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Peres experiment using photons: No test for hypercomplex (quaternionic) quantum theories

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Assuming the standard axioms for quaternionic quantum theory and a spatially localized scattering interaction, the S matrix in quaternionic quantum theory is complex valued, not quaternionic. Using the standard connections between the S matrix, the forward scattering amplitude for electromagnetic wave scattering, and the index of refraction, we show that the index of refraction is necessarily complex, not quaternionic. This implies that the recent optical experiment of Procopio *et al.* [Nat. Commun. **8**, 15044 (2017)] based on the Peres proposal does not test for hypercomplex or quaternionic quantum effects arising within the standard Hilbert space framework. Such a test requires looking at near zone fields, not radiation zone fields.

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Textbook quantum theory is based on complex numbers of the form $a_0 + a_1 i$, with i the imaginary unit $i^2 = -1$. It has long been known (see [1] for a comprehensive review) that an alternative quantum mechanics can be based on the quaternion or hypercomplex numbers of the form $a_0 + a_1 i + a_2 j + a_3 k$, with i, j, k three noncommuting imaginary units obeying the algebra

$$\begin{aligned} i^2 &= j^2 = k^2 = -1, \\ ij &= -ji = k, \\ jk &= -kj = i, \\ ki &= -ik = j. \end{aligned} \tag{1}$$

The standard axioms for quaternionic quantum theory posit [1] a Hilbert space structure with quaternion scalars, a positive semidefinite inner product, and an inner product-preserving time development. In complex quantum theory time development is generated by a complex unitary evolution operator $U = e^{iHt}$ with H self-adjoint, and thus iH anti-self-adjoint. In the quaternionic generalization of quantum theory, this is replaced by a quaternion unitary evolution operator $U = e^{\tilde{H}}$, with the "quaternionic Hamiltonian" $\tilde{H}$ a general quaternion anti-self-adjoint operator. Because the quaternion algebra is noncommutative, the operator $\tilde{H}$ cannot be reduced to a self-adjoint operator by multiplication by a fixed quaternion unit. Quaternionic quantum mechanics is a distinctly different physical theory from complex quantum mechanics, unlike what is sometimes termed "quaternion electrodynamics", which is a rewriting of the Maxwell equations using quaternions in place of the vector calculus, but with the same physical content.

Whether nature chooses complex or quaternionic quantum theory is ultimately an experimental issue, and Peres [2] many years ago proposed a test for quaternionic effects. His idea was to set up an interferometer with two branches, one containing materials A and B through which the beam passes, and the other containing B and A in the opposite order. This would, in principle, test for noncommutativity of the phases α and β induced by passage of the beam through the respective materials A and B. Such a noncommutativity could be present in quaternionic quantum theory, but is zero in complex quantum theory. Experiments of this type were carried out with neutrons [3], with a null result for the phase noncommutativity. However, in the book [1], Adler showed[1] by detailed calculations using the nonrelativistic Schrödinger equation, and more general arguments using the Möller wave operator formalism, that quaternionic effects in scattering decay asymptotically, with the far-zone scattering amplitude containing only commmuting phases lying in a complex subalgebra of the quaternions. So the null results of the neutron scattering experiments do not in fact place useful bounds on possible quaternionic effects.

Recently, Procopio *et al.* [5] reported a version of the Peres experiment using photons, for which the nonrelativistic calculations of [1] do not apply. They used photons of wavelength 790 nm, three to four orders of magnitude larger than a typical atomic dimension, and so the optical properties of materials are adequately described by a frequency ω dependent refractive index $n(\omega)$, which is the zero wave number limit of the more general frequency and wave number dependent refractive index or dielectric constant [6]. In the two branches of their interferometer, they used optical materials with different optical responses; for A they used a material with positive refractive index, and for B they used a material with negative refractive index. For nondissipative quaternionic scattering, the phases induced by A and B, respectively, are assumed to be

$$\alpha = e^{i\phi_A^1 + j\phi_A^2 + k\phi_A^3}, \quad \beta = e^{i\phi_B^1 + j\phi_B^2 + k\phi_B^3}, \tag{2}$$

and quaternionic effects (nonzero $\phi_A^{2,3}$ and/or nonzero $\phi_B^{2,3}$) would be signaled by noncommutativity of the phases α and β induced in opposite orders in the two branches of the interferometer. The experiment gave a null result, showing identity of the overall phases $\alpha\beta$ and $\beta\alpha$ in the two branches of the interferometer to less than 0.03° of arc. Does this result for photons give a test of quaternionic quantum theory?

To answer this question, we first reexpress the phase α in terms of the refractive index n. For light of wave number $q = \omega/c$ passing through a thickness w of optical material

[1]See Secs. 6.1, 6.3, 7.2, and 8.3 of [1] for the arguments that the S matrix in quaternionic quantum theory is complex. For related results for non-Abelian scattering theory, see Soffer [4].

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with refractive index n, the induced phase is $\alpha = e^{inqw}$. For a quaternionic index of refraction $n = n_0 + n_1 i + n_2 j + n_3 k$, we thus have $\alpha = e^{(n_0 i - n_1 + n_2 k - n_3 j)qw}$. When the material is nondissipative $n_1 = 0$ and this phase then has the form of Eq. (2), with the presence of quaternionic effects signaled by nonzero values for n_2 and n_3. So the question we are asking is whether the index of refraction in quaternionic quantum theory is complex valued (only n_0 and n_1 nonzero) or quaternionic (components n_2 and/or n_3 present).

The next step is to invoke the Rayleigh relation [7] between the index of refraction $n(\omega)$ and the coherent forward scattering amplitude $f(\omega)$,

$$n(\omega) = 1 + \frac{2\pi c^2}{\omega^2} N f(\omega), \tag{3}$$

with N the number of scattering centers per unit volume. A concise derivation of this relation is given in the Fermi lecture notes [8].[2] Thus the question of whether n can be quaternionic reduces to that of whether the photon scattering amplitude $f(\omega)$ can be quaternionic. A detailed analysis of photon scattering is given in the electrodynamics text of Akhiezer and Berestetskii [10], which gives in Eq. (5.6) the formula[3] for the asymptotic photon $\vec{E}$ field for an incident wave propagating along the z direction toward a scatterer at the coordinate origin,

$$\sqrt{2}\vec{E} = \hat{e} e^{iqz} + \vec{F}(\hat{n}) \frac{e^{iqr}}{r}, \tag{4}$$

with $\hat{e}$ the polarization unit vector and $\hat{n}$ a unit vector defined by $\vec{r} = \hat{n} r$. Denoting by $\hat{n}_0$ the unit vector along the z axis, Ref. [10] gives in Eq. (5.10) a formula for $\vec{F}(\hat{n})$, the amplitude for elastic (i.e., nondissipative) scattering by a spherically symmetric scatterer,

$$\vec{F}(\hat{n}) = \frac{2\pi}{iqr} \sum_{jM\lambda} \hat{e} \cdot \vec{Y}^{\lambda}_{jM}(\hat{n}_0)(S_{j\lambda} - 1)\vec{Y}^{\lambda}_{jM}(\hat{n}), \tag{5}$$

with $\vec{Y}^{\lambda}_{jM}$ vector spherical harmonics describing the photon fields, and with $S_{j\lambda}$ the S-matrix element in the scattering channel labeled by $j\lambda$ [which is related to the phase shift $\delta_{j\lambda}(\omega)$ by $S_{j\lambda}(\omega) = e^{2i\delta_{j\lambda}(\omega)}$]. The question of whether the scattering amplitude $f(\omega)$, which is a component of $\vec{F}(\hat{n}_0)$, is complex or quaternionic therefore reduces to that of whether the S matrix in quaternionic quantum theory is complex or quaternionic.

[2]Equation (3) holds for $n - 1$ small. When n is of order unity, and the reduction in the transmitted wave amplitude in the medium by a factor $2/(n+1)$ is taken into account in a self-consistent way, the formula becomes [9]

$$n(\omega)^2 = 1 + \frac{4\pi c^2}{\omega^2} N f(\omega).$$

This change has no effect on our argument, since it still implies that if $f(\omega)$ is complex, then so is n.

This question was settled in Sec. 8.3 of [1] by a very general argument within the standard quaternionic Hilbert space framework, using the Möller wave operator formalism, which applies for scattering by massless as well as massive particles. The derivation assumes that the total anti-self-adjoint quaternionic Hamiltonian $\tilde{H}$ is the sum of an asymptotic part $\tilde{H}_0$ and a part $\tilde{V}$ that is spatially localized, $\tilde{H} = \tilde{H}_0 + \tilde{V}$. No other specific assumptions about the structure of $\tilde{V}$ are made. The general Möller formalism implies that the S matrix commutes with the asymptotic Hamiltonian,

$$[\tilde{H}_0, S] = 0. \tag{6}$$

We can use the ray representative freedom of quaternionic quantum theory states to choose an energy eigenstate basis for $\tilde{H}_0$ in which the energy eigenvalues are all of the form iE, that is, the quaternionic phase lies in the complex subspace of the quaternion algebra spanned by 1 and i. Sandwiching Eq. (6) between $\tilde{H}_0$ eigenstates with eigenvalues iE and iE' then implies that the S-matrix element $S(E,E')$ is nonvanishing only when $E = E'$, and when $E \neq 0$ the on-energy-shell S matrix commutes with i,

$$[i, S] = 0, \tag{7}$$

and so is complex. By the chain of equalities established in the preceding paragraphs, this shows that the index of refraction in quaternionic quantum theory is complex, and not quaternionic.

Thus, our conclusion is that the experiment of [5] does not test for quaternionic effects, unless these are assumed to arise from a hypothetical theory that does not obey the standard axioms of quaternionic quantum mechanics, for example, by having an indefinite inner product. To test for quaternionic effects arising within the standard Hilbert space framework, one would need an experiment accessing near-zone scattering, not just the asymptotic or radiation zone scattering amplitude. The derivation we have given does not show how rapidly near-zone quaternionic effects decay with distance r from the scattering center. We conjecture, based on our computations in the nonrelativistic case, that even for massless photons quaternionic scattering effects will decay exponentially as e^{-kr}, rather than as a power of r, but a detailed calculation using a model for the interaction term $\tilde{V}$ would be needed to check this.

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[3]We make the minor changes in notation from [10] of using a caret to denote unit vectors, and denoting the wave number by q instead of k.

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