Quantum resource theories in the single-shot regime

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One of the main goals of any resource theory such as entanglement, quantum thermodynamics, quantum coherence, and asymmetry, is to find necessary and sufficient conditions that determine whether one resource can be converted to another by the set of free operations. Here we find such conditions for a large class of quantum resource theories which we call *affine* resource theories. Affine resource theories include the resource theories of athermality, asymmetry, and coherence, but not entanglement. Remarkably, the necessary and sufficient conditions can be expressed as a family of inequalities between resource monotones (quantifiers) that are given in terms of the conditional min-entropy. The set of free operations is taken to be (1) the maximal set (i.e., consists of all resource nongenerating quantum channels) or (2) the self-dual set of free operations (i.e., consists of all resource nongenerating maps for which the dual map is also resource nongenerating). As an example, we apply our results to quantum thermodynamics with Gibbs preserving operations, and several other affine resource theories. Finally, we discuss the applications of these results to resource theories that are not affine and, along the way, provide the necessary and sufficient conditions that a quantum resource theory consists of a resource destroying map.

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I. INTRODUCTION

A few of the key hallmarks of quantum information science are characterized with the recognition that certain properties of quantum systems, such as entanglement, can be viewed as resources for quantum information processing tasks [1–3]. These realizations have initially sparked the development of entanglement theory [4,5], and later on the development of other quantum resource theories (QRTs) [6–8]. Today, in addition to entanglement, QRTs provides an ideal platform to study many properties of quantum systems including (but not limited to) athermality [9–16], asymmetry [6,17–20], coherence [21–27], contextuality [28,29], non-Markovianity [30], knowledge [31], and incompatibility [32]. Other properties, such as non-Gaussianity (see, e.g., Ref. [33]) or total correlations (e.g., mutual information) between two parties, can also be formulated in the framework of resource theories. However, not much work has been done on them, since these models are lacking certain convenient mathematical properties, such as convexity.

All QRTs have in common three ingredients: free states, free operations, and quantum resources. These components are not independent of each other, because with free operations alone it is not possible to convert free states into resource states. This general structure suggests the existence of general theorems that can be applied to a large class of QRTs. Indeed, recently such a theorem was proved in Ref. [34], showing that many QRTs are asymptotically reversible if the set of free operations is maximal (i.e., consists of all possible operations that cannot generate a resource from free states).

In the single-copy regime, where the law of large numbers does not apply, there are no known such theorems that can be applied to all QRTs. This is, in part, due to the fact that the set of free states and free operations can be very different from one QRT to another. Even the asymptotic reversibility result of Ref. [34] holds only if the set of free states satisfy certain conditions and the set of free operations is maximal. Therefore, in order to better understand QRTs, it is essential to classify them according to some general properties that add an additional structure and then obtain general theorems that apply to QRTs with this additional structure.

In this paper, we consider one of the core problems of any QRT in the single-shot regime: given two resource states ρ and ρ', what are the necessary and sufficient conditions (NSCs) that determine whether it is possible to convert ρ to ρ' by *free* quantum operations? We answer this question for QRTs with the property that any density matrix that can be expressed as an *affine* combination of free states is itself a free state. We call such QRTs *affine* resource theories (ARTs). We show that QRTs of athermality, asymmetry, and coherence are all ARTs, while entanglement theory is not an ART. Remarkably, our NSC can be expressed in terms of resource monotones (i.e., functions from the set of density matrices to the nonnegative real numbers that behave monotonically under free operations). Specifically, we find that

$$\rho \xrightarrow{\substack{\text{free}\\ \text{operations}}} \rho'$$

if and only if for any $t \in [0,1]$, and any density matrix η,

$$R_{\eta,t}(\rho) \leqslant R_{\eta,t}(\rho'), \tag{1}$$

where $R_{\eta,t}$ are functions on the set of density matrices that are given in terms of the conditional min-entropy [35–40] of a certain mixture of $\eta \otimes \rho$ with another separable state (see Definition 2 for the precise definition of $R_{\eta,t}$).

Our results can be applied to two sets of free operations: (1) The maximal set of all resource nongeneration (RNG) maps (quantum channels), and (2) the set consisting of all RNG maps with a dual map that is also RNG (for example, in the QRT of coherence, this is the set of all dephasing covariant operations [23,24]). We discuss the applications of our results particularly to the QRT of thermodynamics with Gibbs preserving operations, and to quantum coherence with maximal operations or dephasing covariant operations [23,24]. In addition, we show that QRTs with a resource destroying map

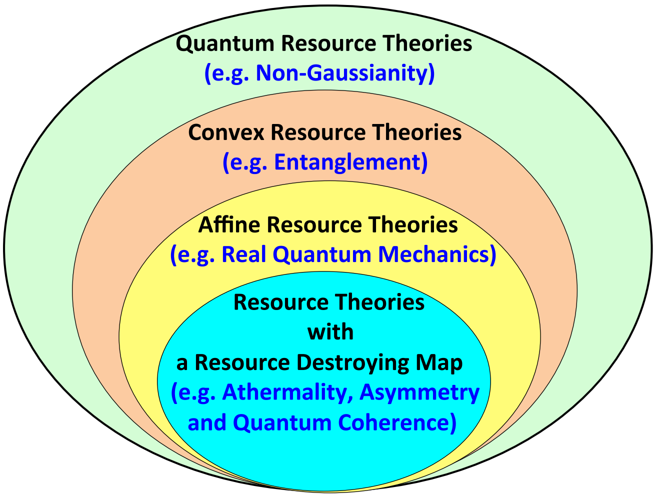

FIG. 1. An heuristic diagram of QRTs, classified according to the properties of their set of free states. Non-Gaussianity is an example of a QRT with a nonconvex set of free states. Entanglement theory is an example of a QRT that is convex but not affine. Real (vs complex) quantum mechanics (see below) is an example of an affine QRT that does not have a RDM, and athermality, asymmetry, and coherence are examples of QRTs with a RDM.

(RDM) [41] form a strict subset of ARTs (see Fig. 1), and we provide the NSC that a QRT consists of a RDM.

This paper is organized as follows: In Sec. II we define and discuss the properties of affine resource theories. In Sec. III we present and prove the main theorem of this paper, which is applicable to affine resource theory. Then, in Sec. IV we apply the main theorem to two important examples; namely, the resource theories of athermality and coherence. In Sec. V we move to discuss ART under a smaller set of operations that we call self-dual RNG operations. Finally, in Sec. VI we study ARTs with a RDM, and provide NSCs that a QRT consists of a RDM. We end with conclusions in Sec. VII.

II. AFFINE RESOURCE THEORIES

Let $\mathcal{H}_d$ be the real vector space of $d\times d$ Hermitian matrices, $\mathcal{H}_{d,+}\subset\mathcal{H}_d$ be the cone of positive-semidefinite matrices, and $\mathcal{H}_{d,+,1}\subset\mathcal{H}_{d,+}$ be the set of all $d\times d$ density matrices. Denote by $\mathcal{R}(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}},\mathcal{O})$ a QRT consisting of input and output free sets $\mathcal{F}_{\text{in}}\subset\mathcal{H}_{d,+,1}$ and $\mathcal{F}_{\text{out}}\subset\mathcal{H}_{d',+,1}$, respectively, and a set of free operations $\mathcal{O}$. The set $\mathcal{O}$ consists all free completely positive and trace preserving (CPTP) maps from the input space $\mathcal{H}_{d,+,1}$ to the output space $\mathcal{H}_{d',+,1}$. By the definition of a QRT, any free operations $\mathcal{E}\in\mathcal{O}$ cannot generate a resource from a free state. Mathematically, if $\sigma\in\mathcal{F}_{\text{in}}$ and $\mathcal{E}\in\mathcal{O}$ then $\mathcal{E}(\sigma)\in\mathcal{F}_{\text{out}}$. We call the set of all such CPTP maps *resource nongenerating* (RNG) operations and denote it by $\mathcal{O}_{\max}$. Note that $\mathcal{O}\subset\mathcal{O}_{\max}$. The main results of this paper can be applied to a class of resource theories that we call *affine* resource theories (ARTs):

Definition 1. A set of quantum states $\mathcal{F}\subset\mathcal{H}_{d,+,1}$ is said to be *affine* if any affine combination of states in $\mathcal{F}$ that is positive semidefinite is itself in $\mathcal{F}$. That is, if

$$\rho=\sum_i t_i\sigma_i\in\mathcal{H}_{d,+,1} \tag{2}$$

for some $\sigma_i\in\mathcal{F}$ and $t_i\in\mathbb{R}$, then $\rho\in\mathcal{F}$. Moreover, a QRT, $\mathcal{R}(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}},\mathcal{O})$, is said to be *affine* if both $\mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$ are affine.

The QRTs of athermality, asymmetry, and coherence are all ARTs. The QRT of athermality is affine since the set of free states contains only the Gibbs state, while the QRT of coherence is affine since the set of free states contains only diagonal elements. On the other hand, entanglement theory is not affine. We know it since the set of bipartite separable states is not of measure zero and in particular contains a ball with the maximally mixed state at its center. Hence, entanglement theory is not an ART and, in fact, it can be viewed as "maximally nonaffine" in the sense that *all* states can be written as an affine combination of free (even pure product) states.

A. Properties of affine sets

The affine condition also implies that $\mathcal{F}$ is convex but, as we show below, convexity of $\mathcal{F}$ does not necessarily imply that $\mathcal{F}$ is affine. Moreover, note that, if $\mathcal{F}$ is affine and $\mathcal{V}\equiv\text{span}_{\mathbb{R}}\{\mathcal{F}\}$ is the subspace of $\mathcal{H}_d$ consisting of all the linear combinations of the elements in $\mathcal{F}$, then the *only* positive semidefinite matrices in $\mathcal{V}$ are the elements of $\mathcal{F}$. Therefore, $\mathcal{F}$ is affine if and only if it satisfies the following condition:

$$\mathcal{F}=\mathcal{V}\cap\mathcal{H}_{d,+,1},\quad \mathcal{V}:=\text{span}_{\mathbb{R}}\{\mathcal{F}_d\}, \tag{3}$$

where $\mathcal{V}$ is the subspace of $\mathcal{H}_d$ consisting of all the linear combinations of the elements in $\mathcal{F}$.

Lemma 1. Let $\mathcal{F}\subset\mathcal{H}_{d,+,1}$ be an affine set with $\mathcal{V}$ as above, with $\dim\mathcal{V}=n$. Then $\mathcal{V}$ has a basis consisting of n density matrices $\sigma_1,\ldots,\sigma_n\in\mathcal{F}$, such that $\mathcal{V}=\text{span}_{\mathbb{R}}\{\sigma_1,\ldots,\sigma_n\}$.

