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Fast forward of adiabatic control of tunneling states

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By developing the preceding work on the fast forward of transient phenomena of quantum tunneling by Khujakulov and Nakamura [Phys. Rev. A **93**, 022101 (2016)], we propose a scheme of the exact fast forward of adiabatic control of stationary tunneling states with use of the electromagnetic field. The idea allows the acceleration of both the amplitude and phase of wave functions throughout the fast-forward time range. The scheme realizes the fast-forward observation of the transport coefficients under the adiabatically changing barrier with the fixed energy of an incoming particle. As typical examples, we choose systems with (1) Eckart's potential with tunable asymmetry and (2) double δ-function barriers under tunable relative height. We elucidate the driving electric field to guarantee the stationary tunneling state during a rapid change of the barrier and evaluate both the electric-field-induced temporary deviation of transport coefficients from their stationary values and the modulation of the phase of complex scattering coefficients.

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I. INTRODUCTION

Various methods to control quantum states have been reported in Bose-Einstein condensates (BEC), quantum computations, and many other fields of applied physics. It is important to consider the speed-up of such manipulations of quantum states for manufacturing purposes and for innovation of technology, because the coherence of systems is degraded by their interaction with the environment.

Masuda and Nakamura [1–3] investigated a way to accelerate quantum dynamics with use of a characteristic driving potential determined by the additional phase of a wave function. This kind of acceleration is called the fast forward, which means to reproduce a series of events or a history of matters in a shortened time scale, like a rapid projection of movie films on the screen.

The fast forward theory applied to quantum adiabatic dynamics [2,3] assumes that a product of the mean value $\bar{\alpha}$ of an infinitely large time-scaling factor $\alpha(t)$ and an infinitesimally small growth rate ϵ in the quasiadiabatic parameter should satisfy the constraint $\bar{\alpha} \times \epsilon = \mathit{finite}$ in the asymptotic limit $\bar{\alpha} \to \infty$ and $\epsilon \to 0$. The scheme needs no knowledge of spectral properties of the system and is free from the initial and boundary value problem. Therefore, it constitutes one of the promising ways of shortcuts to adiabaticity (STA) devoted to tailor excitations in nonadiabatic processes [4–9,11–13]. Some papers [14,15] made clear the relationship between the fast-forward approach and other STA protocols. Recent interesting application of the fast-forward theory can be found in acceleration of Dirac dynamics [16] and in accelerated construction of classical adiabatic invariant under nonadiabatic circumstances [17].

Although Masuda and Nakamura's works guarantee the exact target state at the fast-forward final time $t = T_{\mathrm{FF}}$, in the intermediate time range $0 \leqslant t \leqslant T_{\mathrm{FF}}$ they accelerate only the amplitude of the wave function and fail to accelerate its phase because of the nonvanishing additional phase on the way.

Up to now the adiabatic states to be fast forwarded are limited to bound states. If one wants to accelerate the current-carrying scattering states, one must innovate the scheme so as to keep the original phase exactly in the intermediate time range until $t = T_{\mathrm{FF}}$.

Recently, in the context of the transient phenomena of quantum tunneling, Khujakulov and Nakamura [18] found a way of fast-forwarding of quantum dynamics for charged particles by applying the electromagnetic field, which exactly accelerates both amplitude and phase of the wave function throughout the fast-forward time range. This means the fast forward with complete fidelity. The scheme suggests a possibility to accelerate the adiabatic control of stationary scattering states under the fixed energy of an incoming particle. The scheme of Khujakulov and Nakamura as it stands, however, is not useful and must be innovated so as to be suitable to the adiabatic dynamics characterized by infinitesimally slowly changing control parameters, such as the height and shape of potential barriers.

In this paper, we develop Khujakulov and Nakamura's scheme so that it can be applicable to the fast forward of stationary tunneling states under the adiabatically changing potential barrier. To make the paper self-sustained, we shall sketch the general theory of fast forward with complete fidelity [18] in Sec. II. In Sec. III, the theory is extended to the fast forward of stationary tunneling dynamics through adiabatically changing barriers under the fixed energy of an incoming particle. In Sec. IV we show the time-dependent transport coefficients during fast forwarding. In Sec. V typical examples are presented, where we choose systems with (1) Eckart's potential with tunable asymmetry and (2) double δ-function barriers with tunable relative height. Conclusion is given in Sec. VI. Appendix A is devoted to the gauge transformation of the present scheme to Masuda-Nakamura's one with incomplete fidelity. Appendix B and C treat the technical details to derive some relevant equations.

II. GENERAL FAST-FORWARD THEORY WITH COMPLETE FIDELITY

The Schrödinger equation for a charged particle in standard time with a nonlinearity constant c_0 (appearing in macroscopic quantum dynamics) is represented as

$$\imath\hbar\frac{\partial\psi_0}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_0 + V_0(\mathbf{x},t)\psi_0 - c_0|\psi_0|^2\psi_0, \quad (1)$$

where the coupling with electromagnetic field is assumed to be absent. $\psi_0 \equiv \psi_0(\mathbf{x},t)$ is a known function of space $\mathbf{x}$ and time t under a given potential $V_0(\mathbf{x},t)$ and is called a standard state. For any long time T called as a standard final time, we choose $\psi_0(t=T)$ as a target state that we are going to generate in a shorter time.

Let $\Lambda(t)$ be the advanced time defined by

$$\Lambda(t) = \int_0^t \alpha(t')\,dt', \quad (2)$$

where t is a time scale shorter than the standard one. $\alpha(t)$ is a magnification time-scale factor given by $\alpha(0)=1$, $\alpha(t) > 1(0<t<T_{\rm FF})$ and $\alpha(t)=1(t\geqslant T_{\rm FF})$. We consider the fast-forward dynamics with a new time variable, which reproduces the target state $\psi_0(T)$ in a shorter final time $T_{\rm FF}(<T)$ defined by

$$T = \int_0^{T_{\rm FF}} \alpha(t)dt. \quad (3)$$

The explicit expression for $\alpha(t)$ in the fast-forward range $(0 \leqslant t \leqslant T_{\rm FF})$ is typically given by Refs. [1–3] as

$$\alpha(t) = \bar{\alpha} - (\bar{\alpha}-1)\cos\left(\frac{2\pi}{T_{\rm FF}}t\right), \quad (4)$$

where $\bar{\alpha}$ is the mean value of $\alpha(t)$ and is given by $\bar{\alpha} = T/T_{\rm FF}$. Besides the time-dependent scaling factor in Eq. (4) in the fast-forward time range, we can also have recourse to the uniform scaling factor $\alpha(t)=\bar{\alpha}(0\leqslant t\leqslant T_{\rm FF})$, which is useful in the quantitative analysis of fast forward.

The fast-forward wave function $\psi_{\rm FF}$ in this paper does not include the additional phase and is given by

$$\psi_{\rm FF}(\mathbf{x},t) = \psi_0(\mathbf{x},\Lambda(t)) \equiv \tilde{\psi}_0(\mathbf{x},t). \quad (5)$$

$\psi_{\rm FF}$ is just like a movie film projected on the screen in a shortened time scale. Equation (5) guarantees the complete fidelity, namely $\langle\psi_{\rm FF}|\tilde{\psi}_0\rangle = 1$ throughout the fast-forward time range. We shall realize $\psi_{\rm FF}$ by applying the electromagnetic field $\mathbf{E}_{\rm FF}$ and $\mathbf{B}_{\rm FF}$, which are related to vector $\mathbf{A}_{\rm FF}(\mathbf{x},t)$ and scalar $V_{\rm FF}(\mathbf{x},t)$ potentials through

$$\mathbf{E}_{\rm FF} = -\frac{\partial\mathbf{A}_{\rm FF}}{\partial t} - \nabla V_{\rm FF}, \quad \mathbf{B}_{\rm FF} = \nabla\times\mathbf{A}_{\rm FF}. \quad (6)$$

Let's assume $\psi_{\rm FF}$ to be the solution of the Schrödinger equation for a charged particle in the presence of $\mathbf{A}_{\rm FF}(\mathbf{x},t)$ and $V_{\rm FF}(\mathbf{x},t)$, as given by

$$\begin{aligned}\imath\hbar\frac{\partial\psi_{\rm FF}}{\partial t} &= \hat{H}_{\rm FF}\psi_{\rm FF}\\ &\equiv \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}_{\rm FF}\right)^2 + qV_{\rm FF} + V_0\right]\psi_{\rm FF}\\ &\quad - c_0|\psi_{\rm FF}|^2\psi_{\rm FF}. \end{aligned} \quad (7)$$

For simplicity, we shall hereafter employ the unit velocity of light $c=1$ and the prescription of a positive unit charge $q=1$. $V_{\rm FF}$ in Eq. (7) is introduced independently from a given potential V_0, in contrast to the preceding work [1]. The electromagnetic field investigated in Refs. [3,19] was not used to suppress the additional phase.

