\hat{\tilde{c}}(0),\\ \hat{\alpha}_2 &= \left(\frac{1}{2} + \frac{\hat{R}_3}{2\hat{\Omega}}\right)\hat{\tilde{\Sigma}}_-(0) + \frac{i\hat{\chi}}{2K\hat{\Omega}}\hat{\tilde{c}}(0).\end{aligned} \tag{A11}$$

Finally, after some manipulations the operators of system are obtained as in Eqs. (29) and (30).

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