Proof. Let γ be a state in $\mathcal{F}$ with maximal rank. That is, the support space of any state $\sigma\in\mathcal{F}$ is a subspace of the support of γ. Such a state exists since $\mathcal{F}$ is convex. Now, let $X_1,\ldots,X_n\in\mathcal{V}$ be a basis of $\mathcal{V}$. Then, for each $j=1,\ldots,n$, let $t_j>0$ be a small enough number such that $\gamma+t_jX_j\geqslant 0$. Denoting by

$$\sigma_j\equiv\frac{\gamma+t_jX_j}{1+t_j\text{Tr}[X_j]}, \tag{4}$$

we conclude that $\sigma_j\in\mathcal{F}$ since $\mathcal{F}$ is affine, and

$$\text{span}_{\mathbb{R}}\{\sigma_1,\ldots,\sigma_n\}=\text{span}_{\mathbb{R}}\{X_1,\ldots,X_n\}=\mathcal{V}. \tag{5}$$

This completes the proof. ■

Lemma 2. A set $\mathcal{F}\subset\mathcal{H}_{d,+,1}$ is affine if it is convex, and for any pair of distinct free states $\sigma_1,\sigma_2\in\mathcal{F}$ and any $t\in[0,2^{-D_{\max}(\sigma_1\|\sigma_2)}]\subset[0,1]$, there exists a free state $\omega_t\in\mathcal{F}$ such that σ_2 is the convex combination $\sigma_2=t\sigma_1+(1-t)\omega_t$. Here,

$$D_{\max}(\sigma_1\|\sigma_2)=\log_2\min_{\lambda\in\mathbb{R}_+}\{\lambda|\lambda\sigma_2\geqslant\sigma_1\}. \tag{6}$$

Proof. Suppose first that $\mathcal{F}$ is affine. Then, for any distinct $\sigma_1,\sigma_2\in\mathcal{F}$ and $t\in[0,2^{-D_{\max}(\sigma_1\|\sigma_2)}]$, the matrix $\sigma_1-t\sigma_2\geqslant 0$. Hence, since $\mathcal{F}$ is affine the matrix $\omega_t\equiv\frac{\sigma_1-t\sigma_2}{1-t}$ is free.

Conversely, let $\omega=\sum_j s_j\omega_j$ be an affine combination of free states $\omega_j\in\mathcal{F}$, with $\sum_j s_j=1$, and suppose $\omega\geqslant 0$. Then, ω can be written as

$$\begin{aligned}\omega=\sum_j s_j\omega_j&=\sum_{\{j:s_j\geqslant 0\}}s_j\omega_j-\sum_{\{j:s_j\leqslant 0\}}|s_j|\omega_j\\&=(1+s)\sigma_2-s\sigma_1,\end{aligned}\tag{7}$$

where

$$\begin{aligned}s&=\sum_{\{j:s_j\leqslant 0\}}|s_j|\geqslant 0,\\ \sigma_2&\equiv\frac{1}{1+s}\sum_{\{j:s_j\geqslant 0\}}s_j\omega_j,\\ \sigma_1&\equiv\frac{1}{s}\sum_{\{j:s_j\leqslant 0\}}|s_j|\omega_j.\end{aligned}\tag{8}$$

Since σ_1 and σ_2 are given as a convex combination of the free states ω_j they themselves are free. Moreover, since $\omega\geqslant 0$ we must have $t\equiv\frac{s}{1+s}\leqslant 2^{-D_{\max}(\sigma_1\|\sigma_2)}$. Therefore, from the assumption of the lemma, $\omega=(1+s)\sigma_2-s\sigma_1=\frac{\sigma_1-t\sigma_2}{1-t}$ is free. This completes the proof. ■

B. The dual of affine sets

The following notion of *duality* of a set of density matrices plays an important role in ARTs. The *dual* set $\mathcal{F}^\star$ of a set of states $\mathcal{F}\in\mathcal{H}_{d,+,1}$ is defined here as

$$\mathcal{F}^\star\equiv\{\omega\in\mathcal{H}_{d',+,1}|\mathrm{Tr}[\omega\sigma]=\mathrm{Tr}[\omega\sigma']\ \forall\,\sigma,\sigma'\in\mathcal{F}\}.\tag{9}$$

Note that this dual set is affine (and therefore convex) even if $\mathcal{F}$ is not affine, and the maximally mixed state $u_d\equiv\frac{1}{d}I_d\in\mathcal{F}^\star$. In particular, we show now that if $\mathcal{F}$ is affine and $u_d\in\mathcal{F}$ then $\mathcal{F}^{\star\star}=\mathcal{F}$.

Theorem 1. Let $\mathcal{F}\subset\mathcal{H}_{d,+,1}$ be an affine set of density matrices, $\mathcal{V}\equiv\mathrm{span}_{\mathbb{R}}\{\mathcal{F}\}$, and $\mathcal{V}_0\subset\mathcal{V}$ be the subspace of traceless matrices in $\mathcal{V}$.

(1) $\mathcal{F}^\star$ is an affine set and $u_d\in\mathcal{F}^\star$.

(2) If $u_d\in\mathcal{F}$ then $\mathcal{F}^{\star\star}=\mathcal{F}$ (and consequently $\mathcal{F}^{\star\star\star}=\mathcal{F}^\star$ even if $u_d\notin\mathcal{F}$).

(3) If $u_d\notin\mathcal{F}$ then

$$\mathcal{F}^{\star\star}=\{u_d+Y|-u_d\leqslant Y\in\mathcal{V}_0\},$$

and in particular $\mathcal{F}^{\star\star}\cap\mathcal{F}=\emptyset$.

Remark 1. Note that $\mathcal{F}$ can be written as

$$\mathcal{F}=\{\gamma+Y|-\gamma\leqslant Y\in\mathcal{V}_0\},$$

where γ is a state in $\mathcal{F}$ with a maximal rank. Therefore, roughly speaking, $F^{\star\star}$ is a shifted version of $\mathcal{F}$ that contains the maximally mixed state.

Proof 1. Property 1 follows directly from the definitions. We therefore move to prove property 2. Indeed, if $u_d\in\mathcal{F}$ then

$$\mathcal{F}^\star\equiv\left\{\omega\in\mathcal{H}_{d',+,1}|\mathrm{Tr}[\omega\sigma]=\frac{1}{d}\ \forall\,\sigma\in\mathcal{F}\right\},\tag{10}$$

and since we always have $u_d\in\mathcal{F}^\star$ we conclude

$$\mathcal{F}^{\star\star}\equiv\left\{\gamma\in\mathcal{H}_{d',+,1}|\mathrm{Tr}[\gamma\omega]=\frac{1}{d}\ \forall\,\omega\in\mathcal{F}^\star\right\}.\tag{11}$$

Hence, if $\gamma\in\mathcal{F}$ we must have $\mathrm{Tr}[\gamma\omega]=1/d$ for all $\omega\in\mathcal{F}^\star$, so that $\gamma\in\mathcal{F}^{\star\star}$. This proves $\mathcal{F}\subset\mathcal{F}^{\star\star}$. To prove the converse, note that if $\gamma\in\mathcal{F}^{\star\star}$ then we must have

$$\mathrm{Tr}[\gamma(\omega-u_d)]=0\tag{12}$$

for all ω that satisfy

$$\mathrm{Tr}[\omega(\sigma-\sigma')]=0\ \forall\,\sigma,\quad\sigma'\in\mathcal{F}.\tag{13}$$

The condition above is equivalent to $\omega\in\mathcal{V}_0^\perp$. Note also that all matrices in $\mathcal{V}^\perp$ have a zero trace since we assume $u_d\in\mathcal{F}\subset\mathcal{V}$. Now, any $\omega\in\mathcal{V}_0^\perp$ can be written as $\omega=u_d+tX$ for some arbitrary $X\in\mathcal{V}^\perp$ and small enough $t>0$ so that $\omega\geqslant 0$. Combining this with Eq. (12) we get $\mathrm{Tr}[\gamma X]=0$ for all $X\in\mathcal{V}^\perp$. This implies that $\gamma\in\mathcal{V}$ and since $\mathcal{F}$ is affine we get $\gamma\in\mathcal{F}$. This completes the proof of property 2.

Finally, we prove property 3. As before, suppose $\gamma\in\mathcal{F}^{\star\star}$ so that Eq. (12) holds for all ω that satisfy Eq. (13) or, equivalently, for all $\omega\in\mathcal{V}_0^\perp$. Similarly to the above argument, any $\omega\in\mathcal{V}_0^\perp$ can be written as $\omega=(1-t\mathrm{Tr}[X])u_d+tX$ for some arbitrary $X\in\mathcal{V}^\perp$ and small enough $t>0$ so that $\omega\geqslant 0$. Combining with Eq. (12) we conclude that

$$\mathrm{Tr}[\gamma X]=\frac{1}{d}\mathrm{Tr}[X]\ \forall\ X\in\mathcal{V}^\perp.\tag{14}$$

Defining $Y=u_d-\gamma$ we get from the above equation that $\mathrm{Tr}[XY]=0$ for all $X\in\mathcal{V}^\perp$. That is, $\gamma=u_d+Y$ with $Y\in\mathcal{V}_0$. ■

Consider the function $g:\mathcal{F}^\star\to[0,1]$ defined by $g(\omega)=\mathrm{Tr}[\omega\sigma]$, where σ is any state in $\mathcal{F}$. Note that the range of this function $g:\mathcal{F}^\star\to[0,1]$ provides further characterization of $\mathcal{F}^\star$. For example, if the maximally mixed state $u_d\in\mathcal{F}$, than $g(\omega)=\frac{1}{d}$ for *all* $\omega\in\mathcal{F}^\star$. In the other extreme, if $\mathcal{F}$ consists of only one state γ, then $g(\mathcal{F}^\star)=[\lambda_{\min}(\gamma),\lambda_{\max}(\gamma)]$. Particularly, if γ is a pure state then $g(\mathcal{F}^\star)=[0,1]$.

III. SINGLE-SHOT TRANSFORMATIONS UNDER MAXIMAL RESOURCE NONGENERATING OPERATIONS

One of the main results of this paper (Theorem 2 below) is expressed in terms of the conditional min-entropy. The conditional min-entropy is defined by

$$H_{\min}(A|B)_\Omega=-\log_2\min_{\tau\geqslant 0}\{\mathrm{Tr}[\tau]|I\otimes\tau\geqslant\Omega^{AB}\},\tag{15}$$

where the minimum is over all positive semidefinite matrices τ. It is known to be a single-shot analog of the conditional quantum entropy $S(A|B)\equiv S(A,B)-S(B)$, where S is the von Neumann entropy defined by $S(\rho)=-\mathrm{Tr}[\rho\log_2\rho]$. This analogy is particularly motivated by the fully quantum asymptotic equipartition property [38], which states that, in the asymptotic limit of many copies of Ω^{AB}, the smooth version of $H_{\min}(A|B)$ approaches the conditional (von Neumann) entropy. The conditional min-entropy has numerous applications in single-shot quantum information (e.g., Refs. [35–38]) and quantum hypothesis testing (e.g., Refs. [39,40] and references therein). We first use it to define a class of functions that behave monotonically under maps in $\mathcal{O}_{\max}$.

Definition 2. Let $\mathcal{R}(\mathcal{F}_{\mathrm{in}},\mathcal{F}_{\mathrm{out}},\mathcal{O})$ be an ART as above, and set $n\equiv\dim\mathcal{V}_{\mathrm{in}}$. For any $t\in g(\mathcal{F}^\star_{\mathrm{out}})\subset[0,1]$, let $\mathcal{S}_t^{\mathrm{in}}$ and $\mathcal{S}_t^{\mathrm{out}}$

be the set of all states Θ^{AB} of the form

$$\Theta^{AB} = \frac{1}{n}\sum_{\ell=1}^{n}\omega_\ell^T \otimes \sigma_\ell. \quad (16)$$

Here, $\omega_\ell \in \mathcal{F}_{\rm out}^\star$, $r(\Theta^A) = t$, and for $\mathcal{S}_t^{\rm in}$, $\sigma_\ell \in \mathcal{F}_{\rm in}$, whereas for $\mathcal{S}_t^{\rm out}$, $\sigma_\ell \in \mathcal{F}_{\rm out}$. With this notation, for any $t \in g(\mathcal{F}_{\rm out}^\star)$, and $\eta \in \mathcal{H}_{d,+,1}$, we define the functions $R_{\eta,t} : \mathcal{H}_{d,+,1} \to [0,1]$ by

$$R_{\eta,t}(\rho) \equiv \min_{\Theta^{AB}\in\mathcal{S}_t^{\rm in}} 2^{-H_{\min}(A|B)_{\Omega_{\eta,\Theta}(\rho)}}, \quad (17)$$

where

$$\Omega_{\eta,\Theta}^{AB}(\rho) \equiv \frac{1}{n+1}(\eta^T \otimes \rho + n\Theta^{AB}). \quad (18)$$

Similarly, for $\rho' \in \mathcal{H}_{d',+,1}$ in the output space, $R_{\eta,t}(\rho')$ is defined exactly as above with $\mathcal{S}_t^{\rm out}$ replacing $\mathcal{S}_t^{\rm in}$.

Remark 2. We will see in the theorem below that the functions $R_{\eta,t}$ form a complete set of resource monotones, determining whether there exists a RNG map converting a state in the input space to a state in the output space. Since $2^{-H_{\min}(A|B)_{\Omega_{\boldsymbol\omega}(\rho)}}$ quantify the amount of correlations in the the state $\Omega_{\boldsymbol\omega}^{AB}(\rho)$ [36], the quantities $R_{\eta,t}(\rho)$ quantify the minimum amount of correlations in separable states obtained by mixing the product state $\eta \otimes \rho$ with the separable states Θ^{AB} as in Eq. (18).

Theorem 2. Let $\mathcal{R}(\mathcal{F}_{\rm in},\mathcal{F}_{\rm out},\mathcal{O})$ be an ART as above, $\rho \in \mathcal{H}_{d,+,1}$ and $\rho' \in \mathcal{H}_{d',+,1}$ be two states, and $\mathcal{O}_{\max}$ be the set of RNG operations. Assuming that both $\mathcal{F}_{\rm in}$ and $\mathcal{F}_{\rm out}$ are nonempty, let n be the dimension of the input subspace $\mathcal{V}_{\rm in} \equiv {\rm span}_{\mathbb{R}}\{\mathcal{F}_{\rm in}\} = {\rm span}_{\mathbb{R}}\{\sigma_1,\ldots,\sigma_n\}$, where $\sigma_1,\ldots,\sigma_n \in \mathcal{F}_{\rm in}$. Denote $\boldsymbol\omega \equiv \{\eta,\omega_1,\ldots,\omega_n\}$ by $\eta \in \mathcal{H}_{d',+,1}$ and by $\omega_j \in \mathcal{F}_{\rm out}^\star$ for $j = 1,\ldots,n$, where $\mathcal{F}_{\rm out}^\star$ is the dual of $\mathcal{F}_{\rm out}$. Finally, for any such $\boldsymbol\omega$, denote by $\Omega_{\boldsymbol\omega}^{AB}(\rho)$ the state $\Omega_{\eta,\Theta}^{AB}(\rho)$ as defined in Eq. (3) for this fixed choice of $\sigma_1,\ldots,\sigma_n$. Then, the following are equivalent:

(1) There exists $\mathcal{E} \in \mathcal{O}_{\max}$ such that $\rho' = \mathcal{E}(\rho)$.

(2) For any $\boldsymbol\omega$ as above, with $\omega \equiv \frac{1}{n}\sum_{j=1}^n \omega_j$,

$$2^{-H_{\min}(A|B)_{\Omega_{\boldsymbol\omega}(\rho)}} \geqslant \frac{{\rm Tr}[\eta\rho'] + ng(\omega)}{n+1}. \quad (19)$$

(3) For any $\eta \in \mathcal{H}_{d',+,1}$ and $t \in g(\mathcal{F}_{\rm out}^\star) \subset [0,1]$,

$$R_{\eta,t}(\rho) \geqslant R_{\eta,t}(\rho'). \quad (20)$$

(4) For all $\boldsymbol\omega$ as above,

$$f_{\boldsymbol\omega}(\rho) \geqslant f_{\boldsymbol\omega}(\rho'), \quad (21)$$

with

$$f_{\boldsymbol\omega}(\rho) \equiv \min_{\{\sigma_\ell\}_{\ell=1}^n\subset\mathcal{F}_{\rm in}} 2^{-H_{\min}(A|B)_{\Omega_{\eta,\Theta}(\rho)}}, \quad (22)$$

where the minimization is over all separable states $\Omega_{\eta,\Theta}^{AB}(\rho)$ as defined in Definition 2, while keeping $\boldsymbol\omega$ fixed.

Remark 3. Note that both $R_{\eta,t}(\rho)$ and $f_{\boldsymbol\omega}(\rho)$ obtained by optimizing $2^{-H_{\min}(A|B)_{\Omega_{\eta,\Theta}(\rho)}}$. The first one fixes η with the optimization carried over all $\omega_1,\ldots,\omega_n \in \mathcal{F}_{\rm out}^\star$ with $t = r(\omega)$ (and $\sigma_1,\ldots,\sigma_n$ are taken to be a fixed basis of $\mathcal{V}_{\rm in}$), while the second one fixes $\boldsymbol\omega = \{\eta,\omega_1,\ldots,\omega_n\}$ with the optimization carried over *any* $\sigma_1,\ldots,\sigma_n \in \mathcal{V}_{\rm in}$.

Remark 4. The set $\mathcal{F}_{\rm out}^\star$ is convex, and since $\omega_j \in \mathcal{F}_{\rm out}^\star$ for $j = 1,\ldots,n$ we conclude that $\omega \in \mathcal{F}_{\rm out}^\star$. Therefore, $r(\omega)$ is a well-defined function from $\mathcal{F}_{\rm out}^\star$ to [0,1]. Since the right-hand side (RHS) of Eq. (19) depends only on $r(\omega)$ and ω_0, we can minimize the left-hand side (LHS) over all matrices with the same value of $r(\omega)$. In fact, note that from the above theorem, the function

$$W(\rho,\rho') \equiv \min_{\boldsymbol\omega}\left(2^{-H_{\min}(A|B)_{\Omega_{\boldsymbol\omega}(\rho)}} - \frac{{\rm Tr}[\eta\rho'] + nr(\omega)}{n+1}\right),$$

where the minimum is over all $\boldsymbol\omega$ as defined above, is nonnegative if and only if ρ can be converted to ρ' by RNG operations.

Proof of Theorem 2

Our proof of Theorem 2 relies heavily on the semidefinite programming (SDP) version of the Farkas lemma. The Farkas lemma provides a strong-duality relation, stating that, out of two systems of equations (or inequalities), one or the other has a solution, but not both nor none. Several versions of this lemma can be found in standard textbooks on SDP.

Lemma 3.(Farkas) Let $H_1,\ldots,H_n$ be $d\times d$ Hermitian matrices. Then, the system

$$r_1H_1 + \cdots + r_nH_n > 0 \quad (23)$$

has no solution in $r_1,\ldots,r_n \in \mathbb{R}$ if and only if there exists a positive-semidefinite matrix $\sigma \neq 0$ such that

$${\rm Tr}[H_j\sigma] = 0\ \forall\ j = 1,\ldots,n. \quad (24)$$

Proof. Suppose there is no $x_1,\ldots,x_n$ in $\mathbb{R}$ such that Eq. (23) holds, and recall that the set of positive-semidefinite matrices $\mathcal{H}_{d,+}$ is a convex closed cone in $\mathcal{H}_d$. From our assumption, its interior ${\rm int}\mathcal{H}_{d,+}$ is disjoint from the linear subspace $\mathcal{W} \equiv {\rm Span}_{\mathbb{R}}\{H_1,\ldots,H_n\}$. Therefore, there exists a hyperplane $\mathcal{K} \subset \mathcal{H}_{d,+}$ containing $\mathcal{W}$ such that $\mathcal{K}\cap{\rm int}\mathcal{H}_{d,+} = \emptyset$. The hyperplane is characterized by $\mathcal{K} = \{X : {\rm Tr}[X\sigma] = 0\}$, where σ is some nonzero matrix in $\mathcal{H}_d$. Furthermore, the hyperplane can be chosen such that $\mathcal{H}_{+,d}$ is in one of its half-spaces. We can therefore assume that ${\rm Tr}[X\sigma] \geqslant 0$ for all $X \in \mathcal{H}_{+,d}$. This in turn implies that $\sigma \geqslant 0$. Finally, since $H_i \in \mathcal{W} \subseteq \mathcal{K}$ for all $i = 1,\ldots,n$, we have ${\rm Tr}[H_i\sigma] = 0$ for all $i = 1,\ldots,n$. ■

Remark 5. The positive-definite condition in Eq. (23) can be replaced with a negative-definite one. In particular, one can replace the condition that Eq. (23) has no solution with the condition that

$$W_{\mathbf{r}}(H_1,\ldots,H_n) := \lambda_{\max}(r_1H_1 + \cdots + r_nH_n) \geqslant 0 \quad (25)$$

for all $\mathbf{r} \in \mathbb{R}^n$. Moreover, since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, one can restrict $\mathbf{r} \in \mathbb{Q}^n$. Since the set $\mathbb{Q}^n$ is countable, the condition above can be replaced further with

$$W_k(H_1,\ldots,H_n) \geqslant 0\ \forall\ k \in \mathbb{N}, \quad (26)$$

where $W_k \equiv W_{r_k}$ with $\{r_k\}_{k\in\mathbb{N}} = \mathbb{Q}^n$.

