

## Resource theory of coherence: Beyond states

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We generalize the recently proposed resource theory of coherence (or superposition) [T. Baumgratz *et al.*, *Phys. Rev. Lett.* **113**, 140401 (2014); A. Winter and D. Yang, *Phys. Rev. Lett.* **116**, 120404 (2016)] to the setting where not just the free (“incoherent”) resources, but also the manipulated objects, are quantum operations rather than states. In particular, we discuss an information theoretic notion of the coherence capacity of a quantum channel and prove a single-letter formula for it in the case of unitaries. Then we move to the coherence cost of simulating a channel and prove achievability results for unitaries and general channels acting on a  $d$ -dimensional system; we show that a maximally coherent state of rank  $d$  is always sufficient as a resource if incoherent operations are allowed, and one of rank  $d^2$  for “strictly incoherent” operations. We also show lower bounds on the simulation cost of channels that allow us to conclude that there exists bound coherence in operations, i.e., maps with nonzero cost of implementing them but zero coherence capacity; this is in contrast to states, which do not exhibit bound coherence.

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### I. INTRODUCTION

Since its discovery, quantum mechanics has provided a mathematical framework for the construction of physical theories. Indeterminism, interference, uncertainty, superposition, and entanglement are concepts of quantum mechanics that distinguish it from classical physics and that have become resources in quantum information processing.

Quantum resource theories aim at capturing the essence of these traits and quantifying them. Recently, quantum resource theories have been formulated in different areas of physics such as the resource theory of athermality in thermodynamics [1–6] and the resource theory of asymmetry [7,8]. Furthermore, general structural frameworks of quantum resource theories have been proposed [9].

Resource theories using concepts of quantum mechanics have been developed since the appearance of the theory of entanglement [10–12]. Very recently, Baumgratz *et al.* [13], following earlier work by Åberg [14], have made quantum coherence itself, i.e., the concept of superposition

$$|\psi\rangle = \sum_i \psi_i |i\rangle,$$

the subject of a resource theory; see also [15].

The present paper is concerned with this resource theory of coherence, and here we briefly recall its fundamental definitions, as well as some important coherence measures; for a comprehensive review, see [16]. Let  $\{|i\rangle : i = 0, \dots, d-1\}$  be a particular fixed basis of the  $d$ -dimensional Hilbert space  $\mathcal{H}$ ; then all density matrices in this basis are “incoherent,” i.e., those of the form  $\tilde{\delta} = \sum_{i=0}^{d-1} \delta_i |i\rangle\langle i|$ . We denote by  $\Delta \subset \mathcal{S}(\mathcal{H})$  the set of such incoherent quantum states.

The definition of coherence monotones requires the identification of operations that are incoherent. These map the set of incoherent states to itself. More precisely, such a completely positive and trace-preserving (CPTP) map is specified by a set of Kraus operators  $\{K_\alpha\}$  satisfying  $\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}$  and  $K_\alpha \Delta K_\alpha^\dagger \subset \Delta$  for all  $\alpha$ . A Kraus operator with this property is called *incoherent*; we call it *strictly incoherent* if both  $K$  and  $K^\dagger$  are incoherent [17,18]. We distinguish two classes of incoherent operations (IOs):

(i) incoherent completely positive and trace-preserving quantum operations (nonselective maps)  $T$ , which act as  $T(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$  (note that this formulation implies the loss of information about the measurement outcome); and

(ii) quantum operations for which measurement outcomes are retained, given by  $\rho_\alpha = \frac{1}{p_\alpha} K_\alpha \rho K_\alpha^\dagger$  occurring with probability  $p_\alpha = \text{Tr} K_\alpha \rho K_\alpha^\dagger$ . The latter can be modeled as a nonselective operation by explicitly introducing a new register to hold the (incoherent) measurement result:

$$\tilde{T}(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger \otimes |\alpha\rangle\langle\alpha|.$$

Here, we have made use of the convention that in the composition of systems, the incoherent states in the tensor product space are precisely the tensor products of incoherent states and their probabilistic mixtures (convex combinations) [14]. An operation that can be written with strictly incoherent Kraus operators is called a strictly incoherent operation (SIO).

We define also the maximally coherent state in a  $d$ -dimensional system by  $|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$ , from which every state in dimension  $d$  can be prepared [13,17,19]. Note that the definition of a maximally coherent state is independent of a specific measure for the coherence [13,20].

A list of desirable conditions for any coherence measure, i.e., a functional from states to nonnegative real numbers, is also presented [13]:

$$(1) \quad C(\rho) = 0 \text{ for all } \rho \in \Delta;$$

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(2) monotonicity under nonselective incoherent maps,  $C(\rho) \geq C(T(\rho))$ ;

(3) (strong) monotonicity under selective incoherent maps,  $C(\rho) \geq \sum_{\alpha} p_{\alpha} C(\rho_{\alpha})$ ; and

(4) convexity,  $\sum_i p_i C(\rho_i) \geq C(\sum_i p_i \rho_i)$ .

The first two are definitely required to speak of a coherence measure, the third is sometimes demanded axiomatically but often is really more of a convenience, and convexity should be thought of as nice if present but not absolutely necessary.

Among the most important examples are the following three measures: For pure states  $\varphi = |\varphi\rangle\langle\varphi|$ , the *entropy of coherence* is defined as

$$C(\varphi) := S(\Delta(\varphi)), \quad (1)$$

where  $\Delta$  is the dephasing (i.e., coherence-destroying) map  $\Delta(\rho) = \sum_i |i\rangle\langle i| \rho |i\rangle\langle i|$ ; for mixed states, it is extended by the convex hull construction to the *coherence of formation* [13,14],

$$C_f(\rho) := \min \sum_i p_i C(\psi_i) \text{ s.t. } \rho = \sum_i p_i \psi_i. \quad (2)$$

Finally, the *relative entropy of coherence* [13,14]

$$C_r(\rho) := \min_{\sigma \in \Delta} D(\rho \| \sigma) = S(\Delta(\rho)) - S(\rho), \quad (3)$$

with the relative entropy  $D(\rho \| \sigma) = \text{Tr} \rho (\log_2 \rho - \log_2 \sigma)$ . Both  $C_r$  and  $C_f$  satisfy all of properties 1 through 4 above.

In the present paper, we expand our view from states as coherent resources to operations, showing how to extract pure-state coherence from a given operation (Sec. II) and how to implement operations using coherent states as a resource (Sec. III). We briefly discuss the case of qubit unitaries as an example in Sec. IV, then conclude in Sec. V, where we observe that in operations, there is bound coherence, something that does not exist for states. Most generally, we propose a definition for the rate of conversion between two channels using only incoherent operations and present many open questions about this concept.

## II. COHERENCE-GENERATING CAPACITY AND COHERENCE POWER OF TRANSFORMATIONS

The free operations in our resource theory are the incoherent ones (IOs), which means that, in some sense, any other CPTP map represents a resource. How do we measure it? or, better, How do we assess its resource character? In the present section, we focus on how much pure-state coherence can be created asymptotically, using a given operation  $T : A \rightarrow B$  a large number of times, when incoherent operations are free.