Replacing t by $\Lambda(t)$ in Eq. (1) and noting Eq. (5), we can eliminate $\frac{\partial\tilde{\psi}_0}{\partial t}$ between Eqs. (1) and (7). The resultant equality is decomposed into real and imaginary parts as respectively given by

$$\nabla\cdot\mathbf{A}_{\rm FF} + 2{\rm Re}\left[\frac{\nabla\tilde{\psi}_0}{\tilde{\psi}_0}\right]\cdot\mathbf{A}_{\rm FF} + \hbar(\alpha-1){\rm Im}\left[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}\right] = 0 \quad (8)$$

and

$$\begin{aligned}V_{\rm FF} = &-(\alpha-1)\frac{\hbar^2}{2m}{\rm Re}\left[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}\right] + \frac{\hbar}{m}\mathbf{A}_{\rm FF}{\rm Im}\left[\frac{\nabla\tilde{\psi}_0}{\tilde{\psi}_0}\right]\\ &- \frac{1}{2m}\mathbf{A}_{\rm FF}^2 + (\alpha-1)V_0 - (\alpha-1)c_0|\tilde{\psi}_0|^2. \end{aligned} \quad (9)$$

Rewriting $\tilde{\psi}_0$ in terms of the real positive amplitude ρ and phase η as

$$\tilde{\psi}_0 = \rho(\mathbf{x},\Lambda(t))\exp[i\eta(\mathbf{x},\Lambda(t))], \quad (10)$$

we find that

$$\mathbf{A}_{\rm FF} = -\hbar(\alpha-1)\nabla\eta \quad (11)$$

satisfies Eq. (8). Using Eq. (11), $V_{\rm FF}$ can be expressed only in terms of η as

$$V_{\rm FF} = -(\alpha-1)\hbar\frac{\partial\eta}{\partial\Lambda(t)} - \frac{\hbar^2}{2m}(\alpha^2-1)(\nabla\eta)^2. \quad (12)$$

Applying the driving vector $\mathbf{A}_{\rm FF}$ and scalar $V_{\rm FF}$ potentials in Eqs. (11) and (12), we can realize the fast-forwarded state $\psi_{\rm FF}$ in Eq. (5), which is now free from the additional phase f used in Ref. [1].

Two points should be noted: (1) The above driving potentials do not explicitly depend on the nonlinearity coefficient c_0: Eqs. (11) and (12) work for the nonlinear Schrödinger equation as well; (2) the magnetic field $\mathbf{B}_{\rm FF}$ is vanishing, because a combination of Eqs. (6) and (11) leads to $\mathbf{B}_{\rm FF} = \nabla\times\mathbf{A}_{\rm FF} = 0$. Therefore, only the electric field $\mathbf{E}_{\rm FF}$ is required to accelerate a given dynamics. With use of Eqs. (6), (11), and (12), we find $\mathbf{E}_{\rm FF} = \hbar\dot{\alpha}\nabla\eta + \hbar\frac{\alpha^2-1}{\alpha}\partial_t\nabla\eta + \frac{\hbar^2}{2m}(\alpha^2-1)\nabla(\nabla\eta)^2$.

A remarkable issue of the present scheme is the enhancement of the current density $\mathbf{j}_{\rm FF}$. Using a generalized momentum, which includes a contribution from the vector potential in Eq. (11), we see

$$\begin{aligned}\mathbf{j}_{\rm FF}(\mathbf{x},t) &\equiv {\rm Re}\left[\psi_{\rm FF}^*(\mathbf{x},t)\frac{1}{m}\left(\frac{\hbar}{i}\nabla - \mathbf{A}_{\rm FF}\right)\psi_{\rm FF}(\mathbf{x},t)\right]\\ &= \frac{\hbar}{m}\alpha(t)\rho^2(\mathbf{x},\Lambda(t))\nabla\eta(\mathbf{x},\Lambda(t))\\ &= \alpha(t)\mathbf{j}(\mathbf{x},\Lambda(t)) \end{aligned} \quad (13)$$

under the prescription of a positive unit charge, where the current density in the standard dynamics is defined by $\mathbf{j}(\mathbf{x},t) \equiv {\rm Re}[\psi_0^*(\mathbf{x},t)\frac{\hbar}{im}\nabla\psi_0(\mathbf{x},t)] = \frac{\hbar}{m}\rho^2(\mathbf{x},t)\nabla\eta(\mathbf{x},t)$. Thus, the standard current density becomes both time-squeezed and magnified by a time-scaling factor $\alpha(t)$ in Eq. (4) as a result of the exact fast forwarding of wave function throughout the time evolution. The present scheme is applicable to the fast forward of diverse quantum-mechanical phenomena.

III. FAST FORWARD OF ADIABATIC CHANGE OF TUNNELING STATES

Section II was concerned with the fast forward of standard dynamics with standard time scale. From now on, we shall investigate the fast forward of very slow dynamics, i.e., of quasiadiabatic dynamics. Confining to one-dimensional (1D) system and suppressing the nonlinear term proportional to c_0, we shall apply the scheme in Sec. II to stationary tunneling states under an adiabatically changeable potential barrier, and show the fast forward of adiabatic control of (1D) tunneling states with use of the electromagnetic field. The goal of this Section is to obtain the driving gauge potentials and electric field to guarantee such fast forwarding.

We shall take the following strategy: (i) A given potential barrier V_0 is assumed to change adiabatically, and we find a stationary state ψ_0, which is a solution of the time-independent Schrödinger equation with the instantaneous Hamiltonian; (ii) then both ψ_0 and V_0 are regularized so that they should satisfy the time-dependent Schrödinger equation; (iii) finally, taking the regularized state as a standard state, we apply the scheme in Sec. II, where the mean value $\bar{\alpha}$ of the infinitely large time scaling factor $\alpha(t)$ will be chosen to cope with the infinitesimally-small growth rate ϵ of the quasiadiabatic parameter and to satisfy $\bar{\alpha} \times \epsilon = finite$.

Let's consider the standard dynamics with a potential barrier characterized by a slowly varying control parameter $R(t)$ given by

$$R(t) = R_0 + \epsilon t, \tag{14}$$

with the growth rate $\epsilon \ll 1$, which means that it requires a very long time $T = O(\frac{1}{\epsilon})$, to see the recognizable change of $R(t)$. The time-dependent 1D Schrödinger equation without the nonlinear term is

$$i\hbar\frac{\partial\psi_0}{\partial t} = -\frac{\hbar^2}{2m}\partial_x^2\psi_0 + V_0(x,R(t))\psi_0. \tag{15}$$

The stationary tunneling state ϕ_0 satisfies the time-independent counterpart given by

$$E\phi_0 = \hat{H}_0\phi_0 \equiv \left[-\frac{\hbar^2}{2m}\partial_x^2 + V_0(x,R)\right]\phi_0. \tag{16}$$

Without loss of generality, we assume that $V_0(x,R)$ is R-independent constant for $x \leqslant x_1$ and $x \geqslant x_2$ and shows a R-dependent variation for $x_1 \leqslant x \leqslant x_2$. In fact, potential barriers are adiabatically controllable in a finite spatial region.

In the case of the bound states, the boundary condition for ϕ_0 is $\phi_0 \to 0$ at $|x| \to \infty$, giving the discrete energy spectra. In the case of scattering states, which includes tunneling states, however, an arbitrary one of the continuum energy is first given, which then determines the stationary scattering state.

Here we investigate the following situation: (1) The potential barrier $V_0(x,R)$ is deformed very slowly through the adiabatic parameter R; (2) during the above adiabatic deformation of $V_0(x,R)$, the energy of a plane-wave type particle incoming from the left is assumed to be R-independent and fixed; i.e.,

$$\frac{\partial E}{\partial R} = 0. \tag{17}$$

Then, with use of the stationary tunneling state ϕ_0 satisfying Eq. (16), one might conceive the corresponding time-dependent state to be a product of ϕ_0 and a dynamical factor as

$$\psi_0 = \phi_0(x,R(t))e^{-\frac{i}{\hbar}Et}. \tag{18}$$

However, ψ_0 as it stands does not satisfy Eq. (15). Therefore, we introduce a regularized state,

$$\psi_0^{\text{reg}} \equiv \phi_0(x,R(t))e^{i\epsilon\theta(x,R(t))}e^{-\frac{i}{\hbar}Et} \equiv \phi_0^{\text{reg}}(x,R(t))e^{-\frac{i}{\hbar}Et}, \tag{19}$$

together with a regularized potential,

$$V_0^{\text{reg}} \equiv V_0(x,R(t)) + \epsilon\tilde{V}(x,R(t)). \tag{20}$$

θ and $\tilde{V}$ will be determined self-consistently so that ψ_0^{reg} should fulfill the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\psi_0^{\text{reg}}}{\partial t} = -\frac{\hbar^2}{2m}\partial_x^2\psi_0^{\text{reg}} + V_0^{\text{reg}}\psi_0^{\text{reg}}, \tag{21}$$

up to the order of ϵ.

Rewriting $\phi_0(x,R(t))$ with use of the real positive amplitude $\bar{\phi}_0(x,R(t))$ and phase $\eta(x,R(t))$ as

$$\phi_0(x,R(t)) = \bar{\phi}_0(x,R(t))e^{i\eta(x,R(t))}, \tag{22}$$

we see θ and $\widetilde{V}$ to satisfy

$$\partial_x(\bar{\phi}_0^2\partial_x\theta) = -\frac{m}{\hbar}\partial_R\bar{\phi}_0^2, \tag{23}$$

$$\frac{\tilde{V}}{\hbar} = -\partial_R\eta - \frac{\hbar}{m}\partial_x\eta\cdot\partial_x\theta. \tag{24}$$

Integrating Eq. (23) over x, we have

$$\partial_x\theta = -\frac{m}{\hbar}\frac{1}{\bar{\phi}_0^2}\int_c^x\partial_R\bar{\phi}_0^2dx', \tag{25}$$

with c an arbitrary R-independent constant. Equation (25) determines $\tilde{V}$ in Eq. (24).

In the stationary (or steady) scattering state, the current density available from Eqs. (18) with (22),

$$\text{Re}\left[\psi_0^*\frac{\hbar}{im}\partial_x\psi_0\right] = \frac{\hbar}{m}\bar{\phi}_0^2(x,R)\partial_x\eta(x,R), \tag{26}$$

is space-independent and non-zero constant. Therefore, $\bar{\phi}_0$ cannot be zero and the right-hand side of Eq. (25) is free from the problem of wave function nodes proper to excited states of bound systems. See also Appendix A.

Applying the scheme in Sec. II, we shall take ψ_0^{reg} as a standard state and define its fast-forward version ψ_{FF} as

$$\psi_{\text{FF}}(x,t) \equiv \phi_0^{\text{reg}}[x,R(\Lambda(t))]e^{-\frac{i}{\hbar}Et} \equiv \tilde{\phi}_0^{\text{reg}}(x,t)e^{-\frac{i}{\hbar}Et}. \tag{27}$$

$\psi_{\text{FF}}(x,t)$ is then assumed to obey the time-dependent Schrödinger equation for a charged particle in the presence of electromagnetic field, as in Eq. (7). Then, $\tilde{\phi}_0^{\rm reg}(x,t)$ satisfies

$$i\hbar\frac{\partial\tilde{\phi}_0^{\rm reg}}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\partial_x - \frac{q}{c}A_{\rm FF}\right)^2\tilde{\phi}_0^{\rm reg} + (qV_{\rm FF} + V_0 - E + \epsilon\tilde{V})\tilde{\phi}_0^{\rm reg}, \tag{28}$$

where $A_{\rm FF}$ and $V_{\rm FF}$ are gauge potentials to guarantee the exact fast forward. Here $V_0 \equiv V_0[x,R(\Lambda(t))]$ and $\tilde{V} \equiv \tilde{V}[x,R(\Lambda(t))]$. The dynamical phase in Eq. (27) has led to the energy shift in the potential in Eq. (28).

In the context of the fast forward of the adiabatic control, it is essential to analyze equalities in Eqs. (8) and (9) directly, because $\tilde{\psi}_0$ and V_0 there should now be read as

$$\tilde{\psi}_0 \to \tilde{\phi}_0^{\rm reg} \equiv \bar{\phi}_0[x,R(\Lambda(t))]e^{i\{\eta[x,R(\Lambda(t))]+\epsilon\theta[x,R(\Lambda(t))]\}} \tag{29}$$

and

$$V_0 \to V_0 - E + \epsilon\tilde{V}, \tag{30}$$

respectively. Then Eqs. (8) and (9) lead to the driving $A_{\rm FF}$ and $V_{\rm FF}$ potentials to realize the fast-forward state $\psi_{\rm FF}$ in Eq. (27):

$$A_{\rm FF} = -\hbar\epsilon(\alpha-1)\partial_x\theta \tag{31}$$

and

$$V_{\rm FF} = -\frac{\hbar^2}{m}\epsilon(\alpha-1)\partial_x\theta\partial_x\eta - \alpha(\alpha-1)\frac{\hbar^2}{2m}\epsilon^2(\partial_x\theta)^2 - \epsilon(\alpha-1)\hbar\partial_R\eta. \tag{32}$$

The derivation of Eqs. (31) and (32) is given in Appendix B.

Now, applying our central strategy to take the limit $\epsilon \to 0$ and $\bar{\alpha} \to \infty$ with $\epsilon\bar{\alpha} = \bar{v}$ being kept finite, we can reach the issue

$$A_{\rm FF} = -\hbar v(t)\partial_x\theta,$$
$$V_{\rm FF} = -\frac{\hbar^2}{m}v(t)\partial_x\theta\partial_x\eta - \frac{\hbar^2}{2m}(v(t))^2(\partial_x\theta)^2 - \hbar v(t)\partial_R\eta, \tag{33}$$

where, with use of $T_{\rm FF}[=\frac{T}{\bar{\alpha}} = O(\frac{1}{\epsilon\bar{\alpha}})] = \textit{finite}$,

$$v(t) \equiv \lim_{\epsilon\to 0,\bar{\alpha}\to\infty}\varepsilon\alpha(t) = \bar{v}\left(1-\cos\frac{2\pi}{T_{\rm FF}}t\right),$$
$$R(\Lambda(t)) = R_0 + \lim_{\epsilon\to 0,\bar{\alpha}\to\infty}\varepsilon\Lambda(t) = R_0 + \bar{v}\left[t - \frac{T_{\rm FF}}{2\pi}\sin\left(\frac{2\pi}{T_{\rm FF}}t\right)\right],$$
$$\text{for}\quad 0 \leqslant t \leqslant T_{\rm FF}, \tag{34}$$

and

$$v(t) = 0,\quad R(\Lambda(t)) = R_0 + \bar{v}T_{\rm FF}\quad \text{for}\quad T_{\rm FF} \leqslant t. \tag{35}$$

$v(t)$ and its mean $\bar{v}$ stand for the time-scaling factors coming from $\alpha(t)$ and $\bar{\alpha}$, respectively.

In the same limiting case as above, $\psi_{\rm FF}$ is explicitly given by

$$\psi_{\rm FF} = \bar{\phi}_0[x,R(\Lambda(t))]e^{i\eta[x,R(\Lambda(t))]}e^{-\frac{i}{\hbar}Et}, \tag{36}$$

and describes the acceleration of the adiabatic control of stationary scattering states throughout the fast-forward time range until $t \leqslant T_{\rm FF}$. It should be emphasized: while $\bar{\alpha} \to +\infty$ is assumed, the gauge potential and electromagnetic field are of finite order [i.e., $O(\bar{v})$ or $O(\bar{v}^2)$].

From Eq. (33), the driving electric field to guarantee the fast-forward state in Eq. (36) is given by

$$E_{\rm FF} = -\frac{\partial A_{\rm FF}}{\partial t} - \partial_x V_{\rm FF} = \hbar\dot{v}\partial_x\theta + \hbar v^2(t)\partial_R(\partial_x\theta) + \frac{\hbar^2}{2m}v(t)\partial_x(\partial_x\theta\partial_x\eta) + \frac{\hbar^2}{2m}(v(t))^2\partial_x(\partial_x\theta)^2 + \hbar v\partial_R(\partial_x\eta). \tag{37}$$

In SI unit for electric field, our dimensionless $E_{\rm FF}$ corresponds to $E_{\rm SI}^{\rm FF} = \frac{m_e c\omega}{e} \times E_{\rm FF} \sim \frac{10^6}{\lambda}E_{\rm FF}$ where m_e,e,c,ω and λ are electron mass, electron charge, velocity of light, frequency of laser light and its wave length, respectively. Typical value $E_{\rm FF} = 1$ in case of IR lasers of wave length $\sim 1\,\mu$m means $E_{\rm SI}^{\rm FF} = 10^{12}$.

Note: (1) We need the space- (and time-)dependent electric field $E_{\rm FF}$ along the 1D target system on x axis, which means that $\partial_x E_{\rm FF}$ is nonzero. On the other hand, the Maxwell's equation (Gauss's law) requires the divergence of electric field $= \partial_x E_x + \partial_y E_y + \partial_z E_z =$ charge density. The experimental setup to be compatible with the Maxwell's equation is to apply the electric field (surrounding the target system), which has three components and exists in 3D space, so that the perpendicular components (E_y,E_z) should satisfy $\partial_y E_y + \partial_z E_z = -\partial_x E_x (\equiv -\partial_x E_{\rm FF})$ along the x axis. An example is to prepare an infinite straight rod that is detached from and perpendicular to the target system and to introduce the inhomogeneous charge distribution along the rod so that $E_x = E_{\rm FF}$ should appear along the x axis. In this case, no charge distribution is necessary along the target system. (2) The time-dependent electric field might induce a magnetic B field due to the Ampere-Maxwell's equation. Since we are concerned with 1D tunneling and the electric field is applied along the x direction, such B field is perpendicular to x axis, and the Lorentz force working on the target particle is perpendicular to both x axis and the direction of B field. Therefore, B field plays no role in the tunneling along x axis.

In closing this section, we should note: the scheme here is the theory of fast forward with complete fidelity, but is compatible with that of the preceding one with the additional phase [2,3], as proved by using the gauge transformation in Appendix A.

IV. FAST FORWARD OF OBSERVATION OF ADIABATICALLY TUNABLE TRANSPORT COEFFICIENTS

Now we shall elucidate the time-dependent transport (i.e., transmission and reflection) coefficients during the accelerated adiabatic control of stationary tunneling states.

With use of the results in Eqs. (33) and (36), the current density $j_{\rm FF}$ during the fast forward time region becomes

$$j_{\rm FF}(x,t) = {\rm Re}\left[\psi^*_{\rm FF}(x,t)\frac{1}{m}\left(\frac{\hbar}{i}\partial_x - A_{\rm FF}\right)\psi_{\rm FF}(x,t)\right] = j_{\rm ad}(x,t) + j_{\rm nad}(x,t), \quad (38)$$

where

$$j_{\rm ad}(x,t) \equiv \frac{\hbar}{m}\bar{\phi}_0^2[x,R(\Lambda(t))]\partial_x\eta[x,R(\Lambda(t))] \quad (39)$$

and

$$j_{\rm nad}(x,t) \equiv v(t)\frac{\hbar}{m}\bar{\phi}_0^2[x,R(\Lambda(t))]\partial_x\theta[x,R(\Lambda(t))] = -v(t)\int_c^x \partial_R\bar{\phi}_0^2 dx'. \quad (40)$$

The last equality of Eq. (40) comes from Eq. (25). The decomposition of $j_{\rm FF}$ into two parts as in Eq. (38) was not seen in the fast forward of the standard dynamics in Sec. II. The adiabatic current $j_{\rm ad}$ guarantees transmission and reflection coefficients to precisely reproduce the stationary values during the period of fast forward because of the complete fidelity of $\psi_{\rm FF}(x,t)$. On the other hand, the nonadiabatic current $j_{\rm nad}$ caused by the driving electric field $E_{\rm FF}(t)$ in Eq. (37) vanishes at the end of fast forward.

The adiabatic potential barrier $V_0(x,R(t))$ is characterized by a slowly-varying control parameter $R(t)$ in Eq. (14). As noted in the previous Section, we shall choose $V_0 = 0$ and $V_0 = V_0^c$ (R-independent constant) for $x \leqslant x_1$ and $x \geqslant x_2$, respectively, assuming that the R-dependent barrier exists only in the range $x_1 \leqslant x \leqslant x_2$.

Before reaching the formula for time-dependent transport coefficients, we shall sketch the stationary state and show the time-independent transport coefficients in $1D$ systems with the barrier in the adiabatic limit $R(t) = R =$ constant. For the electron with R-independent energy E incoming from the left, the wave function for $x \leqslant x_1$ and $x \geqslant x_2$ is given, respectively, by

$$\psi_0 = [e^{ikx} + r_f(R)e^{-ikx}]e^{-\frac{i}{\hbar}Et} \quad (41)$$

and

$$\psi_0 = t_r(R)e^{ik'x}e^{-\frac{i}{\hbar}Et}. \quad (42)$$

Here, both $k = \frac{1}{\hbar}\sqrt{2mE}$ and $k' = \frac{1}{\hbar}\sqrt{2m(E - V_0^c)}$ are R-independent constants. $t_r(R)$ and $r_f(R)$ mean the R-dependent transmission and reflection coefficients, respectively.

The current densities at $x = x_2$ and $x = x_1$ are

$$j(x=x_2,R) = {\rm Re}\left[\psi_0^*\frac{\hbar}{im}\partial_x\psi_0\right] = \frac{\hbar k'}{m}|t_r(R)|^2 \equiv j_t(R),$$
$$j(x=x_1,R) = \frac{\hbar k}{m}[1 - |r_f(R)|^2)] \equiv j_0 - j_r(R), \quad (43)$$

where

$$j_0 \equiv \frac{\hbar k}{m} \quad (44)$$

is R-independent fixed current of the incoming particle. The transmission and reflection probabilities are given by

$$\mathcal{T}(k,R) = \frac{j_t(R)}{j_0} = \frac{k'}{k}|t_r(R)|^2 \quad (45)$$

and

$$\mathcal{R}(k,R) = \frac{j_r(R)}{j_0} = |r_f(R)|^2, \quad (46)$$

respectively. In the stationary state, the current density is space-independent and one can assume $j(x = x_2,R) = j(x = x_1,R)$. Then, we see $j_t(R) + j_r(R) = j_0$ and thereby the unitarity condition

$$\mathcal{T}(k,R) + \mathcal{R}(k,R) = 1 \quad (47)$$

for any value of R.

Now, consider the fast forward of adiabatic change of the potential barrier under the injection of R-independent fixed current density j_0. Then, Eqs. (38), (39), and (40) lead to the current densities on $x = x_2$ and $x = x_1$ at arbitrary time t:

$$j_{\rm FF}(x=x_2,t) = j_t(R) - v(t)\int_c^{x_2}\partial_R\bar{\phi}_0^2 dx,$$
$$j_{\rm FF}(x=x_1,t) = j_0 - j_r(R) - v(t)\int_c^{x_1}\partial_R\bar{\phi}_0^2 dx, \quad (48)$$

where the accelerated adiabatic parameter $R \equiv R(\Lambda(t))$ and the time scaling factor $v(t)$ are given in Eqs. (34) and (35), respectively. By dividing the relevant part of Eq. (48) by j_0, we obtain the formula for the time-dependent transmission and reflection coefficients:

$$\mathcal{T}_{\rm FF}(k,t) \equiv \frac{j_{\rm FF}(x=x_2,t)}{j_0} = \mathcal{T}(k,R) - \frac{m}{\hbar k}v(t)\int_c^{x_2}\partial_R\bar{\phi}_0^2 dx, \quad (49)$$

and

$$\mathcal{R}_{\rm FF}(k,t) \equiv \frac{j_0 - j_{\rm FF}(x=x_1,t)}{j_0} = \mathcal{R}(k,R) + \frac{m}{\hbar k}v(t)\int_c^{x_1}\partial_R\bar{\phi}_0^2 dx, \quad (50)$$

respectively. Equations (49) and (50) are the goal of this section.

The fast forward of adiabatic change of the stationary tunneling state is actually nonstationary dynamics, and Eqs. (49) and (50) together with Eq. (47) lead to the condition

$$\mathcal{T}_{\rm FF}(k,t) + \mathcal{R}_{\rm FF}(k,t) = 1 - \frac{m}{\hbar k}v(t)\int_{x_1}^{x_2}\partial_R\bar{\phi}_0^2 dx \equiv 1 + \delta u. \quad (51)$$

The nonadiabatic correction on the right-hand side of Eq. (51), which is c-independent, shows a deviation δu from the unitarity and vanishes at $t = T_{\rm FF}$. The analysis of the continuity equation of the fast-forward dynamics can also reproduce Eq. (51) (see Appendix C).

The transport coefficients described above are actually transport probabilities. The stationary states at $x \leqslant x_1$ and

$x \geqslant x_2$ can also be characterized by complex scattering coefficients $r_f(R)$ and $t_r(R)$ as in Eqs. (41) and (42). If one wishes to know the deviation of their phase during the fast forward time, it is convenient to construct the $A_{\rm FF}$-field (gauge-field)-free variant of the present theory of fast forward. This can be done by using the Gauge transformation as in Appendix A. Then $\psi_{\rm FF}$ in Eq. (36) acquires the phase that compensates the $A_{\rm FF}$ field and becomes

$$\psi_{\rm FF}^{\rm MN} = \bar{\phi}_0[x,R(\Lambda(t))]e^{i\eta[x,R(\Lambda(t))]} \times e^{iv(t)\theta[\mathbf{x},R(\Lambda(t))]}e^{-\frac{i}{\hbar}Et}. \quad (52)$$

At $x \geqslant x_2$, where $V_0(x,R)$ is R-independent constant, noting $\bar{\phi}_0 e^{i\eta} = t_r(R)e^{ik'x}$, the fast-forward variant of Eq. (42) becomes

$$\psi_{\rm FF}^{\rm MN} = t_r^{\rm FF}[R(\Lambda(t))]e^{ik'x}e^{-\frac{i}{\hbar}Et} \quad (53)$$

with

$$t_r^{\rm FF}[R(\Lambda(t))] = t_r[R(\Lambda(t))]e^{iv(t)\theta[x_2,R(\Lambda(t))]}. \quad (54)$$

The $A_{\rm FF}$-field-free current density at $x = x_2$ is calculated in the same way as in Eq. (43) and is given by

$$j_{\rm FF}^{\rm MN}(x = x_2,t) = {\rm Re}\left[\psi_{\rm FF}^{\rm MN*}\frac{\hbar}{im}\partial_x\psi_{\rm FF}^{\rm MN}\right]_{x=x_2} = \frac{\hbar k'}{m}|t_r[R(\Lambda(t))]|^2 + v(t)\frac{\hbar}{m}\bar{\phi}_0^2\partial_x\theta|_{x=x_2}. \quad (55)$$

Recalling the formula in Eq. (25), Eq. (55) proves to be equal to Eq. (48), and, after its scaling by j_0 in Eq. (44), exactly reproduces the time-dependent transport coefficients in Eq. (49). The shoulder of the exponential of $t_r^{\rm FF}$ in Eq. (54) represents the phase modulation of scattering coefficients during the fast forward time, and, with use of Eq. (25), is explicitly given by

$$v(t)\theta[x_2,R(\Lambda(t))] = -v(t)\frac{m}{\hbar}\int_c^{x_2}\frac{dx}{\bar{\phi}_0^2}\int_c^x \partial_R\bar{\phi}_0^2 dx'. \quad (56)$$

Since $\bar{\phi}_0$ has no nodes as explained below Eq. (26), the double integrals in Eq. (56) are finite and the phase $v(t)\theta[x_2,R(\Lambda(t))]$ vanishes at the end of the fast forward. Similarly, the fast-forward variant of $r_f(R)$ is given by

$$r_f^{\rm FF}[R(\Lambda(t))] = r_f[R(\Lambda(t))]e^{iv(t)\theta[x_1,R(\Lambda(t))]}, \quad (57)$$

where the expression for $v(t)\theta[x_1,R(\Lambda(t))]$ is given by Eq. (56) with the upper integration limit x_2 replaced by x_1.

The important finding in this section is that, throughout the fast-forward time range, the transport coefficients include the nonadiabatic contribution, which vanishes at the goal when $v(t) = 0$, namely both $\mathcal{T}_{\rm FF}(k,t)$ and $\mathcal{R}_{\rm FF}(k,t)$ exactly reproduce the adiabatic counterparts at the end of the fast forward.

V. EXAMPLES

We shall now investigate specific examples and explicitly calculate the time-dependent transport coefficients in Eqs. (49) and (50) together with the driving electric field in Eq. (37). As typical examples of the stationary tunneling, we choose systems with (1) Eckart's potential [20] with tunable asymmetry and (2) double δ-function barriers with tunable relative height [21]. These systems are exactly solvable and allow one to evaluate both adiabatic and nonadiabatic contributions to transport coefficients during the fast forward dynamics.

In our numerical analysis below, we shall use typical space and time units like $L = 10^{-2}\times$ *the linear dimension of a device* and $\tau = 10^{-2}\times$ *the phase coherent time* and put $\frac{\hbar}{m} = 1(\times L^2\tau^{-1})$. The above choice means that space coordinate x (and other length parameters), time t, wave number k and velocity v are scaled by L, τ, L^{-1} and $L\tau^{-1}$, respectively.

A. A system with Eckart's potential under adiabatically tunable asymmetry

This potential has a long history since the work by Eckart [20], and has been used to describe the electron transmission through metal surfaces, nuclear reaction through a Coulomb barrier, etc. With use of length scale l, the potential is written as [20,22]

$$V_0(x,A) = \frac{\hbar^2}{2m}\left[\frac{e^{x/l}}{1+e^{x/l}} + \frac{Ae^{x/l}}{(1+e^{x/l})^2}\right], \quad (58)$$

which tends to 0 and $\frac{\hbar^2}{2m}$ as $x \to -\infty$ and $x \to +\infty$, respectively. The first and second terms on the right-hand side of Eq. (58) are asymmetric and symmetric with respect to $x = 0$, respectively. A is the adiabatic parameter changing very slowly as

$$A = A(t) = \epsilon t, \quad (59)$$

with $0 < \epsilon \ll 1$. Figure 1 shows a profile of $V_0(x,A)$ as function of $x(|x| \leqslant 10l)$ and $A(0 \leqslant A \leqslant 10)$. In the range $A > 1$, V_0 has a maximum $V_0(x_M,A) = \frac{\hbar^2}{2m}\frac{(1+A)^2}{4A}$ at $x_M = l \times \ln\left(\frac{A+1}{A-1}\right)$.

By making a variable change from x to $\xi[= -\exp(x/l)]$, the time-dependent Schrödinger equation with Eckart's potential in Eq. (58) becomes a differential equation for the Gauss' hypergeometric function F. Then, the exact solution for electronic wave function is given by [20,22]

$$\phi_0(x,k,A) = t_r(1-\xi)^{ik'l}\left(-\frac{\xi}{1-\xi}\right)^{ikl} \times F\left[\frac{1}{2} + i(k-k'+\delta)l, \frac{1}{2} + i(k-k'-\delta)l, 1-2ik'l, \frac{1}{1-\xi}\right], \quad (60)$$

with

$$k^2 = \frac{2mE}{\hbar^2},$$
$$k'^2 = k^2 - 1,$$
$$\delta = \sqrt{A - \frac{1}{4l^2}},$$
$$t_r = \frac{\Gamma\left[\frac{1}{2} + i(-k-k'-\delta)l\right]\Gamma\left[\frac{1}{2} + i(-k-k'+\delta)l\right]}{\Gamma(1-2ik'l)\Gamma(-2ikl)}. \quad (61)$$

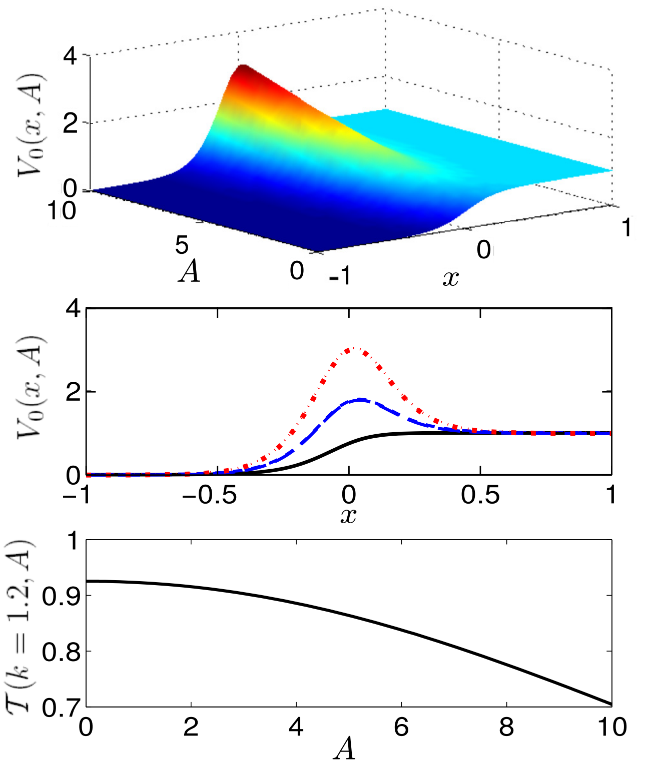

FIG. 1. Upper two panels: Eckart's potential in Eq. (58) as a function of coordinate x and adiabatic parameter A. Vertical axes are scaled by $\frac{\hbar^2}{2m}$; middle panel: Eckart potential for several adiabatic parameters. $A = 1$ (black solid), $A = 5$ (broken blue) and $A = 10$ (dotted red). Lowest panel: Transmission probability in Eq. (45) for the stationary tunneling as a function of A in case of $k = 1.2$. Length scale $l = 0.1$ is used throughout in Figs. 1–4. Units of space, time, and other quantities used in Figs. 1–8 are explained in the beginning of Sec. V and also below Eq. (37).

We should note that the adiabatic parameter A shows up through δ in Eq. (61). In Eqs. (60) and (61), we have corrected the mistakes included in Ref. [20], which was pointed out in Ref. [22].

We can use the linear transformation formula among Gauss' hypergeometric functions [23], which is convenient to see the asymptotic behavior in the region $x \to -\infty(\xi \to 0)$. In fact, we see there a sum of the incoming and reflective waves as

$$\phi_0 = e^{ikx} + r_f e^{-ikx}. \tag{62}$$

In the opposite asymptotic region $x \to \infty(\xi \to -\infty)$, ϕ_0 in Eq. (60) becomes a transmitting wave:

$$\phi_0 = t_r(-\xi)^{ik'l} = t_r \exp(ik'x). \tag{63}$$

In the case of $Al^2 < \frac{1}{4}$, the transition probability becomes

$$\mathcal{T}(k,A) = \frac{k'}{k}|t_r|^2 = \frac{\cosh[2\pi(k+k')l] - \cosh[2\pi(k-k')l]}{\cosh[2\pi(k+k')l] + \cos(2\pi|\delta|l)}. \tag{64}$$

In the case of $Al^2 \geqslant \frac{1}{4}$, on the other hand, we have

$$\mathcal{T}(k,A) = \frac{\cosh[2\pi(k+k')l] - \cosh[2\pi(k-k')l]}{\cosh[2\pi(k+k')l] + \cosh(2\pi\delta l)}. \tag{65}$$

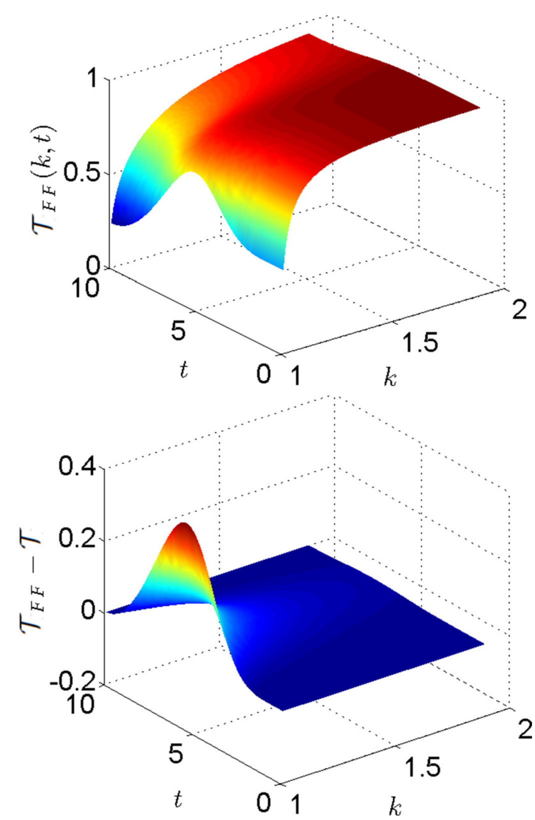

FIG. 2. $\mathcal{T}_{\rm FF}(k,t)$ (upper panel) and its deviation from $\mathcal{T}[k,A(\Lambda(t))]$ (lower panel), as a function of wave number k and time t. We choose $\bar{v} = 1$ and $T_{\rm FF} = 10$ in the accelerated adiabatic parameter $A(\Lambda(t))$ in Eq. (67), which are also used in Figs. 3 and 4.

The reflection probability is given by

$$\mathcal{R}(k,A) = 1 - \mathcal{T}(k,A). \tag{66}$$

In the fast forward of the adiabatic dynamics, the standard time t is replaced by the advanced time $\Lambda(t) = \int_0^t \alpha(t')dt'$, and taking the limit $\bar{\alpha} \to \infty, \epsilon \to 0$ with $\bar{\alpha}\varepsilon = \bar{v}$ kept constant, the accelerated adiabatic parameter is now given by

$$A(\Lambda(t)) = \bar{v}\left[t - \frac{T_{\rm FF}}{2\pi}\sin\left(\frac{2\pi}{T_{\rm FF}}t\right)\right], \tag{67}$$

as given in Eq. (34). Then, using Eqs. (49) and (50), $\mathcal{T}_{\rm FF}(k,t)$ and $\mathcal{R}_{\rm FF}(k,t)$ can be computed.

If we shall confine to the parameter region $0 \leqslant A \leqslant 10$ and employ the length scale $l = 0.1$ as in Fig. 1, we see the saturation of the potential $V_0(x,A)$ for $x \leqslant -1$ and $x \geqslant 1$, as

$$|V_0(x,A)| \leqslant 10^{-3} \quad \text{for} \quad x \leqslant -1,$$
$$\left|V_0(x,A) - \frac{\hbar^2}{2m}\right| \leqslant 10^{-3} \quad \text{for} \quad x \geqslant 1. \tag{68}$$

Then the stationary values $\mathcal{T}(k,A)$ and $\mathcal{R}(k,A)$ do not depend on the choice of x_2 and x_1 so long as $x_2 \geqslant 1$ and $x_1 \leqslant -1$. Therefore, in our numerical calculation of $\mathcal{T}_{\rm FF}(k,t)$ and $\mathcal{R}_{\rm FF}(k,t)$ in Eqs. (49) and (50), we take $x_1 = -1$ and $x_2 = 1$. As for the lower limit of the integration there, we choose $c = 0$.

Figure 2 shows both $\mathcal{T}_{\rm FF}(k,t)$ and its deviation from the stationary counterpart $\mathcal{T}[k,A(\Lambda(t))]$ as a function of $k(1 \leqslant k \leqslant 2)$ and $t(0 \leqslant t \leqslant T_{\rm FF})$. $\mathcal{T}_{\rm FF}$ shows deviation from

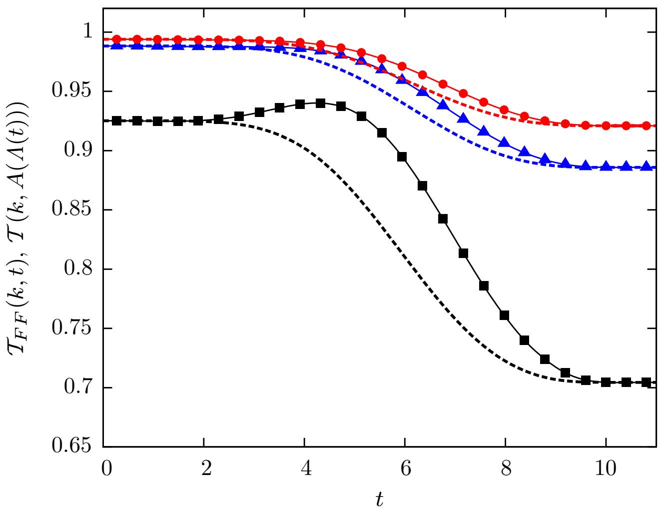

FIG. 3. Cross section of the upper panel of Fig. 2 in the strip between $0.65 \leqslant \mathcal{T}_{\rm FF}(k,t) \leqslant 1$ for input wave numbers $k = 1.2$ (black with squares), 1.6 (blue with triangles) and 1.8 (red with circles). Solid and broken lines correspond to $\mathcal{T}_{\rm FF}(k,t)$ and $\mathcal{T}[k,A(\Lambda(t))]$, respectively.

$\mathcal{T}[k,A(\Lambda(t))]$, but agrees with the latter at $t = T_{\rm FF}$ for any input wave numbers k. Figure 3 is a cross section of the upper panel of Fig. 2 for several input wave numbers k, showing that $\mathcal{T}_{\rm FF}(k,t)$ recovers the stationary value at $t = T_{\rm FF}$.

The electric field $E_{\rm FF}$ to guarantee the fast forward is calculated with use of the formula in Eq. (37), where η and $\partial_x\theta$ are calculable from Eqs. (22) and (25) together with Eq. (60). Figure 4 shows the 3D plots of $E_{\rm FF}$ as a function of $x(|x| < 1)$ and $t(0 \leqslant t \leqslant T_{\rm FF})$ for several input wave numbers k. In SI unit for electric field, typical absolute value $E_{\rm FF} = 0.5$ in ordinates in Fig. 4 means $E_{\rm SI}^{\rm FF} = 5 \times 10^{11}$ in case of IR lasers of wave length $\sim$1 μm [see the argument below Eq. (37)].

B. Double δ-function barriers with adiabatically tunable asymmetry

We shall move to analyze another example: the fast-forward of adiabatic control of double δ-function barriers with tunable asymmetry, which is a simplified variant of the double barrier in semiconductor heterostructures. Assuming the barriers located at $x = \pm a$, the underlying Hamiltonian is given by

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0(x,\Gamma). \tag{69}$$

Here,

$$V_0(x,\Gamma) = \frac{\hbar^2}{2m}[(h_{\rm min} + \Gamma)\delta(x+a) + (h_{\rm max} - \Gamma)\delta(x-a)], \tag{70}$$

with Γ the adiabatic parameter defined by

$$\Gamma = \Gamma(t) = \varepsilon t \quad (\varepsilon \ll 1), \tag{71}$$

which is assumed to vary from $\Gamma(0) = 0$ to $\Gamma(T) = h_{\rm max} - h_{\rm min} \equiv \Delta h$ with $T = \frac{\Delta h}{\varepsilon} \gg 1$.

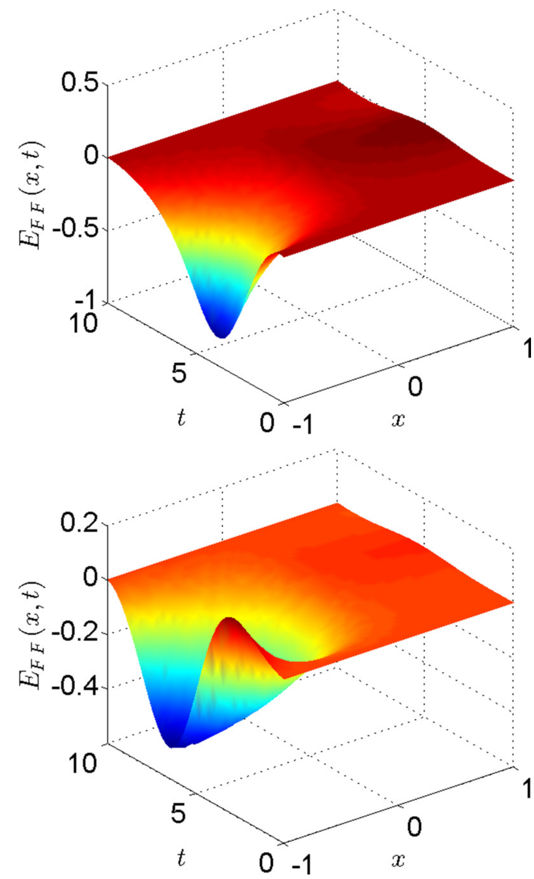

FIG. 4. Electric field as a function of space x and time t for wave numbers $k = 1.2$ (upper panel) and 1.8 (lower panel).

Figure 5 shows a profile of the potential as a function of $x(-a-0 \leqslant x \leqslant a+0)$ with $a = 1$ and $\Gamma(0 \leqslant \Gamma \leqslant \Delta h)$ with $h_{\rm max} = 2$, $h_{\rm min} = 1$, and $\Delta h = 1$.

First, we consider the stationary tunneling state available from the time-independent Schrödinger equation,

$$\hat{H}_0(\Gamma)\phi_0 = E(\Gamma)\phi_0. \tag{72}$$

Let's define three domains, $D_L(: x < -a), D_C(: -a \leqslant x \leqslant a)$, and $D_R(: x > a)$ and suppose the wave functions, respectively, as

$$\begin{aligned} \phi_0^L &= e^{ikx} + r_f(\Gamma)e^{-ikx} && (\text{in} \quad D_L),\\ \phi_0^C &= A(\Gamma)e^{ikx} + B(\Gamma)e^{-ikx} && (\text{in} \quad D_C),\\ \phi_0^R &= t_r(\Gamma)e^{ikx} && (\text{in} \quad D_R), \end{aligned} \tag{73}$$

where ϕ_0^L is a sum of the incident and reflective wave functions. $r_f(t_r)$ means reflection (transmission) coefficient, which is complex.

Unknown coefficients A, B, r_f, and t_r can be obtained by using two constraints: (1) the continuity of the wave-function ϕ_0 at $x = \pm a$; (2) the finite discontinuity of the derivative, $\frac{d}{dx}\phi$, available from the local integration of Eq. (69) in the vicinity of $x = \pm a$. With prescription of $\frac{\hbar^2}{2m} = 1$, the results

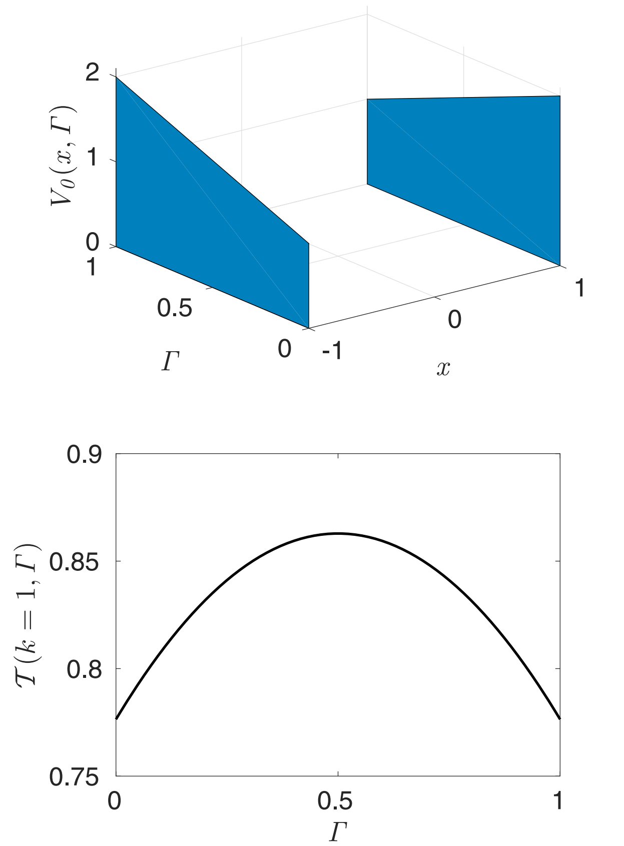

FIG. 5. Upper panel: Asymmetric potential consisting of double δ-functions in Eq. (70), as a function of space variable x and adiabatic parameter Γ. Vertical axis is scaled by $\frac{\hbar^2}{2m}$; lower panel: transmission probability in Eq. (45) for the stationary tunneling in case of $k = 1$. $a = 1$ is used throughout in Figs. 5–8.

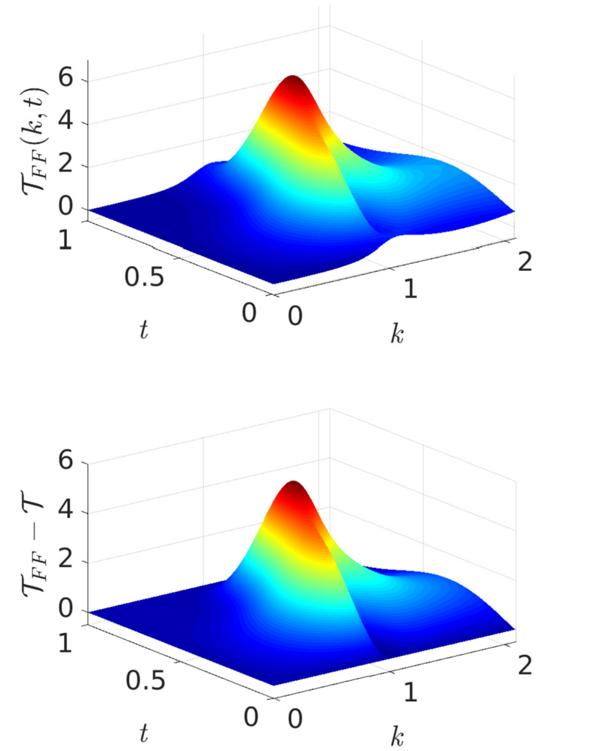

FIG. 6. $\mathcal{T}_{\rm FF}(k,t)$ (upper panel) and its deviation from $\mathcal{T}[k,\Gamma(\Lambda(t))]$ (lower panel), as a function of wave number $k(0 \leqslant k \leqslant 2)$ and time t. We choose $\bar{v} = 1$ and $T_{\rm FF} = 1$ in the accelerated adiabatic parameter $\Gamma(\Lambda(t))$, which are also used in Figs. 7 and 8.

are [21]

$$\begin{aligned}
\Delta(k) &= (h_{\min}+\Gamma)(h_{\max}-\Gamma)(-1+e^{4iak}) \\
&\quad + 4k^2 + 2i(h_{\min}+h_{\max})k, \\
t_r(\Gamma) &= \frac{4k^2}{\Delta(k)}, \\
r_f(\Gamma) &= \frac{e^{2iak}}{\Delta(k)}\{(h_{\min}+\Gamma)(h_{\max}-\Gamma)(-1+e^{-4iak}) \\
&\quad - 2ik[h_{\max}-\Gamma+(h_{\min}+\Gamma)e^{-4ika}]\}, \\
A(\Gamma) &= \frac{2k[2k+i(h_{\max}-\Gamma)]}{\Delta(k)}, \\
B(\Gamma) &= \frac{-2ik(h_{\max}-\Gamma)e^{2iak}}{\Delta(k)}.
\end{aligned} \tag{74}$$

In the fast forward of the adiabatic dynamics, the time t in $\Gamma(t)$ is replaced by $\Lambda = \int_0^t \alpha(t')dt'$ and we take the limit $\bar{\alpha} \to \infty$, $\varepsilon \to 0$ with $\bar{\alpha}\varepsilon = \bar{v}$ fixed. Then the accelerated adiabatic parameter $\Gamma(\Lambda(t))$ has the same form as $A(\Lambda(t))$ in Eq. (67).

Having recourse to the formulas in Eqs. (49) and (50), we can calculate $\mathcal{T}_{\rm FF}(t)$ and $\mathcal{R}_{\rm FF}(t)$ at $x_2 = a + 0$ before the right barrier and at $x_1 = -a - 0$ behind the left barrier, respectively. To evaluate the nonadiabatic correction in Eqs. (49) and (50), we again choose $c = 0$ as the lower limit of integrations and use the following result of integrations:

$$\begin{aligned}
J^{(\pm a)} &\equiv \int_0^{(\pm a)} \partial_\Gamma \bar{\phi}_0^2 dx \\
&= \partial_\Gamma \bigg[\pm a(\bar{A}^2+\bar{B}^2) + \frac{2\bar{A}\bar{B}}{k}\sin(\pm ak) \\
&\quad \times \cos(\pm ak + \alpha - \beta)\bigg],
\end{aligned} \tag{75}$$

where $\bar{A}(\bar{B})$ is the real positive amplitude and $\alpha(\beta)$ is the phase of the complex coefficients $A(\Gamma)(B(\Gamma))$ defined in Eq. (74). In Eq. (75), $+$ and $-$ in the sign $\pm$ correspond to x_2 and x_1, respectively.

Figure 6 shows both $\mathcal{T}_{\rm FF}(k,t)$ (upper panel) and its deviation from the stationary counterpart $\mathcal{T}[k,\Gamma(\Lambda(t))]$ (lower panel) as a function of $k(0 \leqslant k \leqslant 2)$ and $t(0 \leqslant t \leqslant T_{\rm FF})$. $\mathcal{T}_{\rm FF}$ shows to reach the stationary value at $t = T_{\rm FF}$. Figure 7 is a cross section of the upper panel of Fig. 6 for several input wave numbers k, showing that $\mathcal{T}_{\rm FF}(k,t)$ recovers the stationary value at $t = T_{\rm FF}$. The large deviation of $\mathcal{T}_{\rm FF}(k,t)$ from its stationary counterpart in Figs. 6 and 7 is caused by the driving electric field which is stronger in the case of double δ-function barriers than in the case of Eckart's potential (see Fig. 8).

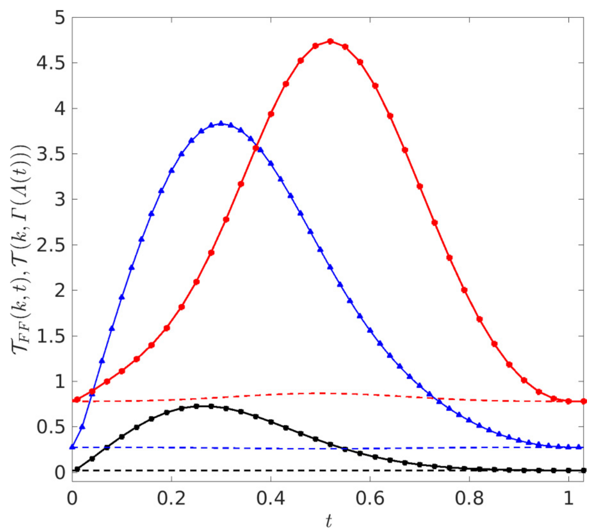

FIG. 7. Cross section of the upper panel of Fig. 6 for input wave numbers $k = 0.4$ (black with squares), 0.8 (blue with triangles), and 1.2 (red with circles). Solid and broken lines correspond to $\mathcal{T}_{\rm FF}(k,t)$ and $\mathcal{T}[k,\Gamma(\Lambda(t))]$, respectively.

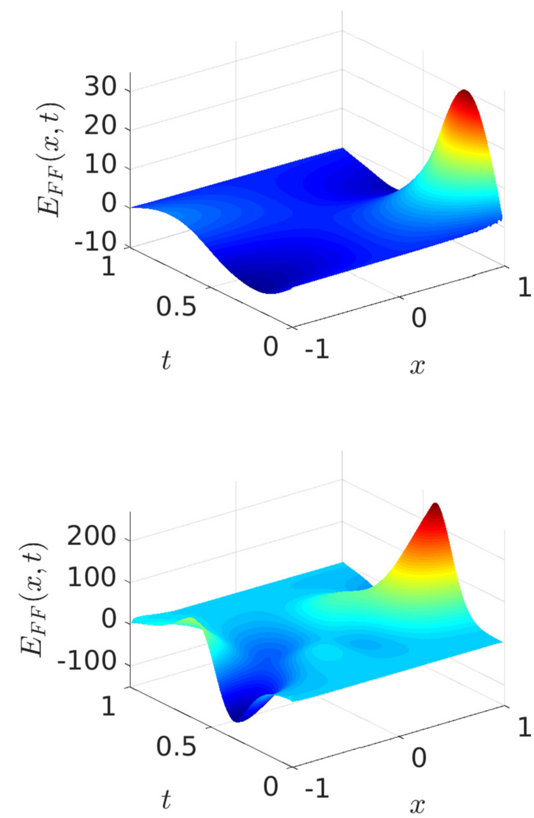

FIG. 8. Electric field as a function of space x and time t for wave numbers $k = 0.4$ (upper panel) and 1.2 (lower panel).

The electric field $E_{\rm FF}$ guarantees the fast forward can be evaluated with use of Eq. (37). Here, $\partial_x \eta$ is available from the wave function in each domain in Eq. (73). On the other hand, $\partial_x \theta$ in Eq. (25) can be available from the following results of the integration:

$$J^{(x)} \equiv \int_0^x \partial_\Gamma \bar{\phi}_0^2 dx' = \begin{cases} J^{(-a)} + \partial_\Gamma[(1+\bar{r}^2)(x+a)] + \frac{2\bar{r}}{k}\sin\{k(x+a)\cos[k(x-a)-\gamma]\} & (x \text{ in } D_L), \\ \partial_\Gamma[(\bar{A}^2+\bar{B}^2)x + \frac{2\bar{A}\bar{B}}{k}\sin(kx)\cos(kx+\alpha-\beta)] & (x \text{ in } D_C), \\ J^{(a)} + \partial_\Gamma(\bar{t}^2(x-a) + \frac{2\bar{t}}{k}\sin\{k(x-a)\cos[k(x+a)+\tau]\} & (x \text{ in } D_R), \end{cases} \tag{76}$$

where $\bar{r}(\bar{t})$ is the real positive amplitude and $\gamma(\tau)$ is the phase of the complex coefficients $r_f(\Gamma)[t_r(\Gamma)]$ defined in Eq. (74). Figure 8 shows $E_{\rm FF}$ as a function of t and x in the range $0 \leqslant t \leqslant T_{\rm FF}$ and $|x| \leqslant 1$ for several input wave numbers k. In SI unit for electric field, typical absolute value $E_{\rm FF} = 100$ in ordinates in Fig. 8 means $E_{\rm SI}^{\rm FF} = 10^{14}$ in case of IR lasers of wave length $\sim 1\ \mu$m. The localized high peaks and deep dips arise when $\bar{\phi}_0^2$ in the denominator on the right-hand side of Eq. (25) takes small but non-zero values due to the interference between a pair of waves in the domain D_C in Eq. (73) that forms an internal structure, i.e., a potential well surrounded by a pair of barriers.

Numerical results in this Section convey some basic features of the fast-forward observation of the transport coefficients under the adiabatically changing barrier. The results will be more-or-less modified by varying the mean time-scaling factor $\bar{v}$, the spatial size of barriers relative to wave length of the incoming particle, etc., which should be investigated separately in due course.

VI. CONCLUSION

We have proposed a scheme of the exact fast forward of adiabatic control of stationary tunneling states with use of the electromagnetic field, which allows the fast forward with complete fidelity, namely the exact acceleration of both the amplitude and phase of wave functions throughout the fast-forward time range. For the incoming particle with fixed energy, the scheme realizes the fast-forward observation of transport coefficients under the adiabatically changing barrier. The fast-forwarded transport coefficients are decomposed into the adiabatic part which satisfies the unitarity and the nonadiabatic one, which vanishes only at the end of the fast forwarding. We have also elucidated the modulation of the phase of complex scattering coefficients.

As typical examples we have investigated systems with (1) Eckart's potential with tunable asymmetry and (2) double δ-function barriers under tunable relative height. The driving electric field is evaluated to guarantee the stationary tunneling state during a rapid change of the barrier. The nonadiabatic contribution to transport coefficients proves to be remarkable in case that barriers have internal structures. Detailed numerical analysis of the dependence on the mean time-scaling factor $\bar{v}$, the spatial size of barriers relative to wave length of the incoming particle, etc., will constitute a future independent subject. The present scheme will be a promising extension of the fast forward of adiabatic dynamics of the bound ground states to that of open tunneling states.

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APPENDIX A: GAUGE TRANSFORMATION BETWEEN SYSTEMS WITH COMPLETE AND INCOMPLETE FIDELITIES

In the context of fast forward of adiabatic dynamics of bound states, the scheme presented here is compatible with the one in Refs. [2,3]. Let us introduce the gauge transformation into Eqs. (7), (33), and (36) (with the dynamical factor replaced by $e^{-\frac{i}{\hbar}\int_0^t E[R(\Lambda(s))]ds}$) as follows:

$$\psi_{FF} \to \psi_{FF}^{MN} e^{if},$$
$$V_{FF} \to V_{FF}^{MN} - \frac{\hbar}{q}\partial_t f, \tag{A1}$$
$$\mathbf{A}_{FF} \to \mathbf{A}_{FF}^{MN} + \frac{\hbar}{q}\nabla f,$$

with the phase defined by

$$f = -v(t)\theta[\mathbf{x}, R(\Lambda(t))]. \tag{A2}$$

Then, we find

$$\psi_{FF}^{MN} = \bar{\phi}_0[\mathbf{x}, R(\Lambda(t))]e^{i\eta[\mathbf{x}, R(\Lambda(t))]}e^{iv(t)\theta[\mathbf{x}, R(\Lambda(t))]} \times e^{-\frac{i}{\hbar}\int_0^t E[R(\Lambda(s))]ds},$$
$$V_{FF}^{MN} = -\frac{\hbar^2}{m}v(t)\nabla\theta\cdot\nabla\eta - \frac{\hbar^2}{2m}[v(t)]^2(\nabla\theta)^2 - \hbar v(t)\partial_R\eta - \hbar\dot{v}(t)\theta - \hbar[v(t)]^2\partial_R\theta,$$
$$\mathbf{A}_{FF}^{MN} = 0, \tag{A3}$$

and ψ_{FF}^{MN} proves to satisfy

$$i\hbar\frac{\psi_{FF}^{MN}}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_0 + qV_{FF}^{MN}\right)\psi_{FF}^{MN}. \tag{A4}$$

Equations (A3) and (A4) together with notion of $q = 1$ reproduces the preceding issue [2,3], which generated the exact adiabatic state only at the final time $t = T_{FF}$, but failed to keep the perfect fidelity in the intermediate time range $0 < t < T_{FF}$.

In fast forward of the particular adiabatic control of bound states, V_{FF}^{MN} in Eq. (A3) has an expression convenient to generate the counterdiabatic potential [4–6], which we shall briefly explain below.

Consider the original potential controlled by the scale-invariant adiabatic expansion and contraction [10–12], as given by

$$V_0 = \frac{1}{R^2}U_0\left(\frac{x}{R}\right), \tag{A5}$$

where R is the adiabatic parameter as in Eq. (14). The corresponding 1D eigenvalue problem for bound systems yields ground and excited states whose normalized forms are commonly given by

$$\phi_0 = \frac{1}{\sqrt{R}}f\left(\frac{x}{R}\right), \tag{A6}$$

where $f = \bar{f}e^{i\eta}$ with real amplitude $\bar{f}$ and phase η. Then, with use of a new variable $X \equiv \frac{x}{R}$, Eq. (25) becomes

$$\partial_x\theta = -\frac{m}{\hbar}\frac{R}{|\bar{f}(X)|^2}\partial_R\int^X|\bar{f}(X')|^2dX'. \tag{A7}$$

Here, the indefinite integral is used because the lower limit of integration is arbitrary. Noting $\partial_R = \frac{\partial X}{\partial R}\frac{\partial}{\partial X} = -\frac{x}{R^2}\frac{\partial}{\partial X}$, Eq. (A7) reduces to

$$\partial_x\theta = \frac{m}{\hbar}\frac{x}{R}\frac{|\bar{f}(X)|^2}{|\bar{f}(X)|^2} = \frac{m}{\hbar R}x. \tag{A8}$$

In the second equality of Eq. (A8), we prescribed $\lim_{X\to X_c}\frac{|\bar{f}(X)|^2}{|\bar{f}(X)|^2} = 1$ if $\bar{f}(X)$ will be $\bar{f}(X_c) = 0$ at $X = X_c$. From Eq. (A8), one finds [3]

$$\theta = \frac{m}{2\hbar R}x^2,$$
$$\partial_R\theta = -\frac{m}{2\hbar R^2}x^2. \tag{A9}$$

In the simple case that ϕ_0 in Eq. (A6) is real, i.e., $\eta = 0$, V_{FF}^{MN} in Eq. (A3) becomes

$$V_{FF}^{MN} = -\frac{m\ddot{R}}{2R}x^2, \tag{A10}$$

where $R = R(\Lambda(t))$, $v(t) = \dot{R}$ and $\dot{v}(t) = \ddot{R}$ in Eq. (34) are used. V_{FF}^{MN} in Eq. (A10) is nothing but the counterdiabatic potential in the scale-invariant bound systems [11,12]. The generalization of the above argument to the case which includes the scale-invariant adiabatic translation is straightforward.

Thus the fast forward approach [1–3] applied to the scale-invariant bound systems is free from the problem of nodes, although such a problem might appear when we shall manage excited states of the bound systems that break the scale invariance. On the other hand, as explained around Eq. (26), the stationary (or steady) tunneling state investigated in the present paper has no nodes and is free from both the problem of nodes and the constraint of scale invariance.

APPENDIX B: DERIVATION OF THE DRIVING A_{FF} AND V_{FF} POTENTIALS IN EQS. (31) AND (32)

As for space derivatives of $\bar{\phi}_0^{reg}$ in Eq. (29), we shall have recourse to the formulas: $\mathrm{Re}[\frac{\partial_x\bar{\phi}_0^{reg}}{\bar{\phi}_0^{reg}}] = \partial_x(\ln\bar{\phi}_0)$, $\mathrm{Im}[\frac{\partial_x\bar{\phi}_0^{reg}}{\bar{\phi}_0^{reg}}] = \partial_x\eta + \epsilon\partial_x\theta$, $\mathrm{Re}[\frac{\partial_x^2\bar{\phi}_0^{reg}}{\bar{\phi}_0^{reg}}] = \frac{\partial_x^2\bar{\phi}_0^{reg}}{\bar{\phi}_0} - (\partial_x\eta + \epsilon\partial_x\theta)^2 = \frac{2m}{\hbar^2}(V_0 - E) - 2\epsilon\partial_x\eta\cdot\partial_x\theta - \epsilon^2(\partial_x\theta)^2$, $\mathrm{Im}[\frac{\partial_x^2\bar{\phi}_0^{reg}}{\bar{\phi}_0^{reg}}] = \frac{2\partial_x\bar{\phi}_0}{\bar{\phi}_0}(\partial_x\eta + \epsilon\partial_x\theta) + (\partial_x^2\eta + \epsilon\partial_x^2\theta) = \epsilon(\partial_x^2\theta + 2\partial_x(\ln\bar{\phi}_0)\partial_x\theta)$. In obtaining the final issue in each of the last two equations, we used the identities,

$$\frac{\partial_x^2 \bar{\phi}_0}{\bar{\phi}_0} - (\partial_x \eta)^2 = \frac{2m}{\hbar^2}(V_0 - E),$$
$$\partial_x^2 \eta + 2\frac{\partial_x \bar{\phi}_0}{\bar{\phi}_0}\partial_x \eta = 0, \quad \text{(B1)}$$

which are available from the adiabatic eigenvalue problem in Eq. (16) for the stationary state in Eq. (22).

Equation (8) now becomes

$$\bar{\phi}_0^2 \partial_x A_{\rm FF} + 2\bar{\phi}_0 \partial_x \bar{\phi}_0 A_{\rm FF} + \hbar(\alpha - 1)\epsilon\left(\bar{\phi}_0^2 \partial_x^2 \theta + 2\bar{\phi}_0 \partial_x \bar{\phi}_0 \partial_x \theta\right) = 0, \quad \text{(B2)}$$

which is found to be satisfied by $A_{\rm FF}$ in Eq. (31). Using Eq. (31) together with spatial derivatives of $\tilde{\phi}_0^{\rm reg}$ described above Eq. (B1), $V_{\rm FF}$ in Eq. (9) turns out to take the form in Eq. (32).

APPENDIX C: ANALYSIS OF CONTINUITY EQUATION OF THE FAST-FORWARD DYNAMICS

Equation (51) is also available from the continuity equation of the fast-forward dynamics:

$$\partial_t |\psi_{\rm FF}|^2 + \partial_x j_{\rm FF}(x,t) = 0, \quad \text{(C1)}$$

where $|\psi_{\rm FF}|^2 = \bar{\phi}_0^2[R(\Lambda(t))]$. By integrating Eq. (C1) from $x = x_1$ to $x = x_2$ and using $\partial_t = \frac{dR}{dt}\partial_R = v(t)\partial_R$, we have

$$j_{\rm FF}(x = x_2,t) - j_{\rm FF}(x = x_1,t) = -v(t)\int_{x_1}^{x_2} \partial_R \bar{\phi}_0^2 dx. \quad \text{(C2)}$$

Dividing the equality in Eq. (C2) by $j_0(= \frac{\hbar}{m}k)$, we can confirm Eq. (51).

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