Next, we use the Farkas lemma to prove the following:

Lemma 4. Let $\mathcal{R}(\mathcal{F}_{\rm in},\mathcal{F}_{\rm out},\mathcal{O})$ be an ART, and let $\mathcal{V}_{\rm in}$, $\mathcal{V}_{\rm out}^\perp$, and $\mathcal{O}_{\max}$ be as above. Assuming that $\mathcal{F}_{\rm out} \neq \emptyset$, let $\gamma \in \mathcal{F}_{\rm out}$ be a free state, and let $\rho \in \mathcal{H}_{d,+,1}$ and $\rho' \in \mathcal{H}_{d',+,1}$ be two density matrices. Denote by $\mathcal{V}_{\rm in}^T := \{X^T | X \in \mathcal{V}_{\rm in}\}$ the set of the transposed matrices of all the matrices in $\mathcal{V}_{\rm in}$. Then, there exists $\mathcal{E} \in \mathcal{O}_{\max}$ such that $\rho' = \mathcal{E}(\rho)$ if and only if the matrix

$$M^{AB} = -\text{Tr}[Y\rho']I_{d'} \otimes \tau + Y \otimes \rho^T + N^{AB} \quad (27)$$

is not positive definite, for any matrix $N^{AB} \in \mathcal{V}_{\text{out}}^{\perp} \otimes \mathcal{V}_{\text{in}}^T \subset \mathcal{H}_{d'} \otimes \mathcal{H}_d$, any $0 < \tau \in \mathcal{H}_{d,+,1}$, and any matrix $Y \in \mathcal{H}_{d'}$ such that $\text{Tr}[Y\gamma] = 0$.

Remark 6. The condition that M^{AB} is not positive definite can be written in terms of the min-eigenvalue; that is, $\rho \xrightarrow{RNG} \rho'$ iff $\lambda_{\min}(M^{AB}) \leqslant 0$ for all N^{AB}, τ, and Y. Therefore, for any choice of of matrices N^{AB}, τ, and Y the condition $-\lambda_{\min}(M^{AB}) \geqslant 0$ is necessary and can be viewed as a "no-go" conversion witness [42,43]. Therefore, the lemma above provides a complete set of no-go conversion witnesses determining whether the transformation $\rho \xrightarrow{RNG} \rho'$ is possible.

Remark 7. From the form of M^{AB} above it is not very obvious why this matrix is never positive definite if $\rho' = \mathcal{E}(\rho)$ and $\mathcal{E} \in \mathcal{O}_{\max}$. To see why, note that any matrix $N^{AB} \in \mathcal{V}_{\text{out}}^{\perp} \otimes \mathcal{V}_{\text{in}}^T$ can be written as $N^{AB} = \sum_{k=1}^{d^2-m} Y_k \otimes A_k^T$, where the Y_k form a basis of $\mathcal{V}_{\text{out}}^{\perp}$ and the A_k are some matrices in $\mathcal{V}_{\text{in}}$. If there exists $\mathcal{E} \in \mathcal{O}_{\max}$ such that $\rho' = \mathcal{E}(\rho)$ then

$$\langle \phi^+ | \mathcal{E}^\dagger \otimes \mathsf{id}(M^{AB}) | \phi^+ \rangle = \sum_{k=1}^{d^2-m} \text{Tr}[Y_k \mathcal{E}(A_k)] - \text{Tr}[Z\mathcal{E}(\rho)] + 1$$
$$= -\text{Tr}[Z\rho'] + 1 = 0, \quad (28)$$

where we used the fact that $\mathcal{E}(A_k) \in \mathcal{V}_{\text{in}}$ [and therefore $\text{Tr}[Y_k \mathcal{E}(A_k)] = 0$] since $A_k \in \mathcal{V}_{\text{in}}$ and $\mathcal{E} \in \mathcal{O}_{\max}$. Hence, in this case $\text{Tr}[M^{AB}[\mathcal{E} \otimes \mathsf{id}(|\phi^+\rangle\langle\phi^+|)]] = 0$, and since $\mathcal{E} \otimes \mathsf{id}(|\phi^+\rangle\langle\phi^+|) \geqslant 0$ we conclude that M^{AB} is not positive definite (as expected).

Proof of Lemma 4. Denoting by $\sigma^{AB} = \mathcal{E} \otimes \mathsf{id}(|\phi^+\rangle\langle\phi^+|) \in \mathcal{H}_{d'd,+}$ the Choi matrix associated with $\mathcal{E}$, where $|\phi^+\rangle = \sum_{j=1}^{d} |jj\rangle$ is the unnormalized maximally entangled state, the condition $\rho' = \mathcal{E}(\rho)$ is equivalent to the existence of such a Choi matrix (of a free operation) that satisfies

$$\rho' = \text{Tr}_B[\sigma^{AB}(I_{d'} \otimes \rho^T)] \quad \text{and } \text{Tr}_A[\sigma^{AB}] = I_d. \quad (29)$$

These equations are equivalent to

$$\text{Tr}[\sigma^{AB}(Y \otimes \rho^T)] = \text{Tr}[Y\rho'] \,\forall\, Y \in \mathcal{H}_{d'}, \quad (30)$$

$$\text{Tr}[\sigma^{AB}(I_{d'} \otimes X)] = \text{Tr}[X] \,\forall\, X \in \mathcal{H}_d. \quad (31)$$

Note that the two equations above are not completely independent. For example, if $Y = I_{d'}$ then Eq. (30) follows from Eq. (31). Hence, without loss of generality (w.l.o.g.) we can assume that $Y \in \mathcal{H}_{d',0}$, where $\mathcal{H}_{d',0} \subset \mathcal{H}_{d'}$ is the subspace of traceless Hermitian matrices. Similarly, denoting by $Z \equiv X - \text{Tr}[X]\frac{1}{d}I_d$ we get that the above two equations are equivalent to

$$\text{Tr}\left[\sigma^{AB}\left(Y \otimes \rho^T - \text{Tr}[Y\rho']I_{d'} \otimes \frac{1}{d}I_d\right)\right] = 0, \quad (32)$$

$$\text{Tr}[\sigma^{AB}(I_{d'} \otimes Z)] = 0, \quad (33)$$

$$\text{Tr}[\sigma^{AB}] = d, \quad (34)$$

for all $Z \in \mathcal{H}_{d,0}$ and $Y \in \mathcal{H}_{d',0}$. Note that the equation $\text{Tr}[\sigma^{AB}] = d$ can be removed since, if there exists a positive semidefinite matrix $\sigma^{AB} \neq 0$ that satisfies conditions (33) and (32), then the matrix $\frac{d}{\text{Tr}[\sigma^{AB}]}\sigma^{AB}$ satisfies all three conditions. Due to the linearity of the above equations with Y and Z, it is enough to consider only $Y \in \{Y_j\}_j$ and $Z \in \{Z_k\}_k$, where $\{Y_j\}_j$ and $\{Z_k\}_k$ are bases of $\mathcal{H}_{d',0}$ and $\mathcal{H}_{d,0}$, respectively. We therefore conclude that for all $j = 1, \ldots, d'^2 - 1$ and for all $k = 1, \ldots, d^2 - 1$,

$$\text{Tr}\left[\sigma^{AB}\left(Y_j \otimes \rho^T - \frac{1}{d}\text{Tr}[Y_j\rho']I_{d'} \otimes I_d\right)\right] = 0,$$
$$\text{Tr}[\sigma^{AB}(I \otimes Z_k)] = 0, \quad (35)$$

The conditions in Eq. (35) can be written as a collection of equalities $\text{Tr}[\sigma^{AB}H_j] = 0$, for some Hermitian matrices $H_j \in \mathcal{H}_{d'} \otimes \mathcal{H}_d$.

In addition to the above conditions, there are constraints on the Choi matrix σ^{AB} that comes from the fact that $\mathcal{E}$ is a *free* operation. Particularly, if $\mathcal{E} \in \mathcal{O}_{\max}$ then $\mathcal{E}(\sigma) \in \mathcal{F}_{\text{out}}$ for all $\sigma \in \mathcal{F}_{\text{in}}$. From the linearity of $\mathcal{E}$, we have $\mathcal{E}(X) \in \mathcal{V}_{\text{out}}$ for all $X \in \mathcal{V}_{\text{in}}$. Therefore, if $\mathcal{E} \in \mathcal{O}_{\max}$, then

$$\text{Tr}[Y\mathcal{E}(X)] = 0 \,\forall\, X \in \mathcal{V}_{\text{in}} \quad \text{and} \,\forall\, Y \in \mathcal{V}_{\text{out}}^{\perp}, \quad (36)$$

where $\mathcal{V}_{\text{out}}^{\perp}$ is the orthogonal complement of $\mathcal{V}_{\text{out}}$ in $\mathcal{H}_{d'}$. In the Choi representation, the condition above take the form

$$\text{Tr}\left[\sigma^{AB}Y_j \otimes X_k^T\right] = 0 \quad (37)$$

for all $j = 1, \ldots, \dim \mathcal{V}_{\text{out}}^{\perp}$ and $k = 1, \ldots, \dim \mathcal{V}_{\text{in}}$, where the set $\{X_k\}$ form a basis of $\mathcal{V}_{\text{in}}$, and $\{Y_j\}$ a basis for $\mathcal{V}_{\text{out}}^{\perp}$. Combining these conditions with those in Eq. (35) we apply the Farkas lemma. To do that, note that a linear combination of the matrices $Y_j \otimes X_k^T$ provides a matrix $N^{AB} \in \mathcal{V}_{\text{out}}^{\perp} \otimes \mathcal{V}_{\text{in}}^T$. Similarly, any linear combination of $Y_k \otimes \rho^T - \frac{1}{d}\text{Tr}[Y_k\rho']I_{d'} \otimes I_d$ is a matrix of the form

$$W \otimes \rho^T - \frac{1}{d}\text{Tr}[W\rho']I_{d'} \otimes I_d,$$

with $W \in \mathcal{H}_{d',0}$, and any linear combination of $I_{d'} \otimes Z_j$ is a matrix of the form $I_{d'} \otimes Z$ with $Z \in \mathcal{H}_{d,0}$. We therefore conclude from the Farkas lemma that there exists a Choi matrix σ^{AB} that satisfies Eqs. (35) and (37) if and only if for any matrices $N^{AB} \in \mathcal{V}_{\text{out}}^{\perp} \otimes \mathcal{V}_{\text{in}}^T$, and $W \in \mathcal{H}_{d',0}$ and $Z \in \mathcal{H}_{d,0}$ the matrix

$$M^{AB} \equiv N^{AB} + W \otimes \rho^T - \frac{1}{d}\text{Tr}[W\rho']I_{d'} \otimes I_d + I_{d'} \otimes Z \quad (38)$$

is not positive definite. Let $\gamma \in \mathcal{F}_{\text{out}}$. Then, M^{AB} is not positive definite if $L^{AB} \equiv (\gamma^{1/2} \otimes I_d)M^{AB}(\gamma^{1/2} \otimes I_d)$ is not positive definite. Moreover, note that the matrix N^{AB} can be expressed as $\sum_\ell H_\ell \otimes \sigma_\ell^T$ with $\sigma_\ell \in \mathcal{F}_{\text{in}} = \text{span}_{\mathbb{R}}\{\sigma_1, \ldots, \sigma_n\}$ and $H_\ell \in \mathcal{V}_{\text{out}}^{\perp}$. With these notations we get

$$L^{AB} = \sum_\ell \gamma^{1/2}H_\ell\gamma^{1/2} \otimes \sigma_\ell^T + \gamma^{1/2}W\gamma^{1/2} \otimes \rho^T$$
$$- \frac{1}{d}\text{Tr}[W\rho']\gamma \otimes I_d + \gamma \otimes Z. \quad (39)$$

In particular, the marginal state takes the form

$$L^B = \text{Tr}[W\gamma]\rho^T - \frac{1}{d}\text{Tr}[W\rho']I_d + Z. \quad (40)$$

Note that, if we choose W and Z such that L^B is not positive definite, then L^{AB} is also not positive definite. We therefore assume w.l.o.g. that $L^B > 0$. In particular, $\mathrm{Tr}[L^B] > 0$ so that $\mathrm{Tr}[W(\gamma - \rho')] > 0$. Denoting by

$$\tau \equiv \frac{\mathrm{Tr}[W\gamma]\rho^T - \frac{1}{d}\mathrm{Tr}[W\rho']I_d + Z}{\mathrm{Tr}[W(\gamma - \rho')]} > 0, \qquad (41)$$

we get that M^{AB} can be expressed as

$$M^{AB} = \sum_\ell H_\ell \otimes \sigma_\ell^T + (W - \mathrm{Tr}[W\gamma]I_{d'}) \otimes \rho^T + \mathrm{Tr}[W(\gamma - \rho')]I_{d'} \otimes \tau. \qquad (42)$$

Next, denoting by $Y \equiv W - \mathrm{Tr}[W\gamma]I_{d'} \in \gamma^\perp$ (here $\gamma^\perp \equiv \{X \in \mathcal{H}_{d'} : \mathrm{Tr}[X\gamma] = 0\}$), we get

$$M^{AB} = N^{AB} + Y \otimes \rho^T - \mathrm{Tr}[Y\rho']I_{d'} \otimes \tau. \qquad (43)$$

This completes the proof of Lemma 4. ■

The condition in the lemma above is given in terms of Hermitian matrices N^{AB} and X. We now prove Theorem 2 by expressing this lemma in terms of density matrices.

Proof of Theorem 2. We start proving the equivalence of 1 and 2. Let M^{AB} be the matrix defined in Lemma 4:

$$M^{AB} = -\mathrm{Tr}[Y\rho']I_{d'} \otimes \tau + Y \otimes \rho^T + \sum_\ell H_\ell \otimes \sigma_\ell^T,$$

with $Y \in \mathcal{H}_{d'}$ such that $\mathrm{Tr}[Y\gamma] = 0$. For all $\ell = 1,\ldots,n$, define the traceless matrices

$$F_\ell = H_\ell - \mathrm{Tr}[H_\ell]u_{d'} \quad \text{and } Z = Y - \mathrm{Tr}[Y]u_{d'}. \qquad (44)$$

where $u_{d'} \equiv \frac{1}{d'}I_{d'}$. Equivalently,

$$H_\ell = F_\ell - \mathrm{Tr}[F_\ell\gamma]I_{d'} \quad \text{and } Y = Z - \mathrm{Tr}[Z\gamma]I_{d'}. \qquad (45)$$

Note that for all $\sigma \in \mathcal{F}_{\rm out}$ we have $\mathrm{Tr}[H_\ell\sigma] = 0$ which is equivalent to $\mathrm{Tr}[F_\ell\sigma] = \mathrm{Tr}[F_\ell\gamma]$. Hence, in terms of these traceless matrices,

$$M^{AB} = \mathrm{Tr}[Z(\gamma - \rho')]I_{d'} \otimes \tau + (Z - \mathrm{Tr}[Z\gamma]I_{d'}) \otimes \rho^T + \sum_\ell (F_\ell - \mathrm{Tr}[F_\ell\gamma]I_{d'}) \otimes \sigma_\ell^T. \qquad (46)$$

Without loss of generality, we can assume that $Z, F_\ell \leqslant u_{d'}$ since rescaling of M^{AB} by a positive factor does not change the signs of its eigenvalues. Therefore, we set for all $\ell = 1,\ldots,n$

$$\omega_\ell = u_d - F_\ell \quad \text{and } \eta = u_d - Z, \qquad (47)$$

with ω_ℓ satisfying $\mathrm{Tr}[\omega_\ell\sigma] = \mathrm{Tr}[\omega_\ell\gamma]$ for all $\sigma \in \mathcal{F}_{\rm in}$. With this notation we get

$$M^{AB} = \mathrm{Tr}[\eta(\rho' - \gamma)]I_{d'} \otimes \tau + (\mathrm{Tr}[\eta\gamma]I_{d'} - \eta) \otimes \rho^T + \sum_\ell (\mathrm{Tr}[\omega_\ell\gamma]I_{d'} - \omega_\ell) \otimes \sigma_\ell^T. \qquad (48)$$

Finally, rescaling $M^{AB} \to \frac{1}{n+1}M^{AB}$, we conclude

$$M^{AB} = \frac{\mathrm{Tr}[\eta(\rho' - \gamma)]}{n+1} I_{d'} \otimes \tau + I_{d'} \otimes \Omega_\gamma^B - \Omega^{AB}, \qquad (49)$$

where for simplicity we denote $\Omega^{AB} \equiv \Omega_{\boldsymbol{\omega}}^{AB}(\rho)$ and

$$\Omega_\gamma^B := \mathrm{Tr}_A[(\gamma^{1/2} \otimes I_d)\Omega^{AB}(\gamma^{1/2} \otimes I_d)]. \qquad (50)$$

Next, note that the conditional min-entropy can be expressed as

$$\begin{aligned} 2^{-H_{\min}(A|B)_\Omega} &= \inf_{\tau \geqslant 0}\{\mathrm{Tr}[\tau] | I_{d'} \otimes \tau \geqslant \Omega^{AB}\} \\ &= \inf_{\tau \geqslant \Omega_\gamma^B}\{\mathrm{Tr}[\tau] | I_{d'} \otimes \tau \geqslant \Omega^{AB}\} \\ &= \inf_{\tau' \geqslant 0}\left\{\mathrm{Tr}\left[\Omega_\gamma^B\right] + \mathrm{Tr}[\tau'] | I_{d'} \otimes \left(\tau' + \Omega_\gamma^B\right) \geqslant \Omega^{AB}\right\}, \end{aligned}$$

where in the second equality we used that fact that, if $I_{d'} \otimes \tau \geqslant \Omega^{AB}$, then $\tau \geqslant \Omega_\gamma^B$, and in the third equality we substitute $\tau' = \tau - \Omega_\gamma^B$. Hence,

$$2^{-H_{\min}(A|B)_\Omega} - \mathrm{Tr}[\Omega^{AB}(\gamma \otimes I_d)] = \inf_{\tau' \geqslant 0}\left\{\mathrm{Tr}[\tau'] | I_{d'} \otimes \tau' + I_{d'} \otimes \Omega_\gamma^B - \Omega^{AB} \geqslant 0\right\}. \qquad (51)$$

Comparing this last equality with the expression for M^{AB} in Eq. (49) we conclude that M^{AB} is not positive definite if and only if

$$2^{-H_{\min}(A|B)_\Omega} - \mathrm{Tr}[\Omega^{AB}(\gamma \otimes I_d)] \geqslant \frac{\mathrm{Tr}[\eta(\rho' - \gamma)]}{n+1}. \qquad (52)$$

Thus,

$$2^{-H_{\min}(A|B)_\Omega} \geqslant \frac{1}{n+1}\left(\sum_{\ell=1}^n \mathrm{Tr}[\omega_\ell\gamma] + \mathrm{Tr}[\eta\rho']\right). \qquad (53)$$

This completes the proof that 1 is equivalent to 2. We now prove that 1 is equivalent to 3. The necessity of Eq. (20) follows from the following monotonicity property of the conditional min-entropy. The conditional min-entropy behaves monotonically under CPTP maps Λ that satisfy

$$\Lambda(u_{d'} \otimes \sigma^B) = u_{d'} \otimes \mathrm{Tr}_A[\Lambda(u_{d'} \otimes \sigma^B)], \qquad (54)$$

for all $\sigma^B \in \mathcal{H}_{d,+,1}$ Therefore, if there exists a RNG map $\mathcal{E}$ that satisfies $\rho' = \mathcal{E}(\rho)$ then the map $\Lambda \equiv \mathsf{id} \otimes \mathcal{E}$ (with T being the transpose map) is a CPTP map that satisfies the above equation. Moreover,

$$\Lambda(\Omega^{AB}) = \frac{1}{n+1}\left(\eta \otimes \rho' + \sum_{\ell=1}^n \omega_\ell^T \otimes \mathcal{E}(\sigma_\ell)\right), \qquad (55)$$

with $\mathcal{E}(\sigma_\ell) \in \mathcal{F}_{\rm out}$ since $\mathcal{E}$ is a RNG map. Therefore, taking Ω^{AB} to be the optimal matrix in Eq. (17) (see Definition 2 in the main text), we conclude that

$$R_{\eta,t}(\rho) = 2^{-H_{\min}(A|B)_\Omega} \geqslant 2^{-H_{\min}(A|B)_{\Lambda(\Omega)}} \geqslant R_{\eta,t}(\rho'),$$

so that the condition (20) is necessary. The sufficiency of the condition follows from the following duality relation of the conditional min-entropy that was proved in Ref. [36]:

$$\begin{aligned} 2^{-H_{\min}(A|B)_{\Omega'}} &= d' \max_{\mathcal{E}}(\langle\phi^+|\mathsf{id} \otimes \mathcal{E}(\Omega'^{AB})|\phi^+\rangle) \\ &\geqslant d'\langle\phi^+|\Omega'^{AB}|\phi^+\rangle \end{aligned} \qquad (56)$$

for any $d'^2 \times d'^2$ separable density matrix Ω'^{AB} of the form (18) with ρ replaced by ρ' and $\Theta'^{AB} \in \mathcal{S}_t^{\rm out}$. Letting Ω'^{AB} be the one that optimizes $R_{\eta,t}(\rho')$, we obtain

$$R_{\eta,t}(\rho) \geqslant R_{\eta,t}(\rho') = 2^{-H_{\min}(A|B)_{\Omega'}} \geqslant d'\langle\phi^+|\Omega'^{AB}|\phi^+\rangle.$$

Thus, for any $\Omega^{AB} = \Omega^{AB}_{\boldsymbol{\omega}}(\rho)$ we have

$$2^{-H_{\min}(A|B)_{\Omega_{\boldsymbol{\omega}}(\rho)}} \geqslant R_{\eta,t}(\rho) \geqslant d'\langle\phi^+|\Omega'^{AB}|\phi^+\rangle = \frac{\text{Tr}[\eta\rho'] + nr(\omega)}{n+1}. \quad (57)$$

Hence, condition (19) holds for any $\boldsymbol{\omega}$ as above, and from the proof of the equivalence of 1 and 2 we get the existence of a RNG map $\mathcal{E}$ that satisfies $\rho' = \mathcal{E}(\rho)$. This completes the proof that 1 and 3 are equivalent. To prove that 1 is equivalent to 4, we follow the exact same lines as we did in the proof of the equivalence of 1 and 3. This is possible since both $R_{\eta,t}(\rho)$ and $f_{\boldsymbol{\omega}}(\rho)$ are obtained by optimizing $2^{-H_{\min}(A|B)_{\Omega_{\eta,\Theta}(\rho)}}$. This completes the proof of Theorem 2. ■

IV. APPLICATIONS OF MAIN RESULT TO SPECIFIC RESOURCE THEORIES

Theorem 2 can take a much simpler form in specific resource theories, such as quantum thermodynamics under Gibbs preserving operations, and the resource theory of coherence under maximally incoherent operations. In the following, we discuss the applications of Theorem 2 to these resource theories.

A. Resource theory of athermality

In the resource theory of athermality, the set of free states consist of only one state (the Gibbs state) $\gamma \in \mathcal{H}_{d,+,1}$, and the set of Gibbs preserving operations consists of all quantum channels, i.e., completely positive and trace preserving (CPTP) maps, $\mathcal{E} : \mathcal{H}_d \to \mathcal{H}_{d'}$ that satisfy $\mathcal{E}(\gamma) = \gamma'$, where $\gamma' \in \mathcal{H}_{d',+,1}$ is the Gibbs state of the output (in the most general case, the output Gibbs state γ' may be associated with a different Hamiltonian than the Hamiltonian that is associated with the input Gibbs state γ). Therefore, in this ART of athermality, $n = 1$, and $\Omega^{AB}_{\eta,\Theta}$ takes the following simple form with $\eta \equiv \omega_0$:

$$\Omega^{AB}_{\boldsymbol{\omega}}(\rho) = \tfrac{1}{2}(\omega_0 \otimes \rho + \omega_1 \otimes \gamma),$$
$$\Omega^{AB}_{\boldsymbol{\omega}}(\rho') = \tfrac{1}{2}(\omega_0 \otimes \rho' + \omega_1 \otimes \gamma'), \quad (58)$$

where $\omega_0, \omega_1 \in \mathcal{H}_{d',+,1}$ are two arbitrary density matrices. Then ρ can be converted to ρ' by Gibbs preserving operations if and only if for all $\omega_0, \omega_1 \in \mathcal{H}_{d',+,1}$

$$H_{\min}(A|B)_{\Omega_{\boldsymbol{\omega}}(\rho)} \leqslant H_{\min}(A|B)_{\Omega_{\boldsymbol{\omega}}(\rho')}. \quad (59)$$

This remarkable result is the quantum generalization of *thermo-majorization* [10]. It demonstrates that the functions $f_{\boldsymbol{\omega}}(\rho) \equiv 2^{-H_{\min}(A|B)_{\Omega_{\boldsymbol{\omega}}(\rho)}}$, which are also known to quantify the amount of correlations in the state $\Omega^{AB}_{\boldsymbol{\omega}}(\rho)$ [36], form a complete set of athermality monotones. For diagonal ω_0 and ω_1 the states $\Omega^{AB}_{\boldsymbol{\omega}}(\rho)$ become classical-quantum states, and in this case $f_{\boldsymbol{\omega}}(\rho)$ can be interpreted as the optimal *guessing probability* (i.e., the optimal probability to guess correctly the classical variable after measuring the quantum system). In the classical case, it is known that the guessing probabilities provide conditions that are equivalent to thermo-majorization (see, e.g., Ref. [40]), however, in the full quantum case, the coherence or off-diagonal terms of ω_0 and ω_1 in Eq. (59) needs also to be considered, so that the guessing probabilities are in general insufficient to determine if ρ can be converted to ρ' by Gibbs preserving operations.

B. Resource theory of coherence

The resource theory of coherence is an ART with a free maximally mixed state. In this case, $u_{d'} \equiv \frac{1}{d'}I_{d'} \in \mathcal{F}_{\text{out}}$ so that

$$\mathcal{F}^{\star}_{\text{out}} := \left\{\omega \in \mathcal{H}_{d',+,1} | \text{Tr}[\omega\sigma] = \frac{1}{d'}\ \forall\, \sigma \in \mathcal{F}_{\text{out}}\right\}, \quad (60)$$

and we therefore have the following:

Corollary 1. Using the same notations of Theorem 2, suppose $u_{d'} \in \mathcal{F}_{\text{out}}$. Then, the map $\rho \to \rho'$ can be achieved by RNG operations if and only if

$$2^{-H_{\min}(A|B)_{\Omega_{\boldsymbol{\omega}}(\rho)}} \geqslant \frac{n + d\text{Tr}[\eta\rho']}{d(n+1)}, \quad (61)$$

for all separable bipartite matrices $\Omega^{AB}_{\boldsymbol{\omega}}(\rho)$ as in Theorem 2 with $\sigma_\ell \in \mathcal{F}_{\text{in}}$, $\omega_\ell \in \mathcal{F}^{\star}_{\text{out}}$, and $\eta \in \mathcal{H}_{d',+,1}$.

Now, in the resource theory of coherence, the set of free states are the diagonal states with respect to some fixed basis. The set $\mathcal{F}^{\star}_{\text{out}}$ becomes

$$\mathcal{F}^{\star}_{\text{out}} = \left\{\omega \in \mathcal{H}_{d',+,1} | \Delta(\omega) = \frac{1}{d'}I_{d'}\right\}, \quad (62)$$

where Δ is the completely dephasing map. Moreover, since every free density matrix is a convex combination of the states $|\ell\rangle\langle\ell|$, the matrix Ω^{AB} can be written as

$$\Omega^{AB} = \frac{1}{d+1}\left(\sum_{\ell=1}^{d} \omega_\ell \otimes |\ell\rangle\langle\ell| + \eta \otimes \rho^T\right), \quad (63)$$

where $\omega_1, \ldots, \omega_d \in \mathcal{F}^{\star}_{\text{out}}$ are density matrices with a uniform diagonal. For any such set of density matrices $\boldsymbol{\omega} \equiv (\omega_1, \ldots, \omega_d)$ and any $\eta \in \mathcal{H}_{d',+,1}$ we define the functions (no-go witnesses)

$$W_{\boldsymbol{\omega},\eta}(\rho,\rho') = 2^{-H_{\min}(A|B)_{\Omega}} - \frac{1 + \text{Tr}[\eta\rho']}{d+1}. \quad (64)$$

We therefore arrive at the following corollary:

Corollary 2. Using the same notations as above, ρ can be converted into ρ' with maximally incoherent operations (MIO) [23] if and only if

$$W_{\boldsymbol{\omega},\eta}(\rho,\rho') \geqslant 0 \quad (65)$$

for all $\boldsymbol{\omega} \equiv (\omega_1, \ldots, \omega_d)$ with $\omega_\ell \in \mathcal{F}^{\star}_{\text{out}}$ and for all $\eta \in \mathcal{H}_{d',+,1}$.

V. THE SELF-DUAL SET OF RESOURCE NONGENERATING OPERATIONS

So far we only considered the maximal set of free operations; namely, the set of all RNG operations $\mathcal{O}_{\max}$. However, in many practical QRTs such as entanglement, athermality, and asymmetry, the operationally and physically motivated set of free operations $\mathcal{O}$ is much smaller than $\mathcal{O}_{\max}$. For example, in entanglement theory, local operations and classical communication is a much smaller set than nonentangling operations [44]. Also, thermal operations form a much smaller set than Gibbs preserving operations. The problem in QRTs such as entanglement theory is that the physically motivated set of operations $\mathcal{O}$ cannot be characterized in the form $\text{Tr}[\sigma^{AB}H_j] = 0$, and therefore the techniques from SDP cannot be applied directly in these important cases. For this reason, it is natural to search for a smaller subset of RNG operations that still can be characterized in a form suitable for SDP and yet contains all the physically motivated free operations. We show here that for ARTs such a natural set exists, and we call it the *self-dual* set of RNG operations. In the context of the QRT of coherence, this set of operations was called *dephasing covariant operations* [23,24].

Definition 3. Let $\mathcal{R}(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}},\mathcal{O})$ be a QRT with $\mathcal{O} \subset \mathcal{O}_{\max}$ a set of free operations. We say that $\mathcal{O}$ is *self-dual* if for any CPTP map $\mathcal{E} : \mathcal{H}_d \to \mathcal{H}_{d'}$ in $\mathcal{O}$, we have

$$\mathcal{E}(\mathcal{V}_{\text{in}}) \subset \mathcal{V}_{\text{out}} \quad \text{and } \mathcal{E}^\dagger(\mathcal{V}_{\text{out}}) \subset \mathcal{V}_{\text{in}}. \tag{66}$$

Moreover, we denote by $\mathcal{O}_{\text{sd}}$ the set of all CPTP maps $\mathcal{E} \in \mathcal{O}_{\max}$ that satisfy Eq. (66).

Equation (66) for $\mathcal{E}$ is equivalent to Eq. (36), and therefore the additional condition that $\mathcal{E}^\dagger(\mathcal{V}_{\text{out}}) \subset \mathcal{V}_{\text{in}}$ can also be expressed in a SDP form. Specifically, Definition (66) can be rewritten as follows:

Definition. Let $\mathcal{R}(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}},\mathcal{O})$ be an ART with $\mathcal{O} \subset \mathcal{O}_{\max}$ a set of free CPTP maps from $\mathcal{H}_d$ to $\mathcal{H}_{d'}$. We say that $\mathcal{O}$ is self-dual if for any $\mathcal{E} \in \mathcal{O}$, with $\mathcal{E} : \mathcal{H}_d \to \mathcal{H}_{d'}$,

$$\text{Tr}[Y'\mathcal{E}(X)] = 0 \quad \text{and } \text{Tr}[X'\mathcal{E}(Y)] = 0 \tag{67}$$

for all $X \in \mathcal{V}_{\text{in}}$, $X' \in \mathcal{V}_{\text{out}}$, $Y \in \mathcal{V}_{\text{in}}^{\perp}$, and $Y' \in \mathcal{V}_{\text{out}}^{\perp}$. Moreover, we denote by $\mathcal{O}_{\text{sd}}$ the set of all CPTP maps $\mathcal{E} \in \mathcal{O}_{\max}$ that satisfy the above equations.

We have shown that, for ARTs, the condition that $\mathcal{E}$ is RNG can be expressed as in Eq. (36). Therefore, the dual map $\mathcal{E}^\dagger$ of a RNG map $\mathcal{E} \in \mathcal{O}_{\max}$ satisfies

$$\text{Tr}[X\mathcal{E}^\dagger(Y)] = 0 \;\forall\; X \in \mathcal{V}_{\text{in}} \quad \text{and} \;\forall\; Y \in \mathcal{V}_{\text{out}}^{\perp}. \tag{68}$$

Hence, in general $\mathcal{E}$ and $\mathcal{E}^\dagger$ satisfy two different conditions. However, if $\mathcal{O}$ is self-dual, and $\mathcal{E} \in \mathcal{O}$, then both $\mathcal{E}$ and $\mathcal{E}^\dagger$ satisfy the same conditions given in Eq. (67).

For ARTs with a resource destroying map [41], $\Delta : \mathcal{H}_d \to \mathcal{H}_d$, the conditions given in Eq. (67) take the following simple form:

$$\Delta \circ \mathcal{E} = \mathcal{E} \circ \Delta. \tag{69}$$

That is, $\mathcal{O}_{\text{sd}}$ in ARTs with a resource destroying map is precisely the set of Δ-commuting maps and in Refs. [23,24] were referred to as Δ-*covariant operations*.

Since the conditions in Eq. (67) can be expressed in the form (24), Lemma 3 implies the following:

Proposition 1. Let $\mathcal{R}(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}},\mathcal{O})$ be an ART with a self-dual set of free operations $\mathcal{O}$, and let $\mathcal{V}_{\text{in}}$, $\mathcal{V}_{\text{in}}^{\perp}$, and $\mathcal{O}_{\text{sd}}$ be as above. Assuming $\mathcal{F}_{\text{in}} \neq \emptyset$, let $\gamma' \in \mathcal{F}_{\text{out}}$ and $\gamma \in \mathcal{F}_{\text{in}}$ be free states, and let $\rho \in \mathcal{H}_{d,+,1}$ and $\rho' \in \mathcal{H}_{d',+,1}$ be two density matrices. Denote by $\mathcal{V}_{\text{in}}^{T} := \{X^T | X \in \mathcal{V}_{\text{in}}\}$ the set of the transposed matrices of all the matrices in $\mathcal{V}_{\text{in}}$. Then there exists $\mathcal{E} \in \mathcal{O}_{\text{sd}}$ such that $\rho' = \mathcal{E}(\rho)$ if and only if the matrix

$$M^{AB} = -\text{Tr}[Y\rho']I_{d'} \otimes \tau + Y \otimes \rho^T + N^{AB} \tag{70}$$

is not positive definite, for any matrix $N^{AB} \in (\mathcal{V}_{\text{out}}^{\perp} \otimes \mathcal{V}_{\text{in}}^{T}) \oplus (\mathcal{V}_{\text{out}} \otimes (\mathcal{V}_{\text{in}}^{\perp})^T)$, any $0 < \tau \in \mathcal{H}_{d,+,1}$, and any matrix $Y \in \mathcal{H}_{d'}$ such that $\text{Tr}[Y\gamma] = 0$.

Note that this proposition is almost identical to Lemma 4 except that, in this case, the space $\mathcal{V}_{\text{out}}^{\perp} \otimes \mathcal{V}_{\text{in}}^{T}$ that N^{AB} belongs to is replaced with the larger space $(\mathcal{V}_{\text{out}}^{\perp} \otimes \mathcal{V}_{\text{in}}^{T}) \oplus (\mathcal{V}_{\text{out}} \otimes (\mathcal{V}_{\text{in}}^{\perp})^T)$. Note that $\mathcal{V}_{\text{out}}^{\perp} \otimes \mathcal{V}_{\text{in}}^{T}$ is a subspace of this larger space, which is consistent with the fact that the self-dual set $\mathcal{O}_{\text{sd}}$ is a subset of $\mathcal{O}_{\max}$. We skip the proof of this proposition because it follows the exact same lines as the proof of Lemma 4.

VI. QUANTUM RESOURCE THEORIES WITH A RESOURCE DESTROYING MAP

Finally, we consider QRTs with a resource destroying map (RDM). Following the terminology of Ref. [41], we call a CPTP map $\Delta : \mathcal{H}_d \to \mathcal{H}_d$ a *resource destroying map* (RDM) if the following two conditions hold:

1. $\Delta(\rho) \in \mathcal{F} \equiv \mathcal{F}_{\text{in}} = \mathcal{F}_{\text{out}} \;\forall\; \rho \in \mathcal{H}_{d,+,1}$;
2. $\Delta(\rho) = \rho \;\forall\; \rho \in \mathcal{F}$.

While there is such a RDM in the QRTs of athermality, asymmetry, and coherence, a RDM does not always exist. For example, a simple consequence of the linearity of Δ implies that, if $\mathcal{F}$ is not convex, then the QRT does not consist of a RDM [41]. However, convexity of $\mathcal{F}$ is not enough to ensure the existence of Δ. Here we provide NSCs on the set of free states $\mathcal{F}$ that ensure the existence of a RDM, and in particular show that a QRT with an RDM *must* be affine. We also demonstrate with an example that not all ARTs have a RDM.

Lemma 5. Consider a QRT with the sets of free states $\mathcal{F}_{\text{in}} = \mathcal{F}_{\text{out}} = \mathcal{F}$ and let $\mathcal{V} := \text{span}_{\mathbb{R}}\{\mathcal{F}\}$. If there exists a RDM $\Delta : \mathcal{H}_d \to \mathcal{H}_d$ associated with the free set $\mathcal{F}$, then $\mathcal{F} = \mathcal{V} \cap \mathcal{H}_{d,+,1}$; i.e., $\mathcal{F}$ is affine.

Proof. From the linearity of Δ we get that $\Delta(A) = A$ for all $A \in \mathcal{V}$. Moreover, since $\mathcal{V} \cap \mathcal{H}_{d,+,1}$ is a subset of $\mathcal{V}$ we get $\Delta(\rho) = \rho$ for all $\rho \in \mathcal{V} \cap \mathcal{H}_{d,+,1}$. Hence, $\mathcal{V} \cap \mathcal{H}_{d,+,1}$ consists of only free states and therefore is a subset of $\mathcal{F}$. On the other hand, $\mathcal{F}$ is a subset of $\mathcal{V}$ and therefore also a subset of $\mathcal{V} \cap \mathcal{H}_{d,+,1}$. We therefore conclude $\mathcal{F} = \mathcal{V} \cap \mathcal{H}_{d,+,1}$. ■

Note that, if there exists a RDM Δ, then we must have $\Delta(X) = X$ for all $X \in \mathcal{V}$ (where $\mathcal{V}$ is defined in Lemma 5), and $\Delta(Y) \in \mathcal{V}$ for all $Y \in \mathcal{V}^{\perp}$, where $\mathcal{V}^{\perp}$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}_d$, so that $\mathcal{H}_d = \mathcal{V} \oplus \mathcal{V}^{\perp}$. If in addition, if $u_d = \frac{1}{d}I_d \in \mathcal{F}$ then Δ must be unital [i.e., $\Delta(I_d) = I_d$], and $\Delta(Y) = 0$ for all $Y \in \mathcal{V}^{\perp}$. To see it, note that for any $Z \in \mathcal{H}_d$ and $Y \in \mathcal{V}^{\perp}$,

$$\text{Tr}[Z\Delta(Y)] = \text{Tr}[\Delta(Z)Y] = 0, \tag{71}$$

since $\Delta(Z) \in \mathcal{V}$. Therefore, if the maximally mixed state u_d is free, then the problem simplifies dramatically:

Theorem 3. Using the same notations as above, let $m := \dim \mathcal{V}$, $n := \dim \mathcal{V}^{\perp} = d^2 - m$, $\{X_1, \ldots, X_m\}$ be an orthonormal basis of $\mathcal{V}$, and $\{Y_1, \ldots, Y_n\}$ be an orthonormal basis of $\mathcal{V}^{\perp}$. Suppose $u_d \equiv \frac{1}{d}I_d \in \mathcal{F}$, and define the linear map $\Delta : \mathcal{H}_d \to \mathcal{H}_d$ by the following action on the basis elements of $\mathcal{H}_d = \mathcal{V} \oplus \mathcal{V}^{\perp}$:

$$\Delta(X_j) = X_j \;\forall\; j \in \{1, \ldots, m\}, \tag{72}$$

$$\Delta(Y_k) = 0 \;\forall\; k \in \{1, \ldots, n\}. \tag{73}$$

Then there exists a RDM associated with the set $\mathcal{F}$ if and only if the following two conditions hold:

$$1. \quad \mathcal{F} = \mathcal{V} \cap \mathcal{H}_{d,+,1}; \tag{74}$$

$$2. \quad \sum_{j,k=1}^{d} \Delta(|j\rangle\langle k|) \otimes |j\rangle\langle k| \geqslant 0. \tag{75}$$

Moreover, in the case in which these two conditions hold, the RDM is unique and is given by Δ.

Proof. From the arguments above, if $\tilde{\Delta}$ is a RDM, then since $u_d \in \mathcal{F}$ we have that $\tilde{\Delta}$ is a unital CPTP map satisfying Eqs. (72) and (73) (with $\tilde{\Delta}$ replacing Δ). We therefore must have $\tilde{\Delta} = \Delta$. This completes the proof. ■

Remark. If the maximally mixed state $\frac{1}{d}I_d \notin \mathcal{F}$, then Δ is not unital. In this case, Δ is not necessarily unique, and the problem of finding the NSC that determines the existence of a RDM can be formulated as a feasibility problem in SDP. Below we use Lemma 3 to find these NSCs for the more general case of nonunital RDM.

The simplest example of a unital RDM can be found in the ART of coherence. There, $\mathcal{F}$ is the set of all diagonal density matrices with respect to some fixed basis $\{|j\rangle\}_{j=1}^{d}$. Thus, $u_d \in \mathcal{F}$ and Δ is the unique completely decohering map $\Delta(\cdot) = \sum_j |j\rangle\langle j|(\cdot)|j\rangle\langle j|$. Note that, in this case, both Eqs. (74) and (75) are satisfied, and the completely dephasing map Δ is the unique CPTP map that satisfys Eqs. (72) and (73).

Example. Real vs complex quantum mechanics. To see why the affine condition in Eq. (74) is not sufficient, consider the following mathematical model of real vs complex quantum mechanics. In this model, $\mathcal{F}$ is the set of all real density matrices with respect to some fixed basis $\{|j\rangle\}_{j=1}^{d}$. That is, $\rho \in \mathcal{F}$ if and only if $\langle j|\rho|k\rangle \in \mathbb{R}$ for all $j,k \in \{1,\dots,d\}$. Thus, $\frac{1}{d}I_d \in \mathcal{F}$, and the affine condition of the theorem holds; namely, $\mathcal{F} = \mathcal{V} \cap \mathcal{H}_{d,+,1}$. Note that

$$\mathcal{V} = \text{span}\{|j\rangle\langle k| + |k\rangle\langle j|\}_{j,k}, \quad j \leqslant k \in \{1,\dots,d\}, \tag{76}$$

and

$$\mathcal{V}^{\perp} = \text{span}\{i(|j\rangle\langle k| - |k\rangle\langle j|)\}, \quad j < k \in \{1,\dots,d\}. \tag{77}$$

According to the theorem above, if there exists a RDM Δ associated with $\mathcal{F}$, then it must satisfy $\Delta(|j\rangle\langle j|) = |j\rangle\langle j|$ for all $j = 1,\dots,d$, and for all $j < k \in \{1,\dots,d\}$,

$$\Delta(|j\rangle\langle k| + |k\rangle\langle j|) = |j\rangle\langle k| + |k\rangle\langle j|, \tag{78}$$

and

$$\Delta(i(|j\rangle\langle k| - |k\rangle\langle j|)) = 0. \tag{79}$$

Thus,

$$\Delta(|j\rangle\langle k|) = \tfrac{1}{2}(|j\rangle\langle k| + |k\rangle\langle j|)\ \forall\ j, \quad k \in \{1,\dots,d\}. \tag{80}$$

The Choi matrix of Δ is therefore given by

$$\sum_{j,k=1}^{d} \Delta(|j\rangle\langle k|) \otimes |j\rangle\langle k| = \frac{1}{2}\sum_{j,k=1}^{d} (|j\rangle\langle k| + |k\rangle\langle j|) \otimes |j\rangle\langle k|,$$

which is not positive semidefinite. Thus, there is no RDM associate with the set of real density matrices.

Existence of nonunital resource destroying map

In this section we discuss the more general case of nonunital RDMs; that is, we do not assume here that the maximally mixed state $\frac{1}{d}I_d$ is in $\mathcal{F}$. This makes the results much more technical since the RDM, if it exists, is not unital. The identity element $I_d \in \mathcal{H}_d$ can therefore be written as $I_d = P + Q$, where $P \in \mathcal{V}$ and $Q \in \mathcal{V}^{\perp}$. Since $\text{Tr}(PQ) = 0$ we must have

$$p := \text{Tr}(P) = \text{Tr}(P^2) \geqslant 0,$$
$$q := \text{Tr}(Q) = \text{Tr}(Q^2) = d - p \geqslant 0. \tag{81}$$

Moreover, note that if a RDM exists, $\mathcal{F}_d$ is nonempty, i.e., $\mathcal{V}$ contains at least one positive-semidefinite matrix with trace 1. In this case, $I \notin \mathcal{V}^{\perp}$ so that $Q \neq I$ and $P \neq 0$; thus, $p > 0$.

Set $m := \dim \mathcal{V}$ (thus $\dim \mathcal{V}^{\perp} = d^2 - m$) and let $\{X_1,\dots,X_m\}$ be an orthonormal basis of $\mathcal{V}$ with $X_1 = \frac{1}{\sqrt{p}}P$. Similarly, let $\{Y_1,\dots,Y_{d^2-m}\}$ be an orthonormal basis of $\mathcal{V}^{\perp}$ with $Y_1 = \frac{1}{\sqrt{q}}Q$ if $q > 0$. Note that $X_2,\dots,X_m$ and $Y_2,\dots,Y_{d^2-m}$ all have zero trace (and if $q = 0$ then Y_1 is also traceless).

Theorem 4. Using the same notation as above, let $\mathcal{W}$ be a subspace of $\mathcal{H}_d \otimes \mathcal{H}_d$ given by

$$\mathcal{W} := \text{span}_{\mathbb{R}}\big\{X_j \otimes Y_k^T\big\}_{j\in\{2,\dots,m\};k\in\{1,\dots,d^2-m\}}, \tag{82}$$

and let $\mathcal{W}^{\perp}$ be the orthogonal complement of $\mathcal{W}$ in $\mathcal{H}_d \otimes \mathcal{H}_d$. Set the matrix $G \in \mathcal{W}^{\perp}$ to be

$$G \equiv \frac{1}{d}\left(\sqrt{\frac{q}{p}}X_1 \otimes Y_1^T + \sum_{j=1}^{m} X_j \otimes X_j^T\right). \tag{83}$$

Finally, let $\mathcal{K} \subset \mathcal{H}_d \otimes \mathcal{H}_d$ be the subspace

$$\mathcal{K} := \{A \in \mathcal{W}^{\perp} | \text{Tr}[AG] = 0\}. \tag{84}$$

Then there exists a RDM corresponding to set of free states $\mathcal{F}$ if and only if the subspace $\mathcal{K}$ does not contain a positive-definite matrix.

Proof. With these notations, a CPTP map $\Delta : \mathcal{H}_d \to \mathcal{H}_d$ is a RDM if and only if for all $i,j \in \{1,\dots,m\}$ and all $k,\ell \in \{1,\dots,d^2 - m\}$

$$1. \quad \text{Tr}[X_i\Delta(X_j)] = \delta_{ij};$$
$$2. \quad \text{Tr}[Y_k\Delta(X_i)] = 0;$$
$$3. \quad \text{Tr}[Y_k\Delta(Y_\ell)] = 0. \tag{85}$$

We denote the Choi matrix of Δ by

$$\sigma^{AB} \equiv \sum_{j,k=1}^{d} \Delta(E_{jk}) \otimes E_{jk} \in \mathcal{H}_{d'} \otimes \mathcal{H}_d, \tag{86}$$

where $E_{jk} = |j\rangle\langle k|$. In the Choi representation, the three conditions above take the form

$$\text{(a)} \quad \text{Tr}\left[\sigma^{AB}\left(X_i \otimes X_j^T - \frac{1}{d}\delta_{ij} I \otimes I\right)\right] = 0;$$
$$\text{(b)} \quad \text{Tr}\big[\sigma^{AB}\big(Y_k \otimes X_i^T\big)\big] = 0;$$
$$\text{(c)} \quad \text{Tr}\big[\sigma^{AB}\big(Y_k \otimes Y_\ell^T\big)\big] = 0. \tag{87}$$

The condition $\text{Tr}_A[\sigma^{AB}] = I$ is equivalent to

$$\text{Tr}[\sigma^{AB}(I \otimes Z)] = 0\ \forall\ Z \in \mathcal{H}_{d,0}. \tag{88}$$

Note that the condition $\text{Tr}[\sigma^{AB}] = d$ was removed since if there exists the positive-semidefinite matrix $\sigma^{AB} \neq 0$ that satisfies all the above conditions, then the matrix $\frac{d}{\text{Tr}[\sigma^{AB}]}\sigma^{AB}$ will satisfy also this last condition. Moreover, note that the condition in Eq. (88) is not independent of the three conditions in Eq. (87). Particularly, conditions (b) and (c) imply that $\text{Tr}[\sigma^{AB}(Q \otimes Z)] = 0$ for all Z in $\mathcal{H}_d$ (and therefore for all $Z \in \mathcal{H}_{d,0}$) so that condition (88) can be replaced with

$$\text{Tr}[\sigma^{AB}(X_1 \otimes Z)] = 0 \,\forall\, Z \in \mathcal{H}_{d,0}. \tag{89}$$

Next, note that if $Z \in \mathcal{H}_{d,0} \cap \mathcal{V}$ then the above condition follows from (a) in Eq. (87). We can therefore assume that Z has the form

$$Z = \alpha\left(\frac{1}{\sqrt{p}}X_1 - \frac{1}{\sqrt{q}}Y_1\right) + Y, \quad \alpha \in \mathbb{R}, \quad Y \in \mathcal{H}_{d,0} \cap \mathcal{V}^{\perp}, \tag{90}$$

assuming $q > 0$. If $q = 0$ we just take $Z \in \mathcal{H}_{d,0} \cap \mathcal{V}^{\perp}$. The above condition is still not completely independent of condition (a) in Eq. (87). To make it independent we replace Eq. (88) with the following condition: For all $k \in \{1, \ldots, d^2 - m\}$,

$$\text{Tr}\left[\sigma^{AB}\left(X_1 \otimes Y_k^T - \frac{r}{d}\delta_{1k} I \otimes I\right)\right] = 0, \tag{91}$$

where $r := \sqrt{q/p}$. Note that the equality $\text{Tr}[\sigma^{AB}(X_1 \otimes W)] = 0$ with $W := \frac{1}{\sqrt{p}}X_1 - \frac{1}{\sqrt{q}}Y_1 \in \mathcal{H}_{d,0}$ follows from Eq. (91) for $k = 1$, together with (a) in Eq. (87) with $i = j = 1$.

To see when there exists such a semidefinite positive matrix σ^{AB} that satisfies Eq. (87) we apply the following generalization of the Farkas lemma from linear programming to SDP:

Thus, from the Farkas lemma, we get that such a σ^{AB} exists if and only if the matrix

$$\begin{aligned} M := &\sum_{i,j=1}^{m} a_{ji}\left(X_i \otimes X_j^T - \frac{1}{d}\delta_{ij} I \otimes I\right) \\ &+ \sum_{k=1}^{d^2-m}\sum_{i=1}^{m} b_{ki} Y_k \otimes X_i^T + \sum_{k,\ell=1}^{d^2-m} c_{k\ell} Y_k \otimes Y_\ell^T \\ &+ \sum_{k=1}^{d^2-m} d_k\left(X_1 \otimes Y_k^T - \frac{r}{d}\delta_{1k} I \otimes I\right) \end{aligned} \tag{92}$$

is not positive definite for all $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{(d^2-m)\times m}$, $C \in \mathbb{R}^{(d^2-m)\times(d^2-m)}$, and $d_k \in \mathbb{R}$. Note that M can be written as $M = M' - \text{Tr}[M'G]I$, where M' is an arbitrary matrix in $\mathcal{W}^{\perp}$. Moreover, since $\text{Tr}[G] = 1$ we have $\text{Tr}[MG] = 0$; that is, M is any matrix in $\mathcal{W}^{\perp}$ satisfying $\text{Tr}[MG] = 0$. This completes the proof. ∎

VII. CONCLUSIONS

To summarize, we studied ARTs in which the set of free states satisfies the condition (2). We used the strong duality of SDP to derive the conditions that determines whether it is possible to convert one resource to another by RNG operations. As an application, we showed particularly how our results can be applied to quantum thermodynamics with Gibbs preserving operations and to quantum coherence with maximally incoherent operations. Remarkably, we were able to express the conditions in the form of a family of resource monotones that are given in terms of the conditional min-entropy.

We were able to apply SDP techniques to ARTs because the conditions in Eq. (36) are linear in $\mathcal{E}$. However, linear conditions are clearly not limited to ARTs. There exists QRTs that are not affine for which techniques similar to those of SDP can also be applied. One such example is the QRT of entanglement with PPT operations [45]. On the other hand, as we have shown, the set of PPT or separable bipartite density matrices does not satisfy (2) and therefore PPT entanglement is not an ART.

It is important to note that SDP feasibility problems are not necessarily computationally easy to solve. In fact, some SDP feasibility problems are known to be NP-hard [46]. In our context, the reason we encounter a SDP *feasibility* problem is that we only considered *exact* transformations. Therefore, the fact that the strong duality leads to an infinite number of conditions [as in Eqs. (19) and (20)] is inevitable for the general case of ARTs. It may be possible to simplify these conditions when considering approximate single-shot transformations.

The implications of the results presented here go far beyond the scope of this paper. They include, for instance, generalizations of the results to approximate transformations, as well as catalysis-assisted transformations. Moreover, some of the techniques we used here can also be applied outside the scope of resource theories. We hope to report soon [47] on their applications in quantum hypothesis testing. Finally, while the work presented here assumes that the free operations are maximal (or self-dual), we believe that similar techniques can also be applied to operations that are not maximal, such as thermal operations in quantum thermodynamics and symmetric operations in the QRT of asymmetry. We leave these investigations for future work [47].

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