The most general protocol for generating coherence must use the resource  $T$  and incoherent operations according to some predetermined algorithm, in some order. We may assume that the channels  $T$  are invoked one at a time; and we can integrate all incoherent operations between one use of  $T$  and the next into one incoherent operation, since the IO is closed under composition. Thus, a mathematical description of the most general protocol is the following: One starts by preparing an incoherent state  $\rho_0$  in  $A \otimes A_0$ , then lets  $T$  act, followed by an incoherent transformation,  $\mathcal{I}_1 : B \otimes A_0 \rightarrow A \otimes A_1$ , resulting in the state

$$\rho_1 = \mathcal{I}_1((T \otimes \text{id})\rho_0).$$

Iterating, given state  $\rho_t$  in  $A \otimes A_t$ , obtained after the action of  $t$  realizations of  $T$  and suitable incoherent operations, we let  $T$  act and the incoherent transformation  $\mathcal{I}_{t+1} : B \otimes A_t \rightarrow A \otimes A_{t+1}$ , resulting in the state

$$\rho_{t+1} = \mathcal{I}_{t+1}((T \otimes \text{id})\rho_t).$$

At the end of  $n$  iterations, we have a state  $\rho_n$  in  $A \otimes A_n$ , and we call the above procedure a *coherence generation protocol* of rate  $R$  and error  $\epsilon$ , if  $|A_n| = 2^{nR}$  and the reduced state  $\rho_n^{A_n} = \text{Tr}_A \rho_n$  has a high fidelity with the maximally coherent state,

$$\langle \Psi_{2^{nR}} | \rho_n^{A_n} | \Psi_{2^{nR}} \rangle \geq 1 - \epsilon.$$

The maximum number  $R$  such that there exist coherence-generating protocols for all  $n$ , with the error going to 0 and rates converging to  $R$ , is called the *coherence-generating capacity* of  $T$  and denoted  $C_{\text{gen}}(T)$ .

*Theorem 1.* For a general CPTP map  $T : A \rightarrow B$ ,

$$C_{\text{gen}}(T) \geq \sup_{|\varphi\rangle \in A \otimes C} C_r((T \otimes \text{id})\varphi) - C(\varphi), \quad (4)$$

where the supremum over all auxiliary systems  $C$  and pure states  $|\varphi\rangle \in A \otimes C$ . Furthermore,

$$C_{\text{gen}}(T) \leq \sup_{\rho \text{ on } A \otimes C} C_r((T \otimes \text{id})\rho) - C_r(\rho), \quad (5)$$

where now the supremum is over mixed states  $\rho$  in  $A \otimes C$ .

If  $T$  is an isometry, i.e.,  $T(\rho) = V\rho V^\dagger$  for an isometry  $V : A \hookrightarrow B$ , the lower bound is an equality and can be simplified:

$$\begin{aligned} C_{\text{gen}}(V \cdot V^\dagger) &= \sup_{|\varphi\rangle \in A \otimes C} C((V \otimes \mathbb{1})\varphi(V \otimes \mathbb{1})^\dagger) - C(\varphi) \\ &= \max_{|\varphi\rangle \in A} C(V\varphi V^\dagger) - C(\varphi). \end{aligned} \quad (6)$$

This result, the main one in the present section, should be compared to the formula, similar in spirit, for the entangling power of a bipartite unitary [21,22]. Furthermore, the above formulas for the coherence-generating capacity are related to the *coherence power* (with regard to the relative entropy measure)

$$P_r(T) = \max_{\rho \text{ on } A} C_r(T(\rho)) - C_r(\rho), \quad (7)$$

investigated by García Díaz *et al.* [23] and Bu *et al.* [24]. Let us also introduce the same maximization restricted to pure input states,

$$\tilde{P}_r(T) = \max_{|\varphi\rangle \in A} C_r(T(\varphi)) - C(\varphi). \quad (8)$$

Note that the only difference from our formulas is that we allow an ancillary system  $C$  of arbitrary dimension. If we consider, for a general CPTP map  $T$ , the extension  $T \otimes \text{id}_k$  and

$$P_r^{(k)}(T) := P_r(T \otimes \text{id}_k), \quad \tilde{P}_r^{(k)}(T) := \tilde{P}_r(T \otimes \text{id}_k), \quad (9)$$

then we have

$$C_{\text{gen}}(V \cdot V^\dagger) = \sup_k \tilde{P}_r^{(k)}(V \cdot V^\dagger) = \tilde{P}_r(V \cdot V^\dagger), \quad (10)$$

and in general,

$$\sup_k \tilde{P}_r^{(k)}(T) \leq C_{\text{gen}}(T) \leq \sup_k P_r^{(k)}(T). \quad (11)$$

*Proof.* We start with the lower bound, Eq. (4): For a given ancilla  $C$  and  $|\varphi\rangle \in A \otimes C$ , let  $R = C_r((T \otimes \text{id})\varphi) - C(\varphi)$ . For any  $\epsilon, \delta > 0$ , we can choose, by the results in [17], a sufficiently large  $n$  such that

$$\begin{aligned} \Psi_2^{\otimes [nC(\varphi)+n\delta]} &\xrightarrow{\text{IO}} \approx \varphi^{\otimes n}, \\ \rho^{\otimes n} &\xrightarrow{\text{IO}} \approx \Psi_2^{\otimes [nC_r(\rho)-n\delta]}, \end{aligned}$$

with  $\rho = (T \otimes \text{id})\varphi$ , and where  $\approx$  refers to approximation of the target state up to  $\epsilon$  in the trace norm. We only have to prove something when  $R > 0$ , which can only arise if  $T$  is not incoherent, meaning that there exists an initial state  $|0\rangle$  mapped to a coherent resource  $\sigma = T(|0\rangle\langle 0|)$ , i.e.,  $C_r(\sigma) > 0$ . In the following, assume that  $R > 2\delta$ . Now, we may assume that  $n$  is large enough so that with  $R_0 = \frac{C(\varphi)+\delta}{C_r(\sigma)} + \delta$ ,

$$\sigma^{\otimes [nR_0]} \xrightarrow{\text{IO}} \approx \Psi_2^{\otimes [nC(\varphi)+n\delta]}.$$

The protocol consists of the following steps.

Step 0: Use  $[nR_0]$  instances of  $T$  to create as many copies of  $\sigma$  and convert them into  $\Psi_2^{\otimes [nC(\varphi)+n\delta]}$  (up to the trace norm  $\epsilon$ ).

Steps 1– $k$ : First, convert  $[nC(\varphi) + n\delta]$  of the already created copies of  $\Psi_2$  into  $n$  copies of  $\varphi$ ; then apply  $T$  to each of them to obtain  $\rho = (T \otimes \text{id})\varphi$  and convert the  $n$  copies of  $\rho$  to  $\Psi_2^{\otimes [nC_r(\rho)-n\delta]}$ , incurring an error of  $2\epsilon$  in the trace norm in each repetition. Repeat.

At the end of the protocol, we have  $(k-1)n(R-2\delta) + nC(\varphi)$  copies of  $\Psi_2$ , up to trace distance  $O(k^2\epsilon)$ , using the channel a total of  $kn + nR_0$  times; i.e., the rate is  $\geq (R-2\delta)\frac{k-1}{k+R_0}$ , which can be made arbitrarily close to  $R$  by choosing  $\delta$  small enough and  $k$  large enough (which in turn can be effected by sufficiently small  $\epsilon$ ).

