

Guided modes in the plane array of optical waveguides

I. Ya. Polishchuk

*National Research Center, Kurchatov Institute, Moscow 123182, Russia;
Max Planck Institute for the Physics of Complex Systems, D-01187 Dresden, Germany;
and Moscow Institute of Physics and Technology, Dolgoprudnii 141700, Russia*

A. A. Anastasiev and E. A. Tsyvkunova

National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), Kashirskoe shosse 31, Moscow 115409, Russia

M. I. Gozman and S. V. Solov'ov

Moscow Institute of Physics and Technology, Dolgoprudnii 141700, Russia

Yu. I. Polishchuk

National Research Center, Kurchatov Institute, Moscow 123182, Russia

(Received 17 January 2017; published 19 May 2017)

It is known that for the isolated dielectric cylinder waveguide, the modes with a certain angular symmetry cannot be guided ones (without loss by radiation) if their frequency is smaller than the cutoff frequency ω_* , which is completely determined by the waveguide refractive index and its radius. It is shown in the paper that the infinite plane periodic array of such waveguides possesses guided modes (with the same angular symmetry) within the frequency domain which is *below* the frequency ω_* . This is due to the inevitable interaction between the waveguides. As far as the finite array is concerned, the modes in this frequency domain are weakly radiating ones, but their quality factor Q increases as $Q(N) \sim N^3$, with N being the number of the waveguides in the array. This dependence is both obtained numerically, using the multiple scattering formalism, and justified within the framework of a simple analytical model.

DOI: [10.1103/PhysRevA.95.053847](https://doi.org/10.1103/PhysRevA.95.053847)**I. INTRODUCTION**

The optical waveguides are the inherent component of the optical and optoelectronic devices, indispensable for the optical signal's transmission between different parts of the system. The interaction between the closely spaced waveguides usually results in the undesirable effects that distort the transmitted signal. However, in some cases, the interaction between the waveguides can be exploited for practical purposes. In particular, this concerns the plane periodic arrays of waveguides, which are a special kind of low-dimensional photonic crystal. The main feature of such systems is a band structure of the optical spectrum which defines its peculiar properties [1–4]. A low-dimensional photonic crystal composed of the parallel rods is of special interest. The band structure of the plane array composed of the semiconductor cylinders was investigated in Ref. [5]. Then a similar metallic structure, which took into account dissipation, was computed in Ref. [6]. Superconductive photonic crystals of such kind were considered in [7]. Photonic crystals composed of parallel cylinders are useful for various applications [8]. In particular, they reveal negative-angle refraction and reflection [9]. On the other hand, the array of parallel interacting cylinders is an ideal system to simulate various physical phenomena inherent in condensed-matter physics, such as Anderson localization, Bloch oscillations, Bloch-Zener tunneling, etc. [10–15].

The electromagnetic field which describes a guided mode in a waveguide is finite inside the waveguide, while it is vanishing at a large distance from it. For the isolated dielectric cylinder waveguide, all the components of the electromagnetic field are expressed via its z components, with z being directed

along the waveguide axis. The angular dependence of these components is determined by the factor $e^{im\varphi}$, with m being an integer (see the next section). It is known that except for the particular case $m = 1$, the modes cannot be guided ones (without loss by radiation) if their frequency is smaller than the cutoff frequency ω_* , which is completely determined by the waveguide refractive index and its radius [16]. This is due to the fact that the conversion of such modes with the frequency below ω_* into a free photon is possible. However, in various applied tasks, it may be necessary to have such guided mode below the frequency ω_* for the given waveguides. In particular, such problem may appear in the following case. Indeed, along with the radiation, there exist losses connected with the absorption by the material itself. If the frequency window is located below the cutoff frequency ω_* , there arises a problem to shift the frequency ω_* into this window region.

It was shown in Refs. [17–21] that the arrays of the interacting identical microspheres manifest the features that are not inherent in a single isolated microsphere. In particular, the quality factor for the whispering-gallery modes reveals a remarkable increase for such arrays. This phenomenon, initially predicted numerically, was explained using the concept of the light-cone constraint [17,19].

In this paper, we investigate a possibility of a formation of high-quality guided modes (with given angular symmetry) in the plane waveguide array which are below the cutoff frequency of the isolated waveguide ω_* . Thus our aim is to suppress the radiation loss using the array of the waveguides. First, we consider the *infinite* plane periodical array of the waveguides and show that taking into account the interaction between them results in the appearance of the guided modes

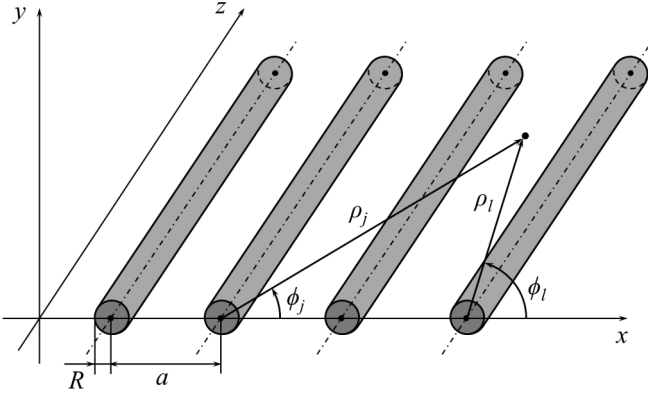


FIG. 1. The optical waveguide array. The polar coordinates of radius vector \mathbf{r} relative to different waveguides.

below ω_* . These modes possess the infinite Q factor. However, actually we deal with the array of a finite size and the modes become low radiating ones. For this reason, we investigate how the Q factor of the low-radiating modes depends on the number of the waveguides in the array N . Using the multiple scattering formalism (MSF), it is shown numerically that $Q(N) \sim N^3$. This formalism is based on the exact description of the electromagnetic waves which are scattered by the infinite cylinder [22]. Besides, we propose a simple model which qualitatively explains this cubic dependence.

