Bipartite entanglement in fermion systems

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We discuss the relation between fermion entanglement and bipartite entanglement. We first show that an exact correspondence between them arises when the states are constrained to have a definite local number parity. Moreover, for arbitrary states in a four-dimensional single-particle Hilbert space, the fermion entanglement is shown to measure the entanglement between two distinguishable qubits defined by a suitable partition of this space. Such entanglement can be used as a resource for tasks like quantum teleportation. On the other hand, this fermionic entanglement provides a lower bound to the entanglement of an arbitrary bipartition, although in this case the local states involved will generally have different number parities. Finally, the fermionic implementation of the teleportation and superdense coding protocols based on qubits with odd and even number parity is discussed, together with the role of the previous types of entanglement.

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I. INTRODUCTION

Entanglement is a fundamental feature of quantum mechanics, and its quantification and characterization have been one of the main goals of quantum information theory for the last decades [1–3]. It is also at the heart of quantum information processing [4], being recognized as the key ingredient for quantum-state teleportation [5] and the resource that makes some pure state-based quantum algorithms exponentially faster than their classical counterparts [6].

Although entanglement has been extensively studied for systems of distinguishable constituents, less attention has been paid to the case of a system of indistinguishable fermions. Only in recent years has the topic gained increasing strength [7–26]. Mainly two approaches may be recognized in the attempts to generalize the definition of entanglement to fermion systems: The first is *entanglement between modes* [13–17], where the system and subsystems consist of some collection of single-particle (SP) modes that can be shared. This approach requires us to fix some basis of the SP state space and then to specify the modes that constitute each subsystem. The other approach is known as *entanglement between particles* [7–12,18–23,25], where the indistinguishable constituents of the system are taken as subsystems and entanglement is defined beyond symmetrization.

In a previous work [24] we defined an entropic measure of mode entanglement in fermion systems which is shown to be a measure of entanglement between particles after an optimization over bases of the SP state space is performed. Moreover, when the SP state-space dimension is four and the particle number is fixed at two, this entanglement measure reduces to the *Slater correlation measure* defined in [7]. In the present work we first show that the entanglement between two distinguishable qubits is the same as that measured by this fermionic entanglement entropy when the fermionic states are constrained to have a fixed local number parity in the associated bipartition of the SP space. Then we use this correspondence to show that, in fact, any state of a fermion system with a four-dimensional SP Hilbert space may be seen as a state of two distinguishable qubits for a suitable bipartition of the SP space, with its entanglement measured by the fermionic entanglement entropy. On the other hand, for an arbitrary bipartition involving no fixed local number parity the fermionic entanglement entropy is shown to provide a lower bound to the associated bipartite entanglement. As an application we use these results to show that qubit-based quantum circuits may be rewritten as mode-based fermionic circuits if we impose the appropriate restriction to the occupation numbers, recovering reversible classical computation when the input states are Slater determinants (in the basis of interest). Two types of fermionic qubit representations, based on odd- or even-number-parity qubits, are seen to naturally emerge. Finally, we show that the extra bipartite entanglement that can be obtained by relaxing this local parity restriction can in principle be used for protocols such as superdense coding.

The formalism and theoretical results are provided in Sec. II, while their applications are discussed in Sec. III. Conclusions are, finally, provided in Sec. IV.

II. FORMALISM

A. Fermionic entanglement entropy and concurrence

We consider a fermion system with a single-particle Hilbert space $\mathcal{H}$. We deal with pure states $|\psi\rangle$ which do not necessarily have a fixed particle number, although the number parity will be fixed, in agreement with the parity superselection rule [27]: $P|\psi\rangle = \pm|\psi\rangle$, with $P = \exp[i\pi \sum_j c_j^\dagger c_j]$ the number parity operator. Here $c_j, c_j^\dagger$ denote fermion annihilation and creation operators satisfying the usual anticommutation relations

$$\{c_i, c_j\} = 0, \quad \{c_i, c_j^\dagger\} = \delta_{ij}. \tag{1}$$

In [24] we defined a *one-body entanglement entropy* for a general pure fermion state $|\psi\rangle$,

$$S^{\rm SP}(|\psi\rangle) = {\rm Tr}\,[h(\rho^{\rm SP})], \tag{2}$$

where $\rho^{\rm SP}_{ij} = \langle c_j^\dagger c_i\rangle \equiv \langle\psi|c_j^\dagger c_i|\psi\rangle$ is the one-body density matrix of the system and $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$. Equation (2) is proportional to the minimum, over all SP bases of $\mathcal{H}$, of the average entanglement entropy between an SP mode and its orthogonal complement (which in turn arises

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from a properly defined measurement of the occupation of an SP mode) and vanishes iff $|\psi\rangle$ is a Slater determinant (SD), i.e., $|\psi\rangle = c_1^\dagger \dots c_k^\dagger|0\rangle$. This definition is easily extended to quasiparticle (qsp) modes, in which case [24]

$$S^{\rm qsp}(|\psi\rangle) = -{\rm Tr}\,[\rho^{\rm qsp}\log_2(\rho^{\rm qsp})], \tag{3}$$

where $\rho^{\rm qsp}$ is now the extended one-body density matrix

$$\rho^{\rm qsp} = 1 - \left\langle \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^\dagger \end{pmatrix} (\boldsymbol{c}^\dagger\, \boldsymbol{c}) \right\rangle = \begin{pmatrix} \rho^{\rm SP} & \kappa \\ -\bar{\kappa} & \mathbb{1} - \bar{\rho}^{\rm SP} \end{pmatrix}, \tag{4}$$

with $\kappa_{ij} = \langle c_j c_i\rangle$, $-\bar{\kappa}_{ij} = \langle c_j^\dagger c_i^\dagger\rangle$, and $(\mathbb{1} - \bar{\rho}^{\rm SP})_{ij} = \langle c_j c_i^\dagger\rangle$. Equation (3) vanishes iff $|\psi\rangle$ is a quasiparticle vacuum or SD and satisfies $S^{\rm qsp}(|\psi\rangle) \leqslant S^{\rm SP}(|\psi\rangle)$, with $S^{\rm qsp}(|\psi\rangle) = S^{\rm SP}(|\psi\rangle)$ iff $\kappa = 0$.

While Eq. (2) is invariant under unitary transformations $c_i \to \sum_k \bar{U}_{ki} c_k$, $UU^\dagger = I$, which lead to $\rho^{\rm SP} \to U^\dagger \rho^{\rm SP} U$, Eq. (4) remains invariant under general Bogoliubov transformations

$$c_i \to a_i = \sum_k \bar{U}_{ki} c_k + V_{ki} c_k^\dagger, \tag{5}$$

where the matrices U and V satisfy $UU^\dagger + VV^\dagger = \mathbb{1}$ and $UV^T + VU^T = 0$ in order that $\{a_i, a_i^\dagger\}$ fulfill the fermionic anticommutation relations [28]. In this case $\rho^{\rm qsp} \to W^\dagger \rho^{\rm qsp} W$, with $W = \left(\begin{smallmatrix} U & V \\ \bar{V} & \bar{U}\end{smallmatrix}\right)$ a unitary matrix. In terms of the operators diagonalizing $\rho^{\rm qsp}$, we then have

$$1 - \left\langle \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{a}^\dagger \end{pmatrix} (\boldsymbol{a}^\dagger\, \boldsymbol{a}) \right\rangle = \begin{pmatrix} f & 0 \\ 0 & 1 - f \end{pmatrix},$$

with $f_{kl} = f_k \delta_{kl}$ and $f_k, 1 - f_k$ the eigenvalues of $\rho^{\rm qsp}$.

For an SP space $\mathcal{H}$ of dimension 4, $\rho^{\rm qsp}$ becomes an 8×8 matrix, and it was shown that its eigenvalues for a pure state $|\psi\rangle$ are *fourfold degenerate* and can be written as [24]

$$f_\pm = \frac{1 \pm \sqrt{1 - C^2(|\psi\rangle)}}{2}, \tag{6}$$

where $C(|\psi\rangle) = 2\sqrt{f_+ f_-} \in [0,1]$ is called *fermionic concurrence*, in analogy with that defined for two-qubits [29]. Equation (3) then becomes an increasing function of $C(|\psi\rangle)$, vanishing iff the latter vanishes. This fermionic concurrence can also be explicitly evaluated: Writing a general even-number-parity pure state in such a space as

$$|\psi\rangle = \left(\alpha_0 + \frac{1}{2}\sum_{i,j} \alpha_{ij} c_i^\dagger c_j^\dagger + \alpha_4 c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger\right)|0\rangle, \tag{7}$$

where $\alpha_{ij} = -\alpha_{ji}$, $i,j = 1,\dots,4$, and $|\alpha_0^2| + |\alpha_4^2| + \frac{1}{2}{\rm tr}\,\alpha^\dagger\alpha = 1$, then $\rho^{\rm SP} = \alpha\alpha^\dagger + |\alpha_4|^2\mathbb{1}$, $\kappa = \alpha_0^*\alpha + \alpha_4\tilde{\alpha}^*$, with $\tilde{\alpha}_{ij} = \frac{1}{2}\sum_{k,l}\epsilon_{ijkl}\alpha_{kl}$ (ϵ_{ijkl} denotes the fully antisymmetric tensor), and it can be shown that [24]

$$C(|\psi\rangle) = 2|\alpha_{12}\alpha_{34} - \alpha_{13}\alpha_{24} + \alpha_{14}\alpha_{23} - \alpha_0\alpha_4|. \tag{8}$$

For two-fermion states ($\alpha_0 = \alpha_4 = 0$) it reduces to the *Slater correlation measure* defined in [7] and [9], for which $\kappa = 0$ and $f_\pm$ become the eigenvalues (twofold degenerate) of $\rho^{\rm SP}$. An expression similar to (8) holds for an odd-number-parity state (see [24] and Sec. II E). Moreover, in such SP space the concurrence and the associated entanglement of formation can also be explicitly determined for arbitrary mixed states [7,24].

A four-dimensional SP space (which generates an eight-dimensional state space for each value of the parity P) then becomes exactly solvable, also being the first nontrivial dimension since for $\dim\mathcal{H} \leqslant 3$ any definite-parity pure state can be written as an SD or quasiparticle vacuum [24], as verified by (8) [$C(|\psi\rangle) = 0$ if one of the SP states is left empty]. It is also physically relevant, since it can accommodate the basic situation of two spin-1/2 fermions at two sites or, more generally, states of spin-1/2 fermions occupying just two orbital states, as in a double-well scenario. The relevant SP space in these cases is $\mathcal{H}_S \otimes \mathcal{H}_O$, with $\mathcal{H}_S$ the spin space and $\mathcal{H}_O$ the two-dimensional subspace spanned by the two orbital states. In particular, just four SP states are essentially used in recent proposals for observing Bell violation from single-electron entanglement [30].

B. Bipartite entanglement as two-fermion entanglement

Let us now consider a system of two distinguishable qubits prepared in a pure state $\alpha_+|00\rangle + \alpha_-|11\rangle$, i.e.,

$$|\psi\rangle_{AB} = \alpha_+|\uparrow\rangle_A \otimes |\uparrow\rangle_B + \alpha_-|\downarrow\rangle_A \otimes |\downarrow\rangle_B, \tag{9}$$

where $|\alpha_+^2| + |\alpha_-^2| = 1$ and the notation indicates a possible realization in terms of two spin-1/2 particles located at different sites A and B, with their spins aligned parallel or antiparallel to a given (z) axis. We can also consider the latter state as a two-fermion state of a spin-1/2 fermion system with SP space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_O$,

$$|\psi\rangle_f = (\alpha_+ c_{A\uparrow}^\dagger c_{B\uparrow}^\dagger + \alpha_- c_{A\downarrow}^\dagger c_{B\downarrow}^\dagger)|0\rangle, \tag{10}$$

with $|0\rangle$ the fermionic vacuum. A measurement of spin "A" or "B" along z can be described in the fermionic representation by the operators $\Pi_{S\mu} = c_{S\mu}^\dagger c_{S\mu}$, $S = A,B$, $\mu = \uparrow, \downarrow$, which satisfy $\Pi_{S\mu}^2 = \Pi_{S\mu}$ and $[\Pi_{S\mu}, \Pi_{S'\mu'}] = 0$, with $\sum_\mu \Pi_{S\mu}|\psi\rangle_f = |\psi\rangle_f$. Furthermore, we can describe any "local" operator on A or B in terms of Pauli operators if we define, for $S = A,B$,

$$\sigma_{Sx} = c_{S\uparrow}^\dagger c_{S\downarrow} + c_{S\downarrow}^\dagger c_{S\uparrow}, \tag{11a}$$

$$\sigma_{Sy} = -i(c_{S\uparrow}^\dagger c_{S\downarrow} - c_{S\downarrow}^\dagger c_{S\uparrow}), \tag{11b}$$

$$\sigma_{Sz} = c_{S\uparrow}^\dagger c_{S\uparrow} - c_{S\downarrow}^\dagger c_{S\downarrow}, \tag{11c}$$

which verify the usual commutation relations $[\sigma_{Sj}, \sigma_{S'k}] = 2i\delta_{SS'}\epsilon_{jkl}$ (ϵ_{jkl} is the antisymmetric tensor), with $\sigma_{Sj}^2|\psi\rangle_f = |\psi\rangle_f$.

It is also apparent that state (9) is separable iff $\alpha_+ = 0$ or $\alpha_- = 0$, which is precisely the condition which ensures that state (10) is an SD. Moreover, the standard concurrence [29] of state (9) is *identical* to the fermionic concurrence, (8), of state (10),

$$C(|\psi\rangle_{AB}) = 2|\alpha_+\alpha_-| = C(|\psi\rangle_f), \tag{12}$$

with $f_\pm = |\alpha_\pm^2|$ in (6). Entangled two-qubit states (9), then correspond to two-fermion states (10) which are not SDs, and vice versa.

Such correspondence remains, of course, valid for any bipartite two-qubit state

$$|\psi\rangle_{AB}=\sum_{\mu,\nu}\alpha_{\mu\nu}|\mu\rangle_A\otimes|\nu\rangle_B, \tag{13}$$

which in the fermionic representation becomes

$$|\psi\rangle_f=\sum_{\mu,\nu}\alpha_{\mu\nu}c^\dagger_{A\mu}c^\dagger_{B\nu}|0\rangle. \tag{14}$$

We now obtain $C(|\psi\rangle_{AB})=2|\det\alpha|=C(|\psi\rangle_f)$, according to the standard and fermionic [Eq. (8)] expressions. These states can in fact be taken to the previous Schmidt forms (9) and (10) (with $|\alpha_\pm|$ the singular values of the matrix α) by means of local unitary transformations, which in the fermionic representation become $c_{S\mu}\to\sum_\nu\bar{U}^S_{\nu\mu}c_{S\nu}$.

Previous considerations remain valid also for *general* bipartite states of systems of *arbitrary* dimension [$\mu=1,\dots,d_A$, $\nu=1,\dots,d_B$ in (13) and (14)], if the SP space of the associated fermionic system (of dimension d_A+d_B) is decomposed as $\mathcal{H}_A\oplus\mathcal{H}_B$. The SP density matrix $\rho^{\rm SP}$ derived from state (14) takes, in the general case, the blocked form

$$\rho^{\rm SP}=\begin{pmatrix}\alpha\alpha^\dagger & 0\\ 0 & \alpha^T\bar{\alpha}\end{pmatrix}=\begin{pmatrix}\rho_A & 0\\ 0 & \rho_B\end{pmatrix}, \tag{15}$$

i.e., $\langle c^\dagger_{S\nu}c_{S'\mu}\rangle=\delta_{SS'}(\rho_S)_{\mu\nu}$, where $\rho_{A(B)}$ are the local reduced density matrices ${\rm Tr}_{B(A)}|\psi\rangle_{AB}\langle\psi|$ of state (13) in the standard basis. Hence, in the fermionic setting $\rho^{\rm SP}$ contains the information of *both* local states and its diagonalization implies that of *both* ρ_A and ρ_B. Its eigenvalues will then be those of these matrices, hence being twofold degenerate and equal to the square of the singular values of the matrix α (becoming $f_\pm=|\alpha_\pm|^2$ in the two-qubit case). In the general case, the entanglement entropy of state (13) can then be written as

$$E(A,B)=S(\rho_A)=S(\rho_B)=\tfrac{1}{2}S(\rho^{\rm SP}), \tag{16}$$

which holds for the von Neumann entropy $S(\rho)=-{\rm Tr}\,[\rho\log_2\rho]$ as well as for any trace form entropy [31] $S(\rho)={\rm Tr}\,[f(\rho)]$ [f concave, $f(0)=f(1)=0$]. Thus, the entanglement entropy of the general bipartite state, (13), is just proportional to the fermionic entanglement entropy [as defined in (2)] of the associated state, (14). Hence, for any dimension there is an exact correspondence between the bipartite states, (13), and the two-fermion states, (14), with local operators represented by linear combinations of one-body local fermion operators $c^\dagger_{S\nu}c_{S\mu}$ (satisfying $[c^\dagger_{A\mu}c_{A\nu},c^\dagger_{B\mu'}c_{B\nu'}]=0$) and $|\psi\rangle_{AB}$ entangled iff $|\psi\rangle_f$ is not an SD.

This equivalence holds also for mixed states i.e., convex combinations of states (13) and (14). The bipartite states will be separable, i.e., convex combinations of product states [32] iff the associated fermionic mixed state can be written as a convex combination of SDs of the form of (14). In particular, for two-qubit states a four-dimensional SP fermion space suffices and the standard mixed-state concurrence [29] will coincide exactly with the fermionic mixed-state concurrence [7,20–22,24] of mixtures of states (14).

C. Bipartite entanglement as quasiparticle fermion entanglement

Other fermionic representations of state (13) with similar properties are also feasible. For instance, in the two-qubit case we can perform a particle-hole transformation of the fermion operators with spin down,

$$c^\dagger_{S\uparrow}\longrightarrow c^\dagger_{S\uparrow},\quad c^\dagger_{S\downarrow}\longrightarrow c_{S\downarrow},\quad S=A,B, \tag{17}$$

such that the aligned state $|\downarrow\rangle_A\otimes|\downarrow\rangle_B$ now corresponds to the vacuum of the new operators ($|0\rangle\longrightarrow c^\dagger_{A\downarrow}c^\dagger_{B\downarrow}|0\rangle$), with the new $c^\dagger_{S\downarrow}$ creating a hole. The remaining states of the standard basis become one and two particle-hole excitations. We can then rewrite state (10) as

$$|\tilde{\psi}\rangle_f=(\alpha_-+\alpha_+c^\dagger_{A\uparrow}c^\dagger_{A\downarrow}c^\dagger_{B\uparrow}c^\dagger_{B\downarrow})|0\rangle, \tag{18}$$

i.e., as a superposition of the vacuum plus two particle-hole excitations, with each "side" now having either zero or two fermions, i.e., an *even* local number parity ($e^{i\pi N_S}=1$ for $S=A,B$, $N_S=\sum_\mu c^\dagger_{S\mu}c_{S\mu}$). It is apparent that state (18) is a quasiparticle vacuum or SD iff $\alpha_+=0$ or $\alpha_-=0$. Moreover, for state (18) Eq. (8) leads again to $C(|\tilde{\psi}\rangle_f)=2|\alpha_+\alpha_-|$, implying equivalence, (12), between the bipartite and the present generalized fermionic concurrrence, invariant under Bogoliubov (and hence particle-hole) transformations.

The local Pauli operators, (11), now become

$$\tilde{\sigma}_{Sx}=c^\dagger_{S\uparrow}c^\dagger_{S\downarrow}+c_{S\downarrow}c_{S\uparrow}, \tag{19a}$$

$$\tilde{\sigma}_{Sy}=-i(c^\dagger_{S\uparrow}c^\dagger_{S\downarrow}-c_{S\downarrow}c_{S\uparrow}), \tag{19b}$$

$$\tilde{\sigma}_{Sz}=c^\dagger_{S\uparrow}c_{S\uparrow}+c^\dagger_{S\downarrow}c_{S\downarrow}-1, \tag{19c}$$

which verify the same SU(2) commutation relations $[\tilde{\sigma}_{Sj},\tilde{\sigma}_{S'k}]=2i\delta_{SS'}\epsilon_{jkl}\tilde{\sigma}_{Sl}$, with $\tilde{\sigma}^2_{Sj}|\tilde{\psi}\rangle_f=|\tilde{\psi}\rangle_f\ \forall j$. Any local operation can be written in terms of these operators, which now represent local particle-hole creation or annihilation and counting.

Similarly, we may write the general two-qubit state, (13), as

$$|\tilde{\psi}\rangle_f=\sum_{\mu,\nu}\alpha_{\mu\nu}(c^\dagger_{A\uparrow}c^\dagger_{A\downarrow})^{n_\mu}(c^\dagger_{B\uparrow}c^\dagger_{B\downarrow})^{n_\nu}|0\rangle, \tag{20}$$

where $\mu,\nu=\pm$ and $n_-=0$, $n_+=1$. This state can be brought back to the "Schmidt" form, (18), by means of "local" Bogoliubov transformations $c_{S\uparrow}\to u_Sc_{S\uparrow}+v_Sc^\dagger_{S\downarrow}$, $c_{S\downarrow}\to u_Sc_{S\downarrow}-v_Sc^\dagger_{S\uparrow}$, $|u_S^2|+|v_S^2|=1$, which will diagonalize $\rho^{\rm qsp}$ (see below) and change the vacuum as $|0\rangle\to[\prod_{S=A,B}(u_S-v_Sc^\dagger_{S\uparrow}c^\dagger_{S\downarrow})]|0\rangle$. It is again verified that for this state Eq. (8) leads to $C(|\tilde{\psi}\rangle_f)=2|\det\alpha|=2|\alpha_+\alpha_-|$, with $|\alpha_\pm|$ the singular values of the matrix α. State (13) is then entangled iff state (20) is not a *quasiparticle* vacuum or SD ($C(|\tilde{\psi}\rangle_f)>0$).

In this case the extended density matrix $\rho^{\rm qsp}$ is to be considered, with elements $\langle c^\dagger_{S\nu}c_{S'\mu}\rangle=\delta_{SS'}\delta_{\mu\nu}p_S$, $\langle c_{S\nu}c_{S'\mu}\rangle=\delta_{SS'}\delta_{\nu,-\mu}(-1)^{n_\mu}q_S$, where $p_{A(B)}=|\alpha_{++}|^2+|\alpha_{+-(-+)}|^2$, $q_{A(B)}=\alpha_{++}\alpha^*_{-+(+-)}+\alpha_{+-(-+)}\alpha^*_{--}$. For the Schmidt form, (20), $\rho^{\rm qsp}$ becomes diagonal [$p_{A(B)}=|\alpha_+|^2$, $q_{A(B)}=0$]. Reduced states $\rho_{A(B)}$ are now to be recovered as particular blocks of $\rho^{\rm qsp}$:

$$\rho_S = \frac{1}{2}\begin{pmatrix} 1+\langle\tilde\sigma_{Sz}\rangle & \langle\tilde\sigma_{Sx}\rangle - i\langle\tilde\sigma_{Sy}\rangle \\ \langle\tilde\sigma_{Sx}\rangle + i\langle\tilde\sigma_{Sy}\rangle & 1-\langle\tilde\sigma_{Sz}\rangle \end{pmatrix} = \begin{pmatrix} \langle c^\dagger_{S\uparrow}c_{S\uparrow}\rangle & \langle c_{S\downarrow}c_{S\uparrow}\rangle \\ \langle c^\dagger_{Si\uparrow}c^\dagger_{S\downarrow}\rangle & \langle c_{S\uparrow}c^\dagger_{S\uparrow}\rangle \end{pmatrix}. \tag{21}$$

Diagonalization of $\rho^{\rm qsp}$ will, nevertheless, still imply that of ρ_A and ρ_B. It is verified that its eigenvalues are $f_\pm = |\alpha_\pm|^2$, fourfold degenerate, with $|\alpha_\pm|$ the singular values of matrix α. We then have

$$E(A,B) = S(\rho_A) = S(\rho_B) = \tfrac{1}{4}S(\rho^{\rm qsp}), \tag{22}$$

again valid for any trace-form entropy $S(\rho) = {\rm Tr}\,[f(\rho)]$. And for convex mixtures of states of the form of (20) (whose rank will be at most 4), the mixed-state fermionic concurrence, as defined in [24], will again coincide exactly with the standard two-qubit concurrence.

The same considerations hold for general bipartite states, (13), of systems of arbitrary dimension if a particle-hole transformation (or, in general, a Bogoliubov transformation) is applied to the original fermion operators in (14). In this case Eq. (22) is valid for entropic functions satisfying $f(p) = f(1-p)$ (a reasonable assumption, as p represents an average occupation number of particle or hole), since $\rho^{\rm qsp}$ will have eigenvalues f_k and $1-f_k$, now twofold degenerate, with f_k those of the local states $\rho_{A(B)}$.

A final remark is that the representations, (11) and (19), of Pauli operators can coexist independently since

$$[\sigma_{Sj},\tilde\sigma_{S'k}] = 0, \tag{23}$$

$\forall\ j,k$, for both $S' \neq S$ *and* $S' = S$ [SU(2) × SU(2) structure [33] at each side A or B]. Moreover, the even-local-parity states, (20), belong to the kernel of operators, (11), while the odd-local-parity states, (14) ($e^{i\pi N_S} = -1$), belong to the kernel of the operators, (19),

$$\sigma_{Sj}|\tilde\psi\rangle_f = \tilde\sigma_{Sj}|\psi\rangle_f = 0, \tag{24}$$

for $S = A,B$ and $j = x,y,z$. Hence, the unitary operators $e^{i\sum_j\lambda_j\sigma_{Sj}}$ ($e^{i\sum_j\lambda_j\tilde\sigma_{Sj}}$) will become identities when applied to states $|\tilde\psi\rangle_f$ ($|\psi\rangle_f$). A fermion system with an SP space of dimension 4 can then accommodate two distinct two-qubit systems, one for each value of the local number parity, keeping the total number parity fixed [$e^{i\pi(N_A+N_B)} = 1$].

D. Bipartite entanglement with no fermion entanglement

Previous examples show an exact correspondence between bipartite and fermion entanglement. The representations considered involve a fixed value not only of the global parity, but also of the *local* number parity. It is apparent, however, that it is also possible to obtain bipartite entanglement from SDs by choosing appropriate partitions of the SP space, although in this case the local parity will not be fixed. For instance, the single-fermion state

$$|\psi\rangle_f = (\alpha c^\dagger_{A\uparrow} + \beta c^\dagger_{B\uparrow})|0\rangle, \tag{25}$$

where the fermion is created in a state with no definite position if $\alpha\beta \neq 0$, leads obviously to $S(\rho^{\rm sp}) = 0$ but corresponds to an entangled state $\alpha|\uparrow\rangle_A \otimes |0\rangle_B + \beta|0\rangle_A \otimes |\downarrow\rangle_B$. However, the local states on each side have different number parities. The same occurs with the two-fermion SD $(\alpha c^\dagger_{A\uparrow} + \beta c^\dagger_{B\uparrow})(\alpha' c^\dagger_{A\downarrow} + \beta' c^\dagger_{B\downarrow})|0\rangle$, which has zero fermionic concurrence but corresponds to the entangled state $\alpha\beta'|\uparrow\rangle_A \otimes |\downarrow\rangle_B - \alpha'\beta|\downarrow\rangle_A \otimes |\uparrow\rangle_B + \alpha\alpha'|\uparrow\downarrow\rangle_A \otimes |0\rangle_B + \beta\beta'|0\rangle_A \otimes |\uparrow\downarrow\rangle_B$.

Hence, although there is entanglement with respect to the (A,B) partition, it is not possible to make arbitrary linear combinations of the eigenstates of ρ_A or ρ_B, since they may not have a definite number parity. While such entanglement may be sufficient for observing Bell inequalities violation, as proposed in [30], it can exhibit limitations for other tasks involving superpositions of local eigenstates, as discussed in Sec. III. This effect will occur whenever one of the fermions is created in a state which is "split" by the chosen partition of the SP space. With the restriction of a fixed number parity on each "side" an equivalence between bipartite and fermionic entanglement can become feasible, as discussed next. Note that such restriction directly implies *blocked* SP density matrices $\rho^{\rm SP}$ and $\rho^{\rm qsp}$, since all contractions $\langle c^\dagger_{Ai}c_{Bj}\rangle$ and $\langle c^\dagger_{Ai}c^\dagger_{Bj}\rangle$ linking both sides do not conserve the local parity and will therefore *vanish* $\forall\ i,j$.

E. Fermion entanglement as two-qubit entanglement

Let us now return to the two-fermion state, (10). The reason the two particles become distinguishable is that the "position" observable allows us to split the SP state space $\mathcal{H}$ as the direct sum of two copies of the spin space $\mathcal{H}_S$, $\mathcal{H} = \mathcal{H}_{S_A} \oplus \mathcal{H}_{S_B}$, with ${}_A\langle\mu|\mu\rangle_B = \langle 0|c_{A\mu}c^\dagger_{B\mu}|0\rangle = 0$ for $\mu = \uparrow$ or $\downarrow$. The latter condition ensures in fact that there is just one fermion on each side ($N_{A(B)}|\psi\rangle_f = |\psi\rangle_f$). However, for a more general two-fermion state, like that considered in the previous section, it is no longer possible to perform a measurement of the spin of only one particle by coupling it with position since both particles may be found at the same site.

But now nothing prevents us from turning back the argument and stating that if, for an arbitrary state $|\psi\rangle_f$, it is possible to split $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ contain just one fermion ($N_A|\psi\rangle_f = N_B|\psi\rangle_f = 1$), then we again recover a system of two distinguishable qubits. The latter feature leads us to the following important result:

Lemma 1. Let $|\psi\rangle_f$ be an arbitrary pure state of a fermion system with a four-dimensional SP space $\mathcal{H}$, having a definite number parity yet a not necessarily fixed fermion number. Then the entropy, (3), of the corresponding density matrix $\rho^{\rm qsp}$ is proportional to the entanglement entropy between the two distinguishable qubits, which can be extracted just by measuring the appropriate observables.

Proof. We start with a general state $|\psi\rangle_f$ with even number parity, which in this space will have the form of (7). For general α_{ij}, α_0, and α_4 in (7), the basis of the SP space $\mathcal{H}$ determined by the fermion operators $\{c_i, c^\dagger_i\}$ cannot be split in order to measure only one particle at each part. This fact remains true even if $\alpha_0 = \alpha_4 = 0$, as α is a general antisymmetric matrix. However, as proven in [24], it is always possible to find another basis of $\mathcal{H}$, determined by fermion operators $\{a_i, a^\dagger_i\}$ related to $\{c_i, c^\dagger_i\}$ through a Bogoliubov transformation, such that state (7) can be rewritten as

$$|\psi\rangle_f = (\alpha_+ a_1^\dagger a_2^\dagger + \alpha_- a_3^\dagger a_4^\dagger)|0\rangle, \quad (26)$$

which is analogous to Eq. (10). Here $|\alpha_\pm|^2 = f_\pm$ are just the distinct eigenvalues, (6), of the extended density matrix $\rho^{\rm qsp}$ determined by state (7), whereas $\{a_i, a_i^\dagger\}$ are suitable quasiparticle operators diagonalizing $\rho^{\rm qsp}$. The concurrence, (8), becomes $C(|\psi\rangle_f) = 2|\alpha_+\alpha_-|$.

We then recognize (26) as the Schmidt decomposition, (9), of a two-qubit state written in the fermionic representation, (10), since, for instance, the sets $\{a_1^\dagger, a_3^\dagger\}$ and $\{a_2^\dagger, a_4^\dagger\}$ (analogous to $\{a_{A\uparrow}^\dagger, a_{A\downarrow}^\dagger\}$ and $\{a_{B\uparrow}^\dagger, a_{B\downarrow}^\dagger\}$) span subspaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with $N_A = N_B = 1$ ($N_{A(B)}|\psi\rangle_f = |\psi\rangle_f$). And because the Schmidt coefficients $|\alpha_\pm|^2$ coincide with the eigenvalues of $\rho^{\rm qsp}$, we again obtain $S(\rho_A) = S(\rho_B) = \frac{1}{4}S(\rho^{\rm qsp})$ [Eq. (22)], with the fermionic concurrence coinciding exactly with the standard one.

The case of general odd-parity states, which in this SP space are linear combinations of states with one and three fermions,

$$|\psi\rangle_f = \sum_{i=1}^{4} \beta_i c_i^\dagger|0\rangle + \tilde{\beta}_i c_i|\bar{0}\rangle, \quad (27)$$

where $|\bar{0}\rangle = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger|0\rangle$ and $c_i|\bar{0}\rangle = \frac{1}{3!}\sum_{j,k,l}\epsilon_{ijkl}c_j^\dagger c_k^\dagger c_l^\dagger|0\rangle$, can be treated in a similar way, as they can be converted to even-parity states of the form of (7) by a particle-hole transformation of one of the states [i.e., $c_1^\dagger \to c_1$, $|0\rangle \to c_1^\dagger|0\rangle$, leading to $\alpha_0 = \beta_1$, $\alpha_4 = -\tilde{\beta}_1$, $\alpha_{1j} = -\beta_j$, and $\alpha_{ij} = \sum_k \epsilon_{ijk1}\tilde{\beta}_k$ for $i,j = 2,3,4$ in Eq. (7)]. They can then be also written in the form (26), in terms of suitable quasiparticle operators diagonalizing $\rho^{\rm qsp}$, so that the previous considerations still hold. The concurrence of the states, (27), given by [24] $C(|\psi\rangle_f) = 2|\sum_{i=1}^4 \beta_i\tilde{\beta}_i|$, again becomes $2|\alpha_+\alpha_-|$. ■

Some further comments are in order here. First, just the subspaces of $\mathcal{H}$ generated by $\{a_1^\dagger, a_2^\dagger\}$ and $\{a_3^\dagger, a_4^\dagger\}$ are defined by (26), since any unitary transformation $a_{1(2)}^\dagger \to \sum_{k=1,2} U_{k,1(2)}a_k^\dagger$ (and similarly for $a_{3(4)}^\dagger$) will leave it unchanged (except for phases in $\alpha_\pm$).

Second, we may also reinterpret state (26) as a two-fermion state with *even* local number parity if side A is identified with operators $\{a_1^\dagger, a_2^\dagger\}$ and side B with $\{a_3^\dagger, a_4^\dagger\}$, such that each side has either zero or two fermions (*even-number-parity qubits*). Still with even local number parity we may as well rewrite it in the form of (18), i.e.,

$$|\psi\rangle_f = (\alpha_- + \alpha_+ a_1^\dagger a_3^\dagger a_2^\dagger a_4^\dagger)|0\rangle, \quad (28)$$

through a transformation $a_i^\dagger \to a_i$ for $i = 3,4$, with $|0\rangle \to a_3^\dagger a_4^\dagger|0\rangle$. Here just the vacuum $|0\rangle$ and the completely occupied state $|\bar{0}\rangle$ are defined, since (28) remains invariant (up to a phase in α_+) by any unitary transformation $a_i^\dagger \to \sum_k U_{ki}a_k^\dagger$ of the operators $a_i^\dagger$.

Finally, if $|\psi\rangle_f$ is a two-fermion state $\frac{1}{2}\sum_{ij}\alpha_{ij}c_i^\dagger c_j^\dagger|0\rangle$, the previous considerations remain valid for an SP space $\mathcal{H}$ of arbitrary dimension. In this case $\kappa = 0$ and it is always possible to rewrite $|\psi\rangle_f$ as [7]

$$|\psi\rangle_f = \sum_k \alpha_k a_k^\dagger a_{\bar{k}}^\dagger|0\rangle,$$

where $|\alpha_k^2|$ are the eigenvalues of $\rho^{\rm sp} = \alpha\alpha^\dagger$ and $\{a_k, a_{\bar{k}}\}$ are suitable fermion operators diagonalizing this matrix, obtained through a unitary transformation $a_{k(\bar{k})} = \sum_i \bar{U}_{ik(\bar{k})}c_i$ (satisfying [7] $U^\dagger\alpha\bar{U} = \alpha'$ with α' a block diagonal matrix with 2×2 blocks $\alpha_k\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$). The SP space can then be written as $\mathcal{H}_A \oplus \mathcal{H}_B$, with $\mathcal{H}_{A(B)}$ the subspaces spanned by the sets $\{a_{k(\bar{k})}^\dagger\}$, each containing one fermion. We thus obtain $S(\rho_A) = S(\rho_B) = \frac{1}{2}S(\rho^{\rm SP})$ [Eq. (16)].

F. Fermion entanglement as minimum bipartite entanglement

We now demonstrate the second general result, concerning the mode entanglement associated with general decompositions $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B$ of a four-dimensional SP space. Any many-fermion state can be written as $|\psi\rangle_f = \sum_{\mu,\nu}\alpha_{\mu\nu}|\mu\nu\rangle$, where $\mu(\nu)$ labels orthogonal SDs on $\mathcal{H}_A$ ($\mathcal{H}_B$) and $|\mu\nu\rangle = [\prod_{i\in\mathcal{H}_A}(c_i^\dagger)^{n_i^\mu}][\prod_{j\in\mathcal{H}_B}(c_j^\dagger)^{n_j^\nu}]|0\rangle$ is an SD on $\mathcal{H}$, with $n_i^\mu = 0,1$ the occupation of SP state i in state μ. The ensuing reduced states $\rho_A = \sum_{\mu,\mu'}(\alpha\alpha^\dagger)_{\mu\mu'}|\mu\rangle\langle\mu'|$ and $\rho_B = \sum_{\nu,\nu'}(\alpha^T\bar{\alpha})_{\nu\nu'}|\nu\rangle\langle\nu'|$ satisfy ${\rm Tr}[\rho_{A(B)}O_{A(B)}] = {}_f\langle\psi|O_{A(B)}|\psi\rangle_f$ for any operator depending just on the local fermions $\{c_i, c_i^\dagger,\ i\in\mathcal{H}_{A(B)}\}$. The entanglement entropy associated with this bipartition is then [26] $E(A,B) = S(\rho_A) = S(\rho_B)$.

In the present case we may have either $2+2$ bipartitions ($\dim\mathcal{H}_A = \dim\mathcal{H}_B = 2$) or $1+3$ bipartitions ($\dim\mathcal{H}_A = 1$, $\dim\mathcal{H}_B = 3$). In the latter the entanglement is determined just by the average occupation of the single state of $\mathcal{H}_A$ [24] and corresponds to the case where A has access to just one of the SP states possibly occupied in $|\psi\rangle_f$. A realization of a $2+2$ partition is just that of spin-1/2 fermions which can be at two different sites (one accessible to Alice and the other to Bob), while a $1+3$ bipartition could be one where Alice has access to one site and just one spin direction, i.e., to the knowledge of the occupation of the SP state $A_\uparrow$. It could also apply to any asymmetric situation like that where spins are all up (i.e., aligned along the field direction) but the fermions can be in four different locations or orbital states, with only one accessible to Alice.

Lemma 2. Let $|\psi\rangle_f$ be a general definite-number-parity fermion state in an SP space $\mathcal{H}$ of dimension 4, and let $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B$ be an arbitrary decomposition of $\mathcal{H}$ with $\mathcal{H}_A$ and $\mathcal{H}_B$ of finite dimension. The entanglement entropy associated with this bipartition satisfies

$$S(\rho_A) = S(\rho_B) \geqslant \tfrac{1}{4}S(\rho^{\rm qsp}). \quad (29)$$

Equation (29) holds for *any* entropic form $S(\rho) = {\rm Tr}[f(\rho)]$ [f concave, $f(0) = f(1) = 0$].

Hence, the fermionic entanglement represents the *minimum* bipartite entanglement that can be obtained in such a space, which is reached for those bipartitions arising from the normal form, (26) or (28). The greater entanglement in a $2+2$ bipartition is obtained at the expense of losing a fixed number parity in the local reduced states. Note that $S(\rho^{\rm qsp})$ vanishes only if $|\psi\rangle_f$ is a quasiparticle vacuum or SD in some SP basis, while $S(\rho_{A(B)})$ does so only when the previous condition holds in a basis compatible with the chosen bipartition.

We actually show the equivalent majorization [34] relation

$$\lambda(\rho_{A(B)}) \prec (f_+, f_-), \quad (30)$$

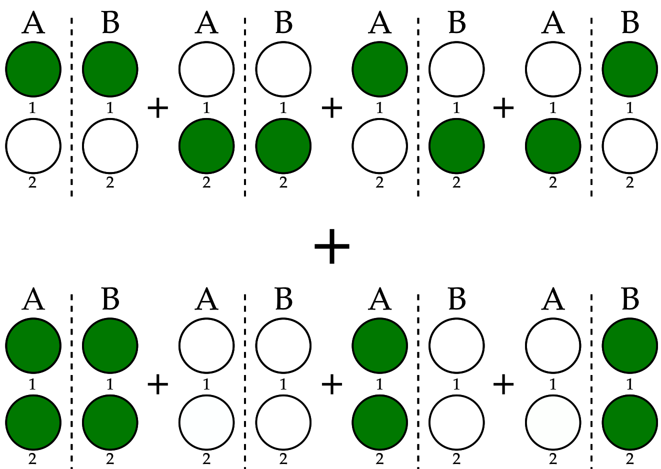

FIG. 1. Depiction of the eight even-number-parity fermion states of four single-particle modes, partitioned such that the two modes on the left of the dashed line belong to Alice, and the two modes on the right to Bob. In the upper row are the states with odd local number parity (one fermion for Alice and one fermion for Bob); in the bottom row, those with even local number parity (Alice and Bob may have zero or two fermions). The states in the bottom row can be formally obtained from those in the top row by performing the particle-hole transformations $c^\dagger_{A2} \leftrightarrow c_{A2}$ and $c^\dagger_{B2} \leftrightarrow c_{B2}$.

where $\lambda(\rho_{A(B)})$ denotes the spectrum of ρ_A or ρ_B sorted in decreasing order and f_+, $f_- = 1 - f_+ \leqslant f_+$ are the distinct eigenvalues, (6) (fourfold degenerate), of $\rho^{\rm qsp}$. Equation (30) is then equivalent to the condition $\lambda_{\max} \leqslant f_+$, with $\lambda_{\max}$ the largest eigenvalue of $\rho_{A(B)}$, and implies (29), while (29) implies (30) if valid for any entropic function f [35].

Proof. Consider first a general even-parity state, (7), and a $2+2$ decomposition $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B$, with $\mathcal{H}_A \equiv \mathcal{H}_{12}$, $\mathcal{H}_B \equiv \mathcal{H}_{34}$, and $\mathcal{H}_{ij}$ the subspace generated by $\{c^\dagger_i, c^\dagger_j\}$. Changing to the notation A_1, A_2, B_1, and B_2 for SP states 1, 2, 3, and 4, we can rewrite (7) as the sum of states of the form of (14) and (20) (Fig. 1),

$$|\psi\rangle_f = \sum_{\mu,\nu} \beta_{\mu\nu} c^\dagger_{A_\mu} c^\dagger_{B_\nu} |0\rangle + \sum_{\mu,\nu} \tilde{\beta}_{\mu\nu} \left(c^\dagger_{A_1} c^\dagger_{A_2}\right)^{n_\mu} \left(c^\dagger_{B_1} c^\dagger_{B_2}\right)^{n_\nu} |0\rangle, \tag{31}$$

where $\mu,\nu = 1,2$, $\beta_{\mu\nu} = \alpha_{\mu,\nu+2}$, $n_\mu = \mu - 1$, $\tilde{\beta}_{11} = \alpha_0$, $\tilde{\beta}_{22} = \alpha_4$, $\tilde{\beta}_{12} = \alpha_{34}$, and $\tilde{\beta}_{21} = \alpha_{12}$. The first (second) sum in (31) is the odd (even) local number-parity component.

After local unitary transformations $c_{S\mu} \to \sum_\nu \bar{U}^S_{\nu\mu} c_{S\nu}$, $S = A,B$, which will not affect the vacuum or the even local parity component (except for phases in $\tilde{\beta}_{\mu\nu}$, determined by $\det U^S$), we can set $\beta_{\mu\nu}$ diagonal. Similarly, after local Bogoliubov transformations $c_{S_1} \to u_S c_{S_1} + v_S c^\dagger_{S_2}$, $c_{S_2} \to u_S c_{S_2} - v_S c^\dagger_{S_1}$, $|u_S^2| + |v_S^2| = 1$, with $|0\rangle \to [\prod_{S=A,B} (u_S - v_S c^\dagger_{S_1} c^\dagger_{S_2})]|0\rangle$, we can set $\tilde{\beta}_{\mu\nu}$ diagonal as discussed in Sec. II C. Though modifying the vacuum, they will not change the form of the odd-local-parity component except for phases in $\beta_{\mu\nu}$. Thus, by local transformations it is possible to rewrite (31) as

$$|\psi\rangle_f = \left(\beta_1 c^\dagger_{A_1} c^\dagger_{B_1} + \beta_2 c^\dagger_{A_2} c^\dagger_{B_2} + \tilde{\beta}_1 + \tilde{\beta}_2 c^\dagger_{A_1} c^\dagger_{A_2} c^\dagger_{B_1} c^\dagger_{B_2}\right)|0\rangle, \tag{32}$$

where $|\beta_\mu|$ and $|\tilde{\beta}_\mu|$ are the singular values of the 2×2 matrices β and $\tilde{\beta}$ in (31). Equation (32) is the Schmidt decomposition for this partition, with $(|\beta_1^2|, |\beta_2^2|, |\tilde{\beta}_1^2|, |\tilde{\beta}_2^2|)$ the eigenvalues of the reduced density matrices ρ_A and ρ_B of modes (A_1, A_2) and (B_1, B_2), respectively.

Now, suppose that $\lambda_{\max} = |\beta_1^2|$. We have

$$|\beta_1|^2 \leqslant |\beta_1|^2 + |\tilde{\beta}_2|^2 = \langle c^\dagger_{A_1} c_{A_1}\rangle. \tag{33}$$

But $\langle c^\dagger_{A_1} c_{A_1}\rangle = \sum_{k=1}^8 |W_{A_1,k}|^2 f_k$, where f_k are the eigenvalues of $\rho^{\rm qsp}$ (equal to f_+ or f_-) and W is the unitary matrix diagonalizing $\rho^{\rm qsp}$ ($\sum_{k=1}^8 |W_{A_1,k}|^2 = 1$). Therefore,

$$f_- \leqslant \langle c^\dagger_{A_1} c_{A_1}\rangle \leqslant f_+. \tag{34}$$

Equations (33) and (34) imply $|\beta_1|^2 \leqslant f_+$, which demonstrates Eq. (30) and hence (29) for a general $2+2$ bipartition $\mathcal{H}_A \oplus \mathcal{H}_B$. For $\lambda_{\max}$ equal to any other coefficient the proof is similar.

Moreover, Eq. (34) also shows that the sorted spectrum $\lambda(\rho_{A_1(A_2,B)}) = (\langle c^\dagger_{A_1} c_{A_1}\rangle, 1 - \langle c^\dagger_{A_1} c_{A_1}\rangle)^\downarrow$ associated with the $1+3$ bipartition $\mathcal{H}_{A_1} \oplus \mathcal{H}_{A_2,B}$ satisfies $\lambda(\rho_{A_1,(A_2,B)}) \prec (f_+, f_-)$. In the latter $S(\rho_{A_1})$ is the entanglement between the SP mode A_1 and its orthogonal complement as defined in [24] and [26], determined by the average occupation $\langle c^\dagger_{A_1} c_{A_1}\rangle$ of the mode. Hence, Eqs. (29) and (30) hold as well for any $1+3$ bipartition.

And equality in (29) for all entropic functions is evidently reached only for those bipartitions arising from the normal forms (26)–(28): Considering the nontrivial case $f_+ < 1$, if equality in (29) is to hold for all entropies, necessarily $\rho_{A(B)}$ should be of rank 2 with $\lambda(\rho_{A(B)}) = (f_+, f_-)$. For a $1+3$ bipartition, this identity directly implies $\langle c^\dagger_{A_1} c_{A_1}\rangle = f_+$ or f_- and hence a bipartition arising from a normal form (26)–(28), where $A \equiv A_1$ is one of the SP states of the normal basis. And for a $2+2$ bipartition, it implies that the two eigenstates of ρ_A with nonzero eigenvalues $f_\pm$ should have the same number parity, since otherwise Eq. (8) would imply $C(|\psi\rangle_f) = 0$ and therefore $f_+ = 1$, in contrast with the assumption. Hence this bipartition must arise from a normal form, (26) or (28).

The demonstration of previous results for odd-global-number-parity states is similar, as they can be rewritten as even-parity states after a particle-hole transformation. ■

Some further comments are also in order. We may rewrite state (32) as

$$|\psi\rangle_f = \sqrt{p_-}|\psi_-\rangle_f + \sqrt{p_+}|\psi_+\rangle_f, \tag{35}$$

where $|\psi_-\rangle_f = \frac{1}{\sqrt{p_-}}(\beta_1 c^\dagger_{A_1} c^\dagger_{B_1} + \beta_2 c^\dagger_{A_2} c^\dagger_{B_2})|0\rangle$, $|\psi_+\rangle_f = \frac{1}{\sqrt{p_+}}(\tilde{\beta}_1 + \tilde{\beta}_2 c^\dagger_{A_1} c^\dagger_{A_2} c^\dagger_{B_1} c^\dagger_{B_2})|0\rangle$ are the normalized odd- and even-local-parity components, and $p_- = |\beta_1^2| + |\beta_2^2|$, $p_+ = |\tilde{\beta}_1^2| + |\tilde{\beta}_2^2| = 1 - p_-$. We then see that for the von Neumann entropy, we obtain

$$S(\rho_A) = S(\rho_B) = p_- S(\rho_A^-) + p_+ S(\rho_A^+) + S(p), \tag{36}$$

where the first two terms represent the average of the entanglement entropies of the odd- and even-local-parity components $[S(\rho_A^-) = -\sum_\mu \frac{|\beta_\mu^2|}{p_-} \log_2 \frac{|\beta_\mu^2|}{p_-}$, $S(\rho_A^+) = -\sum_\mu \frac{|\tilde{\beta}_\mu^2|}{p_+} \log_2 \frac{|\tilde{\beta}_\mu^2|}{p_+}]$ while $S(p) = -\sum_{\nu=\pm} p_\nu \log_2 p_\nu$ is the additional entropy arising from the mixture of both local parities. We then have $0 \leqslant S(\rho_A) \leqslant 2$, with the maximum $S(\rho_A) = 2$ reached iff $S(\rho_A^\pm) = 1$ and $p_\pm = \frac{1}{2}$.

On the other hand, the fermionic concurrence, (8), of state (32) is just

$$C(|\psi\rangle_f) = 2|\beta_1\beta_2 + \tilde{\beta}_1\tilde{\beta}_2|. \qquad (37)$$

It then satisfies

$$|p_-C_- - p_+C_+| \leqslant C(|\psi\rangle_f) \leqslant p_-C_- + p_+C_+, \qquad (38)$$

where $C_\pm = C(|\psi_\pm\rangle_f) = 2(^{|\tilde{\beta}_1\tilde{\beta}_2|/p_+}_{|\beta_1\beta_2|/p_-})$ are the concurrences of the even- and odd-local-parity components. We then see, for instance, that for *maximum* bipartite entanglement $S(\rho_A) = 2$, $C_\pm = 1$ and hence $C(|\psi\rangle_f)$ can take any value between 0 and 1, according to the relative phase between the even- and the odd-local-parity components.

Finally, it is obviously possible to rewrite the Schmidt form, (32), as a two-fermion state by means of suitable local particle-hole transformations (i.e., $c_{B_\mu} \to c^\dagger_{B_\mu}$, $\mu = 1,2$, with $|0\rangle \to c^\dagger_{B_1}c^\dagger_{B_2}|0\rangle$). After some relabeling, we obtain the equivalent form

$$|\psi\rangle_f = \left(\beta_1 c^\dagger_{A_1}c^\dagger_{B_1} + \beta_2 c^\dagger_{A_2}c^\dagger_{B_2} + \tilde{\beta}_2 c^\dagger_{A_1}c^\dagger_{A_2} - \tilde{\beta}_1 c^\dagger_{B_1}c^\dagger_{B_2}\right)|0\rangle, \qquad (39)$$

where terms with two fermions on the same side side are added to the form (10). Therefore, all previous considerations, (35)–(38), can be realized with a fixed total number of fermions, with expression (37) still valid.

III. APPLICATION

The formalism of the previous sections may now be used to rewrite a qubit-based quantum circuit as a circuit based on fermionic modes. It is easy to see by now that any pair of fermionic modes, say i,j, prepared in such a way that their total occupation is constrained to $N_{ij} = c^\dagger_i c_i + c^\dagger_j c_j = 1$, is essentially a qubit. Therefore, a collection of n such pairs of modes constitutes a system of n qubits. Furthermore any single-qubit operation can be performed on each pair just by using unitaries in $\mathcal{H}$ linking only these two modes, and these unitaries can always be written in terms of the effective Pauli operators, (11), i.e., $\sigma^{ij}_x = c^\dagger_i c_j + c^\dagger_j c_i$, $\sigma^{ij}_y = i(c^\dagger_j c_i - c^\dagger_i c_j)$, $\sigma^{ij}_z = c^\dagger_i c_i - c^\dagger_j c_j$. The last ingredient for universal computation is the controlled-NOT (CNOT) gate, which, in the tensor product space $A \otimes B$, can be written as $U_{\text{CNOT}} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \sigma_x = \exp[-i\frac{\pi}{4}(1-\sigma_z)\otimes(\sigma_x - 1)]$. In the fermionic representation, if A is spanned by modes ij and B by the different modes kl, for states having one fermion at each pair of modes it can be written as

$$U^f_{\text{CNOT}} = \exp\left[-i\frac{\pi}{4}\left(1-\sigma^{ij}_z\right)\left(\sigma^{kl}_x - 1\right)\right]. \qquad (40)$$

Since just an even number of fermion operators c per pair is involved, its action is not affected by the state of intermediate pairs. It is then possible to implement any qubit-based quantum circuit using fermion states.

As an example, in Fig. 2 we show the teleportation protocol adapted to be implemented using an entangled fermion state as resource and a two-mode state to be teleported. Alice has the modes $\{|A_1\rangle,|A_2\rangle,|A_3\rangle,|A_4\rangle\}$, while Bob is in possession of $\{|B_1\rangle,|B_2\rangle\}$. The first two modes of Alice are entangled with those of Bob, being in the joint state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(c^\dagger_{A_1}c^\dagger_{B_1} + c^\dagger_{A_2}c^\dagger_{B_2})|0\rangle$, and the remaining modes of Alice are in the state $|\psi\rangle = (\alpha\, c^\dagger_{A_3} + \beta\, c^\dagger_{A_4})|0\rangle$, $|\alpha|^2 + |\beta^2| = 1$. The input state is therefore

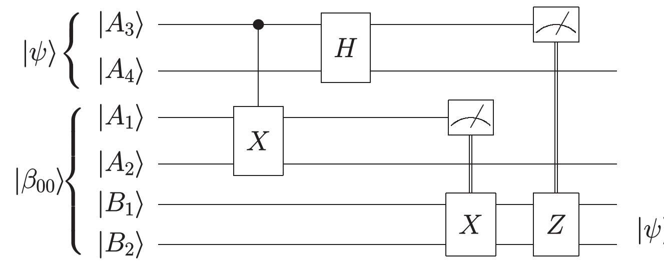

FIG. 2. Teleportation protocol with the present fermionic implementation. Each qubit is represented by a pair of fermionic modes having a total occupation number of 1. The control operation can be realized involving just one of the modes of the pair representing the control qubit due to the occupation number constraint, acting when it is occupied, and similarly, the usual measurement in the standard basis can be implemented by measuring just one of these modes. If the pair occupation number constraint is relaxed so that both local number parities coexist, then control and measurement operations involve both modes.

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}\left(\alpha\, c^\dagger_{A_3} + \beta\, c^\dagger_{A_4}\right)\left(c^\dagger_{A_1}c^\dagger_{B_1} + c^\dagger_{A_2}c^\dagger_{B_2}\right)|0\rangle$$

and it is straightforward to see that the output state is

$$\begin{aligned}|\psi_o\rangle = \tfrac{1}{2}\big[&c^\dagger_{A_4}c^\dagger_{A_2}\left(\alpha\, c^\dagger_{B_1} + \beta\, c^\dagger_{B_2}\right) + c^\dagger_{A_4}c^\dagger_{A_1}\left(\alpha\, c^\dagger_{B_2} + \beta\, c^\dagger_{B_1}\right)\\ &+ c^\dagger_{A_3}c^\dagger_{A_2}\left(-\alpha\, c^\dagger_{B_1} + \beta\, c^\dagger_{B_2}\right)\\ &+ c^\dagger_{A_3}c^\dagger_{A_1}\left(-\alpha\, c^\dagger_{B_2} + \beta\, c^\dagger_{B_1}\right)\big]|0\rangle.\end{aligned}$$

The controlled operations on Bob's modes depicted in Fig. 2 then ensure that his output will be the state $|\psi\rangle$.

Considering now a general circuit, if the input states are restricted to be SDs in the previous basis, with one fermion for each pair, we recover a classical circuit. The CNOT gate in (40) reduces for these states to a classical *controlled swap* or Fredkin gate, which implies that reversible classical computation can be done with SDs as input states.

On the other hand, if the occupation number restriction $N_{ij} = 1$ (i.e., odd number parity for each pair) is relaxed, so that the building blocks of the circuit are no longer single fermions that can be found in two possible states, but rather the fermionic modes themselves, other possibilities arise. For instance, if now the input states contain either zero or two fermions for each pair (even-number-parity qubits), such that modes i,j are either both empty or both occupied, then we should use the $\tilde{\sigma}^{ij}_\mu$ operators as defined in (19), i.e., $\tilde{\sigma}^{ij}_x = c^\dagger_i c^\dagger_j + c_j c_i$, $\tilde{\sigma}^{ij}_y = i(c_j c_i - c^\dagger_i c^\dagger_j)$, $\tilde{\sigma}^{ij}_z = c^\dagger_i c_i + c^\dagger_j c_j - 1$. In this case the operator $\tilde{U}^f_{\text{CNOT}}$ should be constructed as in Eq. (40) with the $\tilde{\sigma}_\mu$ operators, while the operator, (40), and in fact any unitary gate built with the σ^{ij}_μ operators, will become an identity for these states, as previously stated. Hence, by adding the appropriate gates, the same modes can in principle be used for even- and odd-number-parity qubits independently.

For example, in the even-local-parity setting the input state for the teleportation protocol would be

$$|\tilde{\psi}_i\rangle = \frac{1}{\sqrt{2}}\big(\beta + \alpha\, c^\dagger_{A_3} c^\dagger_{A_4}\big)\big(1 + c^\dagger_{A_1} c^\dagger_{A_2} c^\dagger_{B_1} c^\dagger_{B_2}\big)|0\rangle.$$

If $|0\rangle$ stands for a reference SD (Fermi sea), then this state involves zero, one, two, and three particle-hole excitations, with A_4, A_2, and B_2, standing for holes. The output state becomes

$$\begin{aligned}|\tilde{\psi}_o\rangle = \tfrac{1}{2}\big[\big(\beta + \alpha\, c^\dagger_{B_1} c^\dagger_{B_2}\big) + c^\dagger_{A_1} c^\dagger_{A_2}\big(\alpha + \beta\, c^\dagger_{B_1} c^\dagger_{B_2}\big) + c^\dagger_{A_3} c^\dagger_{A_4}\\ \times\big(\beta - \alpha\, c^\dagger_{B_1} c^\dagger_{B_2}\big)\\ + c^\dagger_{A_1} c^\dagger_{A_2} c^\dagger_{A_3} c^\dagger_{A_4}\big(-\alpha + \beta\, c^\dagger_{B_1} c^\dagger_{B_2}\big)\big]|0\rangle,\end{aligned}$$

so that if Alice measures which of her modes are occupied and sends the result to Bob, he can reconstruct the original state by applying the pertinent $\tilde{X} \equiv i e^{-i\frac{\pi}{2}\tilde{\sigma}_x^{12}}$ and $\tilde{Z} \equiv -i e^{-i\frac{\pi}{2}\tilde{\sigma}_z^{12}}$ operators.

Finally, let us consider the case of *superdense coding* [4,36]. It is clear from the previous discussion that it can be implemented with the fermionic $|\beta_{00}\rangle$ state of the teleportation example and performing exactly the same local operations as in the usual case, but now viewed as two-mode operations. Now a general state with even global parity of the four modes $\{|A_1\rangle,|A_2\rangle,|B_1\rangle,|B_2\rangle\}$ is a combination of eight states as in Eq. (7): six two-particle states, the vacuum $|0\rangle$, and the completely occupied state $|\bar{0}\rangle$, as shown in Fig. 1. Four of the six two-particle states (Fig. 1; top) have $N_A = N_B = 1$ and can be used to reproduce the known results of the standard protocol. But the four remaining states, which have even local parity, may be used as well for superdense coding if the proper local operations expressed in terms of the $\tilde{\sigma}_\mu^{AB}$ are performed.

A general even-parity state, (7), may then be thought of as a superposition of states of two different two-qubit systems, as in Eqs. (31) and (35). Defining the maximally entangled orthogonal definite-local-parity states

$$\big|\beta_{\substack{00\\10}}\big\rangle = \frac{1}{\sqrt{2}}\big(c^\dagger_{A_1} c^\dagger_{B_1} \pm c^\dagger_{A_2} c^\dagger_{B_2}\big)|0\rangle,$$

$$\big|\tilde{\beta}_{\substack{00\\10}}\big\rangle = \frac{1}{\sqrt{2}}\big(\pm 1 + c^\dagger_{A_1} c^\dagger_{A_2} c^\dagger_{B_1} c^\dagger_{B_2}\big)|0\rangle, \quad (41)$$

$$\big|\beta_{\substack{01\\11}}\big\rangle = \frac{1}{\sqrt{2}}\big(c^\dagger_{A_1} c^\dagger_{B_2} \pm c^\dagger_{A_2} c^\dagger_{B_1}\big)|0\rangle,$$

$$\big|\tilde{\beta}_{\substack{01\\11}}\big\rangle = \frac{1}{\sqrt{2}}\big(c^\dagger_{A_1} c^\dagger_{A_2} \pm c^\dagger_{B_1} c^\dagger_{B_2}\big)|0\rangle, \quad (42)$$

we may consider, for instance, the state

$$|\Psi_{00}\rangle = \tfrac{1}{\sqrt{2}}(|\beta_{00}\rangle + |\tilde{\beta}_{00}\rangle). \quad (43)$$

By implementing on (43) the identity and the local operations $i e^{-i\frac{\pi}{2}(\sigma_\mu^A + \tilde{\sigma}_\mu^A)} = \sigma_\mu + \tilde{\sigma}_\mu$, $\mu = x,y,z$, and taking into account Eq. (24), Alice can generate four orthogonal states: $|\Psi_{00}\rangle$ and

$$|\Psi_{01}\rangle = i e^{-i\frac{\pi}{2}(\sigma_x^A + \tilde{\sigma}_x^A)}|\Psi_{00}\rangle = \tfrac{1}{\sqrt{2}}(|\beta_{01}\rangle + |\tilde{\beta}_{01}\rangle), \quad (44a)$$

$$|\Psi_{10}\rangle = i e^{-i\frac{\pi}{2}(\sigma_z^A + \tilde{\sigma}_z^A)}|\Psi_{00}\rangle = \tfrac{1}{\sqrt{2}}(|\beta_{10}\rangle + |\tilde{\beta}_{10}\rangle), \quad (44b)$$

$$|\Psi_{11}\rangle = -e^{-i\frac{\pi}{2}(\sigma_y^A + \tilde{\sigma}_y^A)}|\Psi_{00}\rangle = \tfrac{1}{\sqrt{2}}(|\beta_{11}\rangle + |\tilde{\beta}_{11}\rangle). \quad (44c)$$

But she can also perform these operations with a local parity gate $P^A = -\exp[i\pi N_A]$ that changes the sign of local even-parity states. This allows her to locally generate another set of four orthogonal states,

$$|\tilde{\Psi}_{ij}\rangle = P^A|\Psi_{ij}\rangle = \frac{1}{\sqrt{2}}(|\beta_{ij}\rangle - |\tilde{\beta}_{ij}\rangle), \quad i,j = 0,1, \quad (45)$$

which are orthogonal to each other and to states (43) and (44). Hence, by relaxing the occupation number constraint on the partitions it is possible for Alice to send eight orthogonal states to Bob, i.e., three bits of information, using only two modes and local unitary operations that preserve the local parity, while with one type of qubits and the same operations she can send only two bits. Of course, if parity restrictions were absent and she could change the local (and hence the global) parity, she could send four bits (in agreement with the maximum capacity for two $d = 4$ qudits, which is $\log_2 d^2$ [36]). A fixed-global-parity constraint reduces the total number of orthogonal states she can send to Bob by half.

On the other hand, since state (43) does not have a definite local number parity, the ensuing bipartite entanglement is not restricted by the fermionic entanglement as shown in Sec. II F. In fact all eight previous states, (43), (44), and (45), have maximum bipartite entanglement, leading to maximally mixed reduced states $\rho_{A(B)}$, $S(\rho_A) = S(\rho_B) = 2$, while by applying Eq. (8) it is seen that the fermionic concurrence of the previous states is $C(|\Psi_{ij}\rangle) = C(|\tilde{\Psi}_{ij}\rangle) = 1$. The unitary operations applied by Alice are local and hence cannot change the bipartite entanglement, while they are also one-body unitaries (i.e., exponents of quadratic fermion operators) so that they cannot change the fermionic concurrence and entanglement (i.e., the eigenvalues of $\rho^{\rm qsp}$) either. In fact, the fermionic entanglement is not required here. By changing the seed state (i.e., $|\Psi'_{00}\rangle = \frac{1}{\sqrt{2}}(|\beta_{00}\rangle + |\tilde{\beta}_{10}\rangle)$), it is possible for Alice to generate locally eight orthogonal states with the same bipartite entanglement yet no fermion entanglement $[C(|\Psi'_{00}\rangle) = 0]$.

Therefore the entanglement built with local states with different number parities plays the role of a resource for superdense coding. In fact even state (25) with $\alpha = \beta = \frac{1}{\sqrt{2}}$, which obviously has null concurrence, can in principle be used for sending two bits if Alice can perform the parity preserving operations $P^A = -\exp[i\pi N_A]$, $\sigma_x + \tilde{\sigma}_x$ and $P^A(\sigma_x + \tilde{\sigma}_x)$. It is noteworthy, however, that the same state cannot be directly used as a resource for teleportation with the standard protocol without violating the parity superselection rule, since Bob's two local states have opposite parity and cannot be superposed. After a measurement of Alice's modes Bob's reduced state will collapse to a state of definite parity in a realizable protocol, so that it will be impossible for him to recover a general state $|\psi\rangle$.

We have so far considered just the number-parity restriction. If other superselection rules (like charge or fermion number) also apply for a particular realization, they will imply stronger limitations on the capacity of states like (43). Nonetheless, even local-parity qubits with no fixed fermion number remain realizable through particle-hole realizations, i.e., excitations over a reference Fermi sea in a many-fermion system.

We also mention that a basic realization of four-dimensional SP space–based fermionic qubits is that of a pair of spin-1/2 fermions in the two lowest states of a double-well scenario in a magnetic field, which would control the energy gap between the two spin directions and the transitions between them. For single occupation of each well we would have odd-local-parity qubits, while allowing double or zero occupancy through hopping between wells, we could also have even-local-parity qubits.

IV. CONCLUSIONS

We have first shown that there is an exact correspondence between bipartite states and two-fermion states of the form of (14) having a fixed local number parity. Entangled states are represented by fermionic states which are not Slater determinants, and reduced local states correspond to blocks of the SP density matrix. In particular, qubits can be represented by pairs of fermionic modes with the occupation number restricted to 1 (odd-number-parity qubits). This result allows us to rewrite qubit-based quantum circuits as fermionic circuits. But in addition, a fermionic system also enables zero or double occupancy of these pairs, which gives rise to a second type of qubit (the even-number-parity qubit). Dual-type circuits can then be devised, as the gates for one parity become identities for the other parity. And even though both types of qubits cannot be locally superposed due to the parity superselection rule, they can contribute to the entanglement in a global fixed-parity state.

We have then demonstrated rigorous properties of the basic but fundamental case of a four-dimensional SP space. First, there is always a single-particle (or quasiparticle) basis in which any pure state can be seen as a state of two distinguishable qubits, with the fermionic concurrence determining the entanglement between these two qubits [Eq. (22)]. Such entanglement is "genuine," in the sense that the local states involved have a definite parity and can therefore be combined. Second, such fermionic entanglement was shown always to provide a lower bound to the entanglement obtained with any other bipartition of this SP space, although the extra entanglement arises from the superposition of states with different local parities. While its capacity for protocols involving superpositions of local states is limited, such entanglement can still be useful for other tasks such as superdense coding.

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[1] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[3] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).
[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[5] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[6] R. Josza and N. Linden, Proc. R. Soc. London, Sec. A **459**, 2011 (2003); G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003).
[7] J. Schliemann, J. I. Cirac, M. Kuś, M. Lewenstein, and D. Loss, Phys. Rev. A **64**, 022303 (2001).
[8] J. Schliemann, D. Loss, and A. H. MacDonald, Phys. Rev. B **63**, 085311 (2001).
[9] K. Eckert, J. Schliemann, D. Bruß, and M. Lewenstein, Ann. Phys. **299**, 88 (2002).
[10] H. M. Wiseman and J. A. Vaccaro, Phys. Rev. Lett. **91**, 097902 (2003).
[11] G. C. Ghirardi and L. Marinatto, Phys. Rev. A **70**, 012109 (2004).
[12] C. V. Kraus, M. M. Wolf, J. I. Cirac, and G. Giedke, Phys. Rev. A **79**, 012306 (2009).
[13] P. Zanardi, Phys. Rev. A **65**, 042101 (2002).
[14] Y. Shi, Phys. Rev. A **67**, 024301 (2003).
[15] N. Friis, A. R. Lee, and D. E. Bruschi, Phys. Rev. A **87**, 022338 (2013).
[16] F. Benatti, R. Floreanini, and U. Marzolino, Phys. Rev. A **89**, 032326 (2014).
[17] X. M. Puspus, K. H. Villegas, and F. N. C. Paraan, Phys. Rev. B **90**, 155123 (2014).
[18] A. R. Plastino, D. Manzano, and J. Dehesa, Europhys. Lett. **86**, 20005 (2009).
[19] F. Iemini and R. O. Vianna, Phys. Rev. A **87**, 022327 (2013); F. Iemini, T. O. Maciel, and R. O. Vianna, Phys. Rev. B **92**, 075423 (2015); T. Debarba, R. O. Vianna, and F. Iemini, Phys. Rev. A **95**, 022325 (2017).
[20] M. Oszmaniec, J. Gutt, and M. Kuś, Phys. Rev. A **90**, 020302(R) (2014).
[21] M. Oszmaniec and M. Kuś, Phys. Rev. A **90**, 010302(R) (2014); **88**, 052328 (2013).
[22] G. Sárosi and P. Lévay, J. Phys. A **47**, 115304 (2014); Phys. Rev. A **90**, 052303 (2014); **89**, 042310 (2014).
[23] F. Iemini, T. Debarba, and R. O. Vianna, Phys. Rev. A **89**, 032324 (2014).
[24] N. Gigena and R. Rossignoli, Phys. Rev. A **92**, 042326 (2015).
[25] A. P. Majtey, P. A. Bouvrie, A. Valdés-Hernández, and A. R. Plastino, Phys. Rev. A **93**, 032335 (2016).
[26] N. Gigena and R. Rossignoli, Phys. Rev. A **94**, 042315 (2016).
[27] N. Friis, New J. Phys. **18**, 033014 (2016).
[28] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, Berlin, 1980).
[29] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997); W. K. Wootters, *ibid.* **80**, 2245 (1998).
[30] D. Dasenbrook, J. Bowles, J. Bohr Brask, P. P. Hofer, C. Flindt, and N. Brunner, New J. Phys. **18**, 043036 (2016).
[31] N. Canosa and R. Rossignoli, Phys. Rev. Lett. **88**, 170401 (2002); N. Gigena and R. Rossignoli, Phys. Rev. A **90**, 042318 (2014).
[32] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[33] R. Rossignoli and A. Plastino, Phys. Rev. C **32**, 1040 (1985).
[34] R. Bhatia, *Matrix Analysis* (Springer, New York, 1997).
[35] R. Rossignoli and N. Canosa, Phys. Rev. A **67**, 042302 (2003).
[36] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters, Phys. Rev. A **54**, 1869 (1996).