For the upper bound, Eq. (5), consider a generic protocol using the channel  $n$  times, starting from  $\rho_0$  (incoherent) and generating  $\rho_1, \dots, \rho_n$  step by step along the way, such that  $\rho_n$  has fidelity  $\geq 1 - \epsilon$  with  $\Psi_2^{\otimes nR}$ . By the asymptotic continuity of  $C_r$  [17, Lemma 12],  $C_r(\rho_n) \geq nR - 2\delta n - 2$ , with  $\delta = \sqrt{\epsilon(2-\epsilon)}$ , so we can bound

$$\begin{aligned} nR - 2\delta n - 2 &\leq C_r(\rho_n) = \sum_{t=0}^{n-1} C_r(\rho_{t+1}) - C_r(\rho_t) \\ &\leq \sum_{t=1}^{n-1} C_r((T \otimes \text{id})\rho_t) - C_r(\rho_t), \end{aligned}$$

where we have used the fact that  $\rho_0$  is incoherent and that  $\rho_{t+1} = \mathcal{I}_{t+1}((T \otimes \text{id})\rho_t)$ , with an incoherent operation  $\mathcal{I}_{t+1}$ , which can only decrease the relative entropy of coherence. However, each term in the right-hand sum is of the form  $C_r((T \otimes \text{id})\rho) - C_r(\rho)$  for a suitable ancilla  $C$  and a state  $\rho$  in  $A \otimes C$ . Thus, dividing by  $n$  and letting  $n \rightarrow \infty$ ,  $\epsilon \rightarrow 0$  shows that  $R \leq \sup_{\rho \text{ on } A \otimes C} C_r((T \otimes \text{id})\rho) - C_r(\rho)$ .

For an isometric channel  $T(\rho) = V\rho V^\dagger$ , note that the initial state  $\rho_0$  in a general protocol is, without loss of generality, pure and that  $T$  maps pure states to pure states. The incoherent operations  $\mathcal{I}_t$  map pure states to *ensembles* of pure states, so that following the same converse reasoning as above, we end up upper bounding  $R$  by an average of the expressions

$C_r((T \otimes \text{id})\rho) - C_r(\rho)$ , with pure states  $\rho$ , i.e., Eq. (6), since we also have  $C_{\text{gen}}(T) \geq C((T \otimes \text{id})\varphi) - C(\varphi)$  from the other direction. The fact that no ancillary system is needed is an elementary calculation. Indeed, for a pure state  $|\varphi\rangle \in A \otimes C$ ,

$$\begin{aligned} C((T \otimes \text{id})\varphi) - C(\varphi) &= S((\Delta \circ T \otimes \Delta)\varphi) - S((\Delta \otimes \Delta)\varphi) \\ &= S((\Delta \circ T \otimes \text{id})\rho) - S((\Delta \otimes \text{id})\rho) \\ &\quad \left[ \text{with } \rho = (\text{id} \otimes \Delta)\varphi = \sum_i p_i \varphi_i \otimes |i\rangle\langle i| \right] \\ &= \sum_i p_i (S(\Delta(T(\varphi_i))) - S(\Delta(\varphi_i))) \\ &\leq \max_{|\varphi\rangle \in A} S(\Delta(T(\varphi))) - S(\Delta(\varphi)), \end{aligned}$$

and we are done.  $\blacksquare$

*Remark.* We do not know, at this point, whether the suprema over the ancillary systems in the upper and lower bounds in Eq. (11) are necessary in general, i.e., it might be that  $P_r(T) = P_r^{(k)}$  and/or  $\tilde{P}_r(T) = \tilde{P}_r^{(k)}$ ; note that the latter is the case for unitaries, even though it seems unlikely in general (cf. [21] and [22]). We do know, however, that  $P_r$  is convex and nonincreasing under the composition of the channel with IOs [23, Corollaries 1 and 2].

It should be appreciated that even the calculation of  $P_r(T)$  appears to be a hard problem. The investigation of further questions, such as the additivity of  $P_r$ ,  $\tilde{P}_r$ , or  $C_{\text{gen}}$ , may depend on making progress on that problem.

*Remark.* The same reasoning as used in the proof of Theorem 1, replacing  $C_r$  with  $C_f$ , shows that

$$\begin{aligned} C_{\text{gen}}(T) &\leq \sup_{\rho \text{ on } A \otimes C} C_f((T \otimes \text{id})\rho) - C_f(\rho) \\ &= \sup_k P_f(T \otimes \text{id}_k), \end{aligned} \quad (12)$$

with the coherence of formation power given by  $P_f(T) := \max_{\rho} C_f(T(\rho)) - C_f(\rho)$ . Despite the fact that  $C_f(\rho) \geq C_r(\rho)$ , since the upper bound is given by a difference of two coherence measures, it might be that for certain channels, bound, (12), is better than bound (5), and vice versa for others.

Since the suprema over  $k$  of the coherence powers of  $T \otimes \text{id}_k$  play such an important role in our bounds, we introduce notation for them,

$$\mathbb{P}_r(T) := \sup_k P_r(T \otimes \text{id}_k), \quad (13)$$

$$\mathbb{P}_f(T) := \sup_k P_f(T \otimes \text{id}_k), \quad (14)$$

$$\tilde{\mathbb{P}}_r(T) := \sup_k \tilde{P}_r(T \otimes \text{id}_k), \quad (15)$$

and call them the *complete coherence powers* with respect to the relative entropy of coherence and coherence of formation, respectively. For these parameters, and the coherence generation capacity of isometries, we note the following additivity formulas.

*Proposition 2.* For any CPTP maps  $T_1 : A_1 \longrightarrow B_1$  and  $T_2 : A_2 \longrightarrow B_2$ ,

$$\begin{aligned}\mathbb{P}_r(T_1 \otimes T_2) &= \mathbb{P}_r(T_1) + \mathbb{P}_r(T_2), \\ \mathbb{P}_f(T_1 \otimes T_2) &= \mathbb{P}_f(T_1) + \mathbb{P}_f(T_2).\end{aligned}$$

Furthermore, for isometries  $T_i(\rho) = V_i \rho V_i^\dagger$ ,

$$\tilde{\mathbb{P}}_r(T_i) = \tilde{P}_r(T_i) = C_{\text{gen}}(T_i)$$

and

$$\tilde{\mathbb{P}}_r(T_1 \otimes T_2) = \tilde{P}_r(T_1) + \tilde{P}_r(T_2).$$

In other words, the coherence-generating capacity of isometries is additive.

*Proof.* For  $X \in \{r, f\}$ , any ancillary system  $C$ , and any state  $\rho$ ,

$$\begin{aligned}C_X((T_1 \otimes T_2 \otimes \text{id}_C)\rho) - C_X(\rho) &= C_X((T_1 \otimes T_2 \otimes \text{id}_C)\rho) - C_X((T_1 \otimes \text{id}_{B_2} \otimes \text{id}_C)\rho) \\ &\quad + C_X((T_1 \otimes \text{id}_{B_2} \otimes \text{id}_C)\rho) - C_X(\rho) \\ &= C_X((T_2 \otimes \text{id}_{A_1, C})\sigma) - C_X(\sigma) \\ &\quad + C_X((T_1 \otimes \text{id}_{B_2, C})\rho) - C_X(\rho) \\ &\leq \mathbb{P}_X(T_2) + \mathbb{P}_X(T_1),\end{aligned}$$

where we have introduced the state  $\sigma = (T_1 \otimes \text{id}_{B_2} \otimes \text{id}_C)\rho$ . By taking the supremum of the left-hand side over all ancillae  $C$  and all states  $\rho$ , we obtain  $\mathbb{P}_X(T_1 \otimes T_2) \leq \mathbb{P}_X(T_1) + \mathbb{P}_X(T_2)$ ; since the opposite inequality is trivial, using tensor product ancillae and tensor product input states and employing the additivity of  $C_r$  and  $C_f$  [17], we have proved the equality.

In the case of isometries, Eq. (6) in Theorem 1 already shows that  $\tilde{\mathbb{P}}_r(T_i) = \tilde{P}_r(T_i) = C_{\text{gen}}(T_i)$ . For the tensor product, we again have trivially  $\tilde{P}_r(T_1 \otimes T_2) \geq \tilde{P}_r(T_1) + \tilde{P}_r(T_2)$ , by using tensor product input states. To get the opposite inequality, we proceed as above: For any pure state  $|\varphi\rangle \in A_1 \otimes A_2$ ,

$$\begin{aligned}C((T_1 \otimes T_2)\varphi) - C(\varphi) &= C((T_1 \otimes T_2)\varphi) - C((T_1 \otimes \text{id}_{B_2})\varphi) \\ &\quad + C((T_1 \otimes \text{id}_{B_2})\varphi) - C(\varphi) \\ &= C((\text{id}_{A_1} \otimes T_2)\psi) - C(\psi) \\ &\quad + C((T_1 \otimes \text{id}_{B_2})\varphi) - C(\varphi) \\ &\leq \tilde{\mathbb{P}}_r(T_2) + \tilde{\mathbb{P}}_r(T_1) = \tilde{P}_r(T_2) + \tilde{P}_r(T_1),\end{aligned}$$

with the (pure) state  $\psi = (T_1 \otimes \text{id}_{B_2})\varphi$ , and we are done. ■

### III. IMPLEMENTATION OF CHANNELS: COHERENCE COST OF SIMULATION

We have seen that a CPTP map can be a resource for coherence because one can use it to generate coherence from scratch, in the form of maximally coherent qubit states. True to the resource paradigm, we immediately have to ask the opposite question: Is it possible to create the resource using pure coherent states and only incoherent operations? Here we show that the answer is generally yes, and we define the asymptotic coherence cost  $C_{\text{sim}}(T)$  as the minimum

rate of pure-state coherence necessary to implement many independent instances of  $T$  using only IOs otherwise.

We start by recalling the implementation of an arbitrary unitary operation  $U = \sum_{i,j=0}^{d-1} U_{ij}|i\rangle\langle j|$  by means of an incoherent operation with Kraus operators  $\{K_\alpha\}$  and using the maximally coherent state  $|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$  as resource [25, lemma 2]. Since it is important for us that the implementation is in fact by means of a *strictly* incoherent operation, and for self-containedness, we include the proof.

*Proposition 3 (Chitambar/Hsieh [25]).* Consuming one copy of  $\Psi_d$  as a resource, any unitary  $U = \sum_{i,j=0}^{d-1} U_{ij}|i\rangle\langle j|$  acting on  $\mathbb{C}^d$  can be simulated using strictly incoherent operations.

*Proof.* We essentially follow [25]; see also [13] for the qubit case. Let the Kraus operators have the following form:

$$K_\alpha = \sum_{i,j=0}^{d-1} U_{ij}|i\rangle\langle j| \otimes |\alpha\rangle\langle i + \alpha \bmod d|.$$

It is easy to see that these Kraus operators satisfy  $K_\alpha \Delta K_\alpha^\dagger \subset \Delta$ , and that they are indeed strictly incoherent, for all  $\alpha = 0, 1, \dots, d-1$ . Furthermore, with the notation  $q = i + \alpha \bmod d$ ,

$$\begin{aligned}\sum_{\alpha=0}^{d-1} K_\alpha^\dagger K_\alpha &= \sum_{\alpha=0}^{d-1} \sum_{i,j,k,l=0}^{d-1} U_{ij} U_{kl}^* |jq\rangle\langle i\alpha| |k\alpha\rangle\langle lp| \\ &= \sum_{\alpha=0}^{d-1} \sum_{i,j,k,l=0}^{d-1} U_{ij} U_{kl}^* \delta_{i,k} |jq\rangle\langle lp| \\ &= \sum_{\alpha=0}^{d-1} \sum_{i,j,l=0}^{d-1} U_{ij} U_{il}^* |jq\rangle\langle lq| \\ &= \sum_{\alpha=0}^{d-1} \sum_{j,l=0}^{d-1} \delta_{j,l} |jq\rangle\langle lq| = \mathbb{1}.\end{aligned}$$

Now, let  $|\phi\rangle = \sum_{k=0}^{d-1} \phi_k |k\rangle$ ; then

$$K_\alpha(|\phi\rangle \otimes |\Psi_d\rangle) = \frac{1}{\sqrt{d}} U |\phi\rangle \otimes |\alpha\rangle.$$

Thus, under this incoherent operation, and tracing out the ancilla afterwards, the system will be in the desired state  $U|\phi\rangle\langle\phi|U^\dagger$  with certainty. ■

Now we pass to the general case of CPTP maps, which extends the above result for unitaries, with two different protocols.

*Theorem 4.* Any CPTP map  $T : A \longrightarrow B$  can be implemented by incoherent operations, using a maximally coherent resource state  $\Psi_d$ , where  $d = |B|$ .

*Proof.* Let  $T(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$  be a Kraus decomposition of  $T$ , with Kraus operators  $K_\alpha : A \longrightarrow B$ . The idea of the simulation is to use teleportation of the output of  $T$ , which involves a maximally entangled state  $\Phi_D$  in  $B' \otimes B''$ , a Bell measurement on system  $B \otimes B'$  with outcomes  $jk \in \{0, 1, \dots, d-1\}^2$ , and unitaries  $U_{jk}$  in  $B''$ . The unitaries  $U_{jk}$  can be written as  $U_{jk} = Z^j X^k$ , with the phase and cyclic

shift unitaries

$$Z = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{d-1} \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

where  $\omega = e^{\frac{2\pi i}{d}}$  is the  $d$ th root of unity. This scheme can be reduced to a destructive (hence incoherent) positive operator-valued measure (POVM) in  $A \otimes B'$  with outcomes  $jk\alpha$ , followed by the application of the incoherent(!)  $U_{jk}$ . In detail, the probability of getting outcome  $jk$  is

$$\Pr\{jk \mid \sigma\} = \text{Tr} \Phi^{(jk)} (K_\alpha \otimes \mathbb{1}) \sigma (K_\alpha \otimes \mathbb{1})^\dagger,$$

where  $\sigma$  is a state in  $A \otimes B'$  and the

$$|\Phi^{(jk)}\rangle = (\mathbb{1} \otimes Z^j X^k) |\Phi_d\rangle$$

are the Bell states. We can define the POVM elements  $M_{jk\alpha} = (K_\alpha \otimes \mathbb{1})^\dagger \Phi^{(jk)} (K_\alpha \otimes \mathbb{1})$ , so that

$$\begin{aligned} \text{Tr}((T \otimes \text{id})\sigma) \Phi^{(jk)} &= \sum_\alpha \text{Tr} (K_\alpha \otimes \mathbb{1}) \sigma (K_\alpha \otimes \mathbb{1})^\dagger \Phi^{(jk)} \\ &= \sum_\alpha \text{Tr} \sigma (K_\alpha^\dagger \otimes \mathbb{1}) \Phi^{(jk)} (K_\alpha \otimes \mathbb{1}) \\ &= \text{Tr} \left[ \sigma \left( \sum_\alpha (K_\alpha^\dagger \otimes \mathbb{1}) \Phi^{(jk)} (K_\alpha \otimes \mathbb{1}) \right) \right] \\ &= \text{Tr} \left( \sum_\alpha \sigma M_{jk\alpha} \right) = \text{Tr} \sigma M_{jk}, \end{aligned}$$

with  $M_{jk} = \sum_\alpha M_{jk\alpha}$ . This leads to a new equivalent scheme in which, given a state  $\rho$  in  $A$  and a maximally entangled state in  $B' \otimes B''$ , we can apply the measurement  $M_{jk}$  to  $A \otimes B'$  with outcomes  $jk$  and unitaries  $U_{jk}$  acting on  $B''$ . Formally, let us define the Kraus operators of the protocol by letting

$$L_{jk\alpha} := [(\Phi^{(jk)} | (K_\alpha \otimes \mathbb{1})]^{AB'} \otimes U_{jk}^{B''}. \quad (16)$$

It can be checked readily that they satisfy the normalization condition

$$\sum_{jk\alpha} L_{jk\alpha}^\dagger L_{jk\alpha} = \sum_{jk\alpha} M_{jk\alpha} \otimes \mathbb{1} = \mathbb{1} \otimes \mathbb{1}.$$

Applying the Kraus operators  $L_{jk\alpha}$  throughout the system we get

$$\begin{aligned} L_{jk\alpha} |\phi\rangle^A |\Psi_d\rangle^{B'B''} &= \langle \Phi^{(jk)} | (K_\alpha | \phi) \rangle (\mathbb{1} \otimes U_{jk}) |\Psi_d\rangle \\ &= \langle \Phi^{(jk)} | K_\alpha | \phi \rangle |\Psi^{(jk)}\rangle \\ &= \frac{1}{d} K_\alpha | \phi \rangle. \end{aligned}$$

Hence,  $\sum_{jk\alpha} L_{jk\alpha} (\rho \otimes \Phi_d) L_{jk\alpha}^\dagger = \sum_\alpha K_\alpha \rho K_\alpha^\dagger = T(\rho)$ , and the proof is complete.  $\blacksquare$

*Theorem 5.* Any CPTP map  $T : A \rightarrow B$  can be implemented by strictly incoherent operations and a maximally coherent state  $\Psi_d$ , where  $d \leq |A||B|$ .

*Proof.* The channel  $T$  is, first, a convex combination of extremal CPTP maps  $T_\lambda$ , each of which has at most  $|A|$  Kraus operators [26]:  $T = \sum_\lambda p_\lambda T_\lambda$ . Clearly, we only have to prove the claim for the  $T_\lambda$ . Because of the bound on the Kraus operators, each  $T_\lambda$  has a unitary dilation  $U_\lambda : A \otimes B \rightarrow B \otimes A$ , such that  $T_\lambda(\rho) = \text{Tr}_A U_\lambda(\rho \otimes |0\rangle\langle 0|_B) U_\lambda^\dagger$ , with a fixed incoherent state  $|0\rangle \in B$ .

As the  $U_\lambda$  act on a space of dimension  $d = |A||B|$ , we can invoke the simulation according to Proposition 3.  $\blacksquare$

*Remark.* Comparing Theorems 4 and 5, we note that the latter always consumes more resources, but it is guaranteed to be implemented by strictly incoherent operations, a much narrower class than the incoherent operations.

We leave it an open question whether the resource consumption in Theorem 4 can be achieved by strictly incoherent operations or whether there is a performance gap between incoherent and strictly incoherent operations. Note that these two classes, incoherent operations and strictly incoherent operations, are distinct as sets of CPTP maps, although it is known that they induce the same possible state transformations of a given state into a target state for qubits [27] and for pure states in arbitrary dimension [28] (correcting the earlier erroneous proof of the claim in [19]; the SIO part of the pure-state transformations is due to [17]). However, for the distillation of pure coherence at rate  $C_r(\rho)$  [17], IOs are needed and it remains unknown whether SIOs can attain the same rate. Crucially, any destructive measurement, i.e., any POVM followed by an incoherent state preparation, is an IO, but the only measurements allowed under SIOs are of diagonal, i.e., incoherent, observables.

The results so far have been about the resources required for the exact implementation of a single instance of a channel, in the worst case. It is intuitively clear that some channels are easier to implement in the sense that fewer resources are needed; e.g., for the identity of any incoherent channel, no coherent resource is required.

In the spirit of the previous section, we are interested in the minimum resources required to implement many independent instances of  $T$ .

*Definition 6.* An  $n$ -block incoherent simulation of a channel  $T : A \rightarrow B$  with error  $\epsilon$  and coherent resource  $\Psi_d$  (in space  $D$ ) is an incoherent operation  $\mathcal{I} : A^n \otimes D \rightarrow B^n$ , such that  $T'(\rho) := \mathcal{I}(\rho \otimes \Psi_d)$  satisfies

$$\begin{aligned} \epsilon &\geq \|T' - T^{\otimes n}\|_\diamond \\ &= \sup_{|\phi\rangle \in A^n \otimes C} \|(T' \otimes \text{id}_C) \phi - (T^{\otimes n} \otimes \text{id}_C) \phi\|_1. \end{aligned}$$

Here,  $C$  is an arbitrary ancillary system; the error criterion of the simulation is known as the diamond norm [29] or completely bounded trace norm [30]; see also [31].

The rate of the simulation is  $\frac{1}{n} \log_2 d$ , and the simulation cost of  $T$ , denoted  $C_{\text{sim}}(T)$ , is the smallest  $R$  such that there exist  $n$ -block incoherent simulations with the error going to 0 and the rate going to  $R$  as  $n \rightarrow \infty$ .

The best general bounds we have on the simulation cost are contained in the following theorem.

*Theorem 7.* For any CPTP map  $T : A \longrightarrow B$ ,

$$C_{\text{gen}}(T) \leq C_{\text{sim}}(T) \leq \log_2 |B|. \quad (17)$$

Furthermore,

$$C_{\text{sim}}(T) \geq \max\{\sup_k P_r(T \otimes \text{id}_k), \sup_k P_f(T \otimes \text{id}_k)\}, \quad (18)$$

where we recall the definitions of the relative entropy coherence power,  $P_r(T) = \max_\rho C_r(T(\rho)) - C_r(\rho)$ , and of the coherence of formation power,  $P_f(T) = \max_\rho C_f(T(\rho)) - C_f(\rho)$ .

*Proof.* We start with Eq. (17): The upper bound is a direct consequence of Theorem 4. The lower bound follows from the fact that  $T$  is implemented using maximally coherent states at rate  $R = C_{\text{sim}}(T)$  and incoherent operations. The generation of entanglement, on the other hand, uses  $T$  and some more incoherent operations. Since incoherent operations cannot increase the amount of entanglement, the overall process of simulation and generation cannot result in a rate of coherence of more than  $R$ .

Regarding Eq. (18), the idea is that for  $\epsilon > 0$  and  $n$  large enough, since the simulation implements a CPTP map  $T'$  that is within the diamond norm  $\epsilon$  from  $T^{\otimes n}$ , using incoherent operations and  $\Psi_2^{\otimes n(R+\epsilon)}$  as a resource. Applying the simulation to state  $\rho^{\otimes n}$  results in  $(T' \otimes \text{id}_{k^n})\rho^{\otimes n} \approx ((T \otimes \text{id}_k)\rho)^{\otimes n}$ , hence we have an overall incoherent operation:

$$\Psi_2^{\otimes n(R+\epsilon)} \otimes \rho^{\otimes n} \xrightarrow{\text{IO}} (T' \otimes \text{id}_{k^n})\rho^{\otimes n}.$$

By monotonicity of  $C_X$  ( $X \in \{r, f\}$ ) under the IO and  $C_X(\Psi_2) = 1$ , this means that

$$n(R + \epsilon) \geq C_X((T' \otimes \text{id}_{k^n})\rho^{\otimes n}) - C_X(\rho^{\otimes n}),$$

where we have used the additivity of  $C_r$  and  $C_f$  [17]. Since this holds for all  $\rho$ , we obtain

$$\begin{aligned} n(R + \epsilon) &\geq P_X(T' \otimes \text{id}_{k^n}) \\ &\geq P_X(T^{\otimes n} \otimes \text{id}_{k^n}) - n\kappa_X\epsilon - 4 \\ &\geq nP_X(T \otimes \text{id}_k) - n\kappa_X\epsilon - 2, \end{aligned}$$

invoking in the second line Lemma 8 below, with  $\kappa_r = 4 \log_2 |B|$  and  $\kappa_f = \log_2 |B| + \log_2 k$ , and in the third line a tensor-power test state. Since  $\epsilon$  can be made arbitrarily small, and  $n$  as well as  $k$  arbitrarily large, the claim follows. ■

Here we state the technical lemma required in the proof of Theorem 7.

*Lemma 8.* The relative entropy coherence power and the coherence of formation power are asymptotically continuous with respect to the diamond norm metric in channels. To be precise, for  $T_1, T_2 : A \longrightarrow B$  with  $\frac{1}{2}\|T_1 - T_2\|_\diamond \leq \epsilon$ ,

$$|P_r(T_1 \otimes \text{id}_k) - P_r(T_2 \otimes \text{id}_k)| \leq 4\epsilon \log_2 |B| + 2g(\epsilon), \quad (19)$$

$$|P_f(T_1 \otimes \text{id}_k) - P_f(T_2 \otimes \text{id}_k)| \leq \epsilon(\log_2 |B| + \log_2 k) + g(\epsilon), \quad (20)$$

where  $g(x) = (1+x)h_2(\frac{x}{1+x}) = (1+x)\log_2(1+x) - x\log_2 x$ .

*Proof.* For the first bound, observe

$$\begin{aligned} &|P_r(T_1 \otimes \text{id}_k) - P_r(T_2 \otimes \text{id}_k)| \\ &\leq \max_{\rho^{AC}} |C_r((T_1 \otimes \text{id}_k)\rho) - C_r((T_2 \otimes \text{id}_k)\rho)| \\ &= \max_{\rho^{AC}} |S(BC)_{(\Delta T_1 \otimes \Delta)\rho} - S(BC)_{(T_1 \otimes \text{id}_k)\rho} \\ &\quad - S(BC)_{(\Delta T_2 \otimes \Delta)\rho} + S(BC)_{(T_2 \otimes \text{id}_k)\rho}| \\ &= \max_{\rho^{AC}} |S(B|C)_{(\Delta T_1 \otimes \Delta)\rho} - S(B|C)_{(T_1 \otimes \text{id}_k)\rho} \\ &\quad - S(B|C)_{(\Delta T_2 \otimes \Delta)\rho} + S(B|C)_{(T_2 \otimes \text{id}_k)\rho}| \\ &\leq \max_{\rho^{AC}} |S(B|C)_{(\Delta T_1 \otimes \Delta)\rho} - S(B|C)_{(\Delta T_2 \otimes \Delta)\rho}| \\ &\quad + |S(B|C)_{(T_1 \otimes \text{id}_k)\rho} - S(B|C)_{(T_2 \otimes \text{id}_k)\rho}| \\ &\leq 2(2\epsilon \log_2 |B| + g(\epsilon)), \end{aligned}$$

where in the first line we insert the same variable  $\rho^{AC}$  to maximize  $P_r(T_j \otimes \text{id}_k)$  and note that, in this case, the term  $C_r(\rho)$  cancels; then, in the second line, we use the definition of the relative entropy of coherence, and in the third we use the chain rule  $S(BC) = S(B|C) + S(C)$  for the entropy, allowing us to cancel matching  $S(C)$  terms; in the fourth line we invoke the triangle inequality and, finally, the Alicki-Fannes bound for the conditional entropy [32] in the form given in [33, Lemma 2].

For the second bound, we start very similarly:

$$\begin{aligned} &|P_f(T_1 \otimes \text{id}_k) - P_f(T_2 \otimes \text{id}_k)| \\ &\leq \max_{\rho^{AC}} |C_f((T_1 \otimes \text{id}_k)\rho) - C_f((T_2 \otimes \text{id}_k)\rho)| \\ &\leq \epsilon(\log_2 |B| + \log_2 k) + g(\epsilon), \end{aligned}$$

where the last line comes directly from the asymptotic continuity of the coherence of formation [17, Lemma 15]. We close the proof expressing our belief that it is possible to prove a version of Eq. (20) where  $k$  does not appear on the right-hand side. ■

*Remark.* As a consequence, while for a channel  $T$  that is close to an incoherent operation (in the diamond norm), or in fact close to a maximally incoherent operation (MIO), the coherence-generating capacity  $C_{\text{gen}}(T)$  is also close to 0, we do not know at the moment whether the same holds for the simulation cost  $C_{\text{sim}}(T)$ .

#### IV. QUBIT UNITARIES

In this section, we take a closer look at qubit unitaries, for which we would like to find the coherence-generating capacity and simulation cost. To start our analysis, we note that a general  $2 \times 2$  unitary has four real parameters, but we can transform unitaries into each other at no cost by preceding or following them with incoherent unitaries, i.e., combinations of the bit flip  $\sigma_x$  and diagonal (phase) unitaries  $\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}$ . This implies an equivalence relation among qubit unitaries up to incoherent unitaries. A unique representative of each equivalence class is given by

$$U = U(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad (21)$$

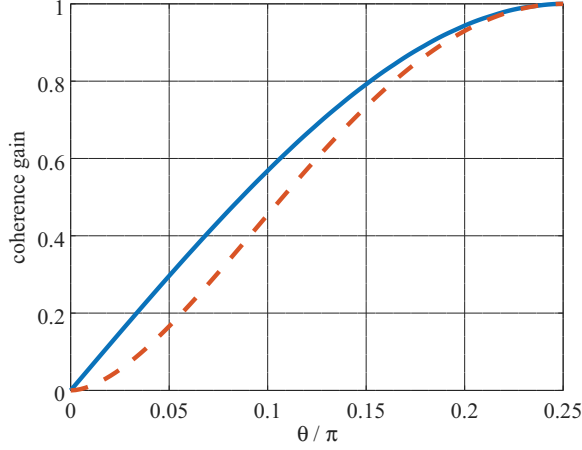


FIG. 1. Plot of  $C_{\text{gen}}(U(\theta)) = \tilde{P}_r(U(\theta))$  as a function of  $\theta \in [0, \frac{\pi}{4}]$  (solid blue line) and comparison with  $h_2(\cos^2 \theta)$  (dashed red line), which is the coherence generated by an incoherent input state. In particular, for  $\theta \approx 0$ , the ratio between the two functions is unbounded. The angle  $\theta$  is plotted as a fraction of  $\pi$ .

where  $c = \cos \theta$  and  $s = \sin \theta$  and with  $0 \leq \theta \leq \frac{\pi}{4}$ , so that  $c \geq s \geq 0$ .

One can calculate  $C_{\text{gen}}(U(\theta))$  using the formula from Theorem 1. Clearly, by choosing the test state  $\varphi$  to be pure incoherent,

$$C_{\text{gen}}(U(\theta)) = \tilde{P}_r(U(\theta)) \geq h_2(c^2) = -c^2 \log_2 c^2 - s^2 \log_2 s^2, \quad (22)$$

with  $h_2(x) = -x \log_2 x - (1-x) \log_2 (1-x)$  the binary entropy. Perhaps surprisingly, however, this is in general not the optimal state [34, corollary 5] (see also [23]), meaning that  $\tilde{P}_r(U(\theta))$  is attained in a coherent test state  $\varphi$ , although no closed-form expression seems to be known. In fact, simple manipulations show that we only need to optimize  $C(U(\theta)\varphi U(\theta)^\dagger) - C(\varphi)$  over states  $|\varphi\rangle = U(\alpha)|0\rangle = \cos \alpha|0\rangle + \sin \alpha|1\rangle$ ,  $0 \leq \alpha \leq \pi$  (i.e., no phases are necessary). The function to optimize becomes  $h_2(\cos^2(\alpha + \theta)) - h_2(\cos^2 \alpha)$ . Its critical points satisfy the transcendental equation

$$\sin(2\alpha + 2\theta) \ln \tan^2(\alpha + \theta) = \sin(2\alpha) \ln \tan^2 \alpha, \quad (23)$$

which can be solved numerically. Figure 1 shows that  $C_{\text{gen}}(U(\theta)) = \tilde{P}_r(U(\theta)) > h_2(\cos^2 \theta)$  across the whole interval, except at the end points  $\theta = 0, \frac{\pi}{4}$ ; in Fig. 2 we plot the optimal  $\alpha$  for  $U(\theta)$ .

On the other hand, regarding the implementation of these unitary channels, all we can say for the moment is that  $C_{\text{sim}}(U(\theta)) \leq 1$ , because we can implement each instance of the qubit unitary using a qubit maximally coherent state  $\Psi_2$ . It is perhaps natural to expect that one could get away with a smaller amount of coherence, but it turns out that with a two-dimensional resource state this is impossible.

**Proposition 9.** The only qubit coherent resource state  $|r\rangle \in \mathbb{C}^2$  that permits the implementation of  $U(\theta)$ ,  $0 < \theta \leq \frac{\pi}{4}$ , is the maximally coherent state.

Furthermore, any two-qubit incoherent operation  $\mathcal{I}$  such that  $\mathcal{I}(\rho \otimes |r\rangle\langle r|) = U(\theta)\rho U(\theta)^\dagger \otimes \sigma$  for general  $\rho$  is such

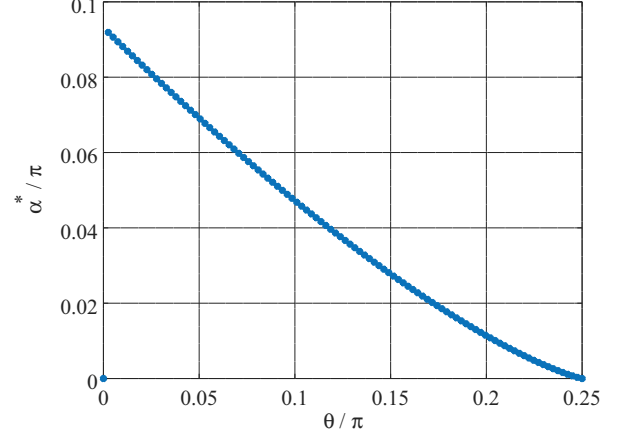


FIG. 2. The optimal value of  $\alpha$  attaining  $C_{\text{gen}}(U(\theta)) = \tilde{P}_r(U(\theta)) = h_2(\cos^2(\alpha + \theta)) - h_2(\cos^2 \alpha)$  as a function of  $\theta \in [0, \frac{\pi}{4}]$ . It is nonzero except at the end points  $\theta = 0, \frac{\pi}{4}$ . Both angles,  $\theta$  and  $\alpha^*$ , are plotted as fractions of  $\pi$ .

that the state  $\sigma$  left behind in the ancilla is necessarily incoherent.

*Proof.* We want to know for which state  $|r\rangle = c'|0\rangle + s'|1\rangle$  the transformation  $|\psi\rangle|r\rangle \xrightarrow{U} (U(\theta)|\psi\rangle)|0\rangle$  is possible, for a general state  $|\psi\rangle$ . Without loss of generality, the incoherent Kraus operators achieving the transformation have the general form

$$K = \lambda(U(\theta) \otimes |0\rangle\langle r| + R \otimes |0\rangle\langle r^\perp|), \quad (24)$$

where  $|r^\perp\rangle = s'|0\rangle - c'|1\rangle$  is the vector orthogonal to  $|r\rangle$ . We now need to find the form of  $R$  such that  $K$  is incoherent. For this, we impose incoherence of  $K$  when tracing out the ancillary part,  $\langle 0|^A K |0\rangle^A =: T_0$  and  $\langle 1|^A K |1\rangle^A =: T_1$ , where  $T_0$  and  $T_1$  must be two-dimensional incoherent operators. We then obtain that  $R = s'T_0 - c'T_1$  and  $\lambda U = c'T_0 + s'T_1$ . The latter condition enforces either that  $T_0 \propto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $T_1 \propto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , or vice versa, or that  $T_0 \propto \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$  and  $T_1 \propto \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ , or vice versa. From these possibilities, we get four possible  $R$  matrices, which define four different Kraus operators  $K_i$  defined, according to Eq. (24), by  $R_i$  matrices as

$$\begin{aligned} R_1 &= \begin{pmatrix} c \frac{s'}{s} & -s \frac{c'}{s} \\ s \frac{c'}{s} & c \frac{s'}{c} \end{pmatrix}, \\ R_2 &= \begin{pmatrix} -c \frac{c'}{s} & s \frac{s'}{c} \\ -s \frac{s'}{c} & -c \frac{c'}{s} \end{pmatrix}, \\ R_3 &= \begin{pmatrix} -c \frac{c'}{s} & -s \frac{c'}{c} \\ -s \frac{s'}{c} & c \frac{s'}{c} \end{pmatrix}, \\ R_4 &= \begin{pmatrix} c \frac{s'}{c} & s \frac{s'}{c} \\ s \frac{c'}{s} & -c \frac{c'}{s} \end{pmatrix}, \end{aligned}$$

and the general incoherent Kraus operator is  $K = \lambda_i K_i$  ( $i = 1, 2, 3, 4$ ). Finally, after imposing  $\sum_i |\lambda_i|^2 K_i^\dagger K_i = \mathbb{1}$ , we obtain

the following conditions on  $R_i$  and  $\lambda_i$ :

$$\begin{aligned} \sum_i |\lambda_i|^2 &= 1, \\ \sum_i |\lambda_i|^2 R_i^\dagger R_i &= \mathbb{1}, \\ \sum_i |\lambda_i|^2 R_i &= 0. \end{aligned}$$

It can be verified that these conditions are only fulfilled when  $|c\rangle = |s\rangle = \frac{1}{\sqrt{2}}$ , i.e.,  $|r\rangle$  is maximally coherent. ■

If the incoherent implementation of the unitary, instead of mapping two qubits (input and resource state) to one qubit (output), maps two qubits to two qubits (output plus residual resource), i.e.,  $\mathcal{I}(\rho \otimes |r\rangle\langle r|) = U(\theta)\rho U(\theta)^\dagger \otimes \sigma$ , then, first,  $\sigma$  has to be the same irrespective of the state  $\rho$ . Otherwise we would be able, by measuring  $\sigma$ , to obtain some information about  $\rho$  without disturbing it. Now consider a pure incoherent input state  $\rho = |0\rangle\langle 0|$ , and note that the desired output state  $U(\theta)|\psi\rangle$  has nontrivial coherence. But now observe that  $\mathcal{I}$  takes in a state of coherence rank 2 [17] and produces a product of a pure state of coherence rank 2 with another state. Since the coherence rank cannot increase, even under individual Kraus operators [17], it must be the case that  $\sigma$  is incoherent.

This result might suggest an irreversibility between simulation and coherence generation for these unitaries, but we point out that it does not preclude the possibility of simulations using a higher rank, yet less coherent, resource state (cf. [35], where the analog is demonstrated for LOCC implementation of bipartite unitaries using entangled resources) or of a simulation of many instances of  $U(\theta)$  at a cost lower than 1 per unitary.

## V. CONCLUSION AND OUTLOOK

We have shown that using a maximally coherent state and strictly incoherent operations, we can implement any unitary in a system (using  $\Psi$  for a  $d$ -dimensional unitary) and, via the Stinespring dilation, any CPTP map (using  $\Psi_{d^2}$  for a channel acting on a  $d$ -dimensional system). By teleportation, we prove that for incoherent operations and a  $d$ -dimensional maximally coherent state any noisy channel can be implemented.

Vice versa, every incoherent operation gives rise to some capacity for generating pure coherence by using it asymptotically many times, and we have given capacity bounds in general and a single-letter formula for the case of unitaries. We also found the additive upper bounds  $\mathbb{P}_r(T)$  and  $\mathbb{P}_f(T)$  on the coherence generation capacity  $C_{\text{gen}}(T)$ , even though we do not know whether these numbers are efficiently computable, due to the presence of the extension  $\otimes \text{id}_k$ , or whether these extensions are even necessary. It is open at the moment whether the coherence generation capacity  $C_{\text{gen}}(T)$  itself is additive for general tensor product channels and, likewise, the lower bound given in Theorem 1,  $\tilde{\mathbb{P}}_r(T) = \sup_k \tilde{\mathbb{P}}_r(T \otimes \text{id}_k)$ ; at least for isometric channels, they are.

The coherence-generating capacity is never greater than the simulation cost, but in general these two numbers will be different. As an extreme case, consider any CPTP map  $T$  that is not incoherent but is a so-called “maximally incoherent operation” (cf. [16]), meaning that  $T(\rho) \in \Delta$  for all  $\rho \in \Delta$ .

This class was considered in [9], and it makes coherence theory asymptotically reversible [17], all states  $\rho$  being equivalent to  $C_r(\rho)$  maximally coherent qubit states. Such maps exist, even acting on a single qubit (cf. [36]), correcting Theorem 21 in [27]. We expect the simulation cost of any such  $T$  to be positive,  $C_{\text{sim}}(T) > 0$ . At the same time,  $C_{\text{gen}}(T) = 0$  by Theorem 1, because the relative entropy of coherence is an MIO monotone, and the tensor product of MIO transformations is an MIO. To obtain an example, we can take any MIO channel for which there exists a state  $\rho$  such that  $C_f(T(\rho)) > C_f(\rho)$ , since by Theorem 7,  $C_{\text{sim}}(T)$  is lower-bounded by the difference between the two. (As an aside, we note that this cannot be realized in qubits, because for qubits, any state transformation possible under an MIO is already possible under an IO, for which  $C_f$  is a monotone [27,37].) Concretely, consider the following states in a  $2d$ -dimensional system  $A$ , which could be called *coherent flower states*, since their corresponding maximally correlated states (cf. [17] and [38]) are the well-known flower states [39]. We write them as  $2 \times 2$ -block matrices,

$$\rho_d = \frac{1}{2d} \begin{pmatrix} \mathbb{1} & U \\ U^\dagger & \mathbb{1} \end{pmatrix}, \quad (25)$$

where  $U$  is the  $d$ -dimensional discrete Fourier transform matrix. Via the correspondence between  $C_r$  and the relative entropy of entanglement, and between  $C_f$  and the entanglement of formation, respectively, of the associated state, we know that  $C_r(\rho_d) = 1$  and  $C_f(\rho_d) = 1 + \frac{1}{2} \log_2 d$  [39]. By the results in [9], however, for every  $\epsilon > 0$  and sufficiently large  $n$ , there exists an MIO transformation  $T^{(n)} : (\mathbb{C}^2)^{\otimes n(1+\epsilon)} \rightarrow A^n$  with

$$\rho^{(n)} := T^{(n)}(\Psi_2^{\otimes n(1+\epsilon)}) \approx \rho_d^{\otimes n}. \quad (26)$$

By the asymptotic continuity of  $C_f$  [17], we have  $C_f(\rho^{(n)}) \geq n(1 + \frac{1}{2} \log_2 d) - n\epsilon \log_2(2d) - g(\epsilon)$ , while of course the preimage  $\rho_0 = \Psi_2^{\otimes n(1+\epsilon)}$  has  $C_f(\rho_0) \leq n + n\epsilon$ , so for  $\epsilon$  small enough and  $n$  large enough, we have a gap,

$$C_{\text{sim}}(T^{(n)}) \geq \frac{n}{2} \log_2 d - n\epsilon(2 + \log_2 d) - g(\epsilon) > 0, \quad (27)$$

invoking Theorem 7. Thus, while *states* in the resource theory of coherence cannot exhibit a bound resource—indeed, it was observed in [17] that vanishing distillable coherence,  $C_r(\rho) = 0$ , implies vanishing coherence cost  $C_f(\rho) = 0$ —operations can have bound coherence, and the gap can be large on the scale of the logarithm of the channel dimension. We observe that this effect can only occur for maximally incoherent operations, which are precisely the ones with  $C_{\text{gen}}(T) = 0$ . We argued already that MIO channels have zero coherence-generating capacity; in the other direction, if  $T$  is not an MIO, it means that there exists an incoherent state  $\rho$  such that  $T(\rho)$  has coherence, and this can be distilled at rate  $C_r(T(\rho))$  [17]. As MIOs are closed upon forming tensor products, it also follows that  $C_{\text{gen}}$  does not exhibit superactivation.

This example and the subsequent considerations raise the question of how our theory would change if we considered all MIO transformations as free operations. By definition, the above example—by virtue of being an MIO—has zero MIO simulation cost, so there is no longer any bound coherence.



It may still be the case that there is in general a difference between MIO simulation cost and MIO coherence-generating capacity, but determining this possibility is beyond the scope of the present investigation. We only note that Theorem 1 gives us a single-letter formula for the MIO coherence-generating capacity, namely,

$$C_{\text{gen}}^{\text{MIO}}(T) = \mathbb{P}_r(T) = \sup_{\rho \text{ on } A \otimes C} C_r((T \otimes \text{id})\rho) - C_r(\rho), \quad (28)$$

the complete relative entropy coherence power of  $T$ . The supremum is over all auxiliary systems  $C$  and mixed states  $\rho$  in  $A \otimes C$ . Indeed, the upper bound of Eq. (5) still applies, because  $C_r$  is MIO monotone, which is all we needed in the proof of Theorem 1. For the lower bound, that it is attainable follows the same idea as the proof of Eq. (4), only we can now use an arbitrary mixed state  $\rho$  in the argument, since its coherence cost under the MIO equals  $C_r(\rho)$  [9,17].

One of the most exciting possibilities presented by the point of view of channels as coherence resources is the transformation of channels into channels by means of pre- and postprocessing a given one by incoherent operations to obtain a different one. The fundamental question one can ask here is how efficiently, i.e., at what rate  $R(T_1 \rightarrow T_2)$  one can

transform asymptotically many instances of  $T_1$  into instances of  $T_2$ , with an asymptotically vanishing diamond norm error. To make nontrivial statements about these rates, one would need to extend some of the various coherence monotones that have been studied for states to CPTP maps.

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