The paper is organized as follows. In Sec. II, we describe the MSF, giving a brief derivation of the principal relations. Based on the relations obtained, the numerical simulation for the infinite and the finite arrays of cylinder waveguides is given in Sec. III. A clear qualitative explanation of the results obtained numerically is given in Sec. IV.

II. MULTIPLE SCATTERING FORMALISM

Let us consider the array of N parallel cylindrical dielectric waveguides (see Fig. 1). The axes of the waveguides are in the xz plane and are parallel to the z axis. The array is equidistant, with a being the distance between the axes of the nearest waveguides. All the waveguides are assumed to have the same radius R and the same refractive index n . The refractive index of the environment is n_0 . The system of units where the speed of light $c = 1$ is used.

Let us consider a guided mode with a frequency ω which is excited in the array. Because of the translation invariance in the z direction, all the components of the electromagnetic field describing the guided mode depend on the coordinate z as $e^{i\beta z}$, with β being a propagation constant. Thus, all the components of the electromagnetic field describing the guided mode are proportional to the factor $e^{-i\omega t + i\beta z}$. First, let us describe the electromagnetic field inside the waveguides. The field inside the j th waveguide, being of a finite value, may be represented as follows:

$$\tilde{\mathbf{E}}_j(t, \mathbf{r}) = e^{-i\omega t + i\beta z} \sum_{m=0, \pm 1, \dots} e^{im\phi_j} [c_{jm} \tilde{\mathbf{N}}_{\omega' \beta m}(\rho_j) - d_{jm} \tilde{\mathbf{M}}_{\omega' \beta m}(\rho_j)],$$

$$\tilde{\mathbf{H}}_j(t, \mathbf{r}) = e^{-i\omega t + i\beta z} n \sum_{m=0, \pm 1, \dots} e^{im\phi_j} [c_{jm} \tilde{\mathbf{M}}_{\omega' \beta m}(\rho_j) + d_{jm} \tilde{\mathbf{N}}_{\omega' \beta m}(\rho_j)], \quad (1)$$

where $\mathbf{r} = (x, y, z) = (\boldsymbol{\rho}, z)$, $\rho_j = |\boldsymbol{\rho} - \mathbf{a}^j| < R$, ϕ_j is the polar angle for the vector $\boldsymbol{\rho} - \mathbf{a}^j$ (see Fig. 1), and $\omega' = n\omega$. The vector cylinder harmonics $\tilde{\mathbf{M}}_{\omega' \beta m}(\rho_j)$ and $\tilde{\mathbf{N}}_{\omega' \beta m}(\rho_j)$ are defined as follows:

$$\tilde{\mathbf{N}}_{\omega' \beta m}(\rho_j) = \mathbf{e}_r \frac{i\beta}{\kappa} J'_m(\kappa\rho_j) - \mathbf{e}_\phi \frac{m\beta}{\kappa^2\rho_j} J_m(\kappa\rho_j) + \mathbf{e}_z J_m(\kappa\rho_j), \quad (2)$$

$$\tilde{\mathbf{M}}_{\omega' \beta m}(\rho_j) = \mathbf{e}_r \frac{m\omega'}{\kappa^2\rho_j} J_m(\kappa\rho_j) + \mathbf{e}_\phi \frac{i\omega'}{\kappa} J'_m(\kappa\rho_j), \quad (3)$$

where $\kappa = \sqrt{n^2\omega^2 - \beta^2}$, $J_m(\kappa\rho_j)$ is the Bessel function, and the prime means the derivative with respect to the argument $\kappa\rho_j$. Thus, the guided mode inside the j th waveguide is determined by the frequency ω , the propagation constant β , and the partial amplitudes c_{jm} , d_{jm} .

Now let us turn to the electromagnetic field for the same guided mode outside of the array. This field is a sum of the contributions induced by all the waveguides,

$$\mathbf{E}(t, \mathbf{r}) = \sum_{j=1}^N \mathbf{E}_j(t, \mathbf{r}), \quad \mathbf{H}(t, \mathbf{r}) = \sum_{j=1}^N \mathbf{H}_j(t, \mathbf{r}). \quad (4)$$

For the guided mode, the contribution of the j th waveguide should vanish at $\rho_j \rightarrow \infty$. Therefore, one can write

$$\mathbf{E}_j(t, \mathbf{r}) = e^{-i\omega t + i\beta z} \sum_m e^{im\phi_j} [a_{jm} \mathbf{N}_{\omega_0 \beta m}(\rho_j) - b_{jm} \mathbf{M}_{\omega_0 \beta m}(\rho_j)], \quad (5)$$

$$\mathbf{H}_j(t, \mathbf{r}) = e^{-i\omega t + i\beta z} n_0 \sum_m e^{im\phi_j} [a_{jm} \mathbf{M}_{\omega_0 \beta m}(\rho_j) + b_{jm} \mathbf{N}_{\omega_0 \beta m}(\rho_j)],$$

where $\rho_j > R$. In (5), another kind of vector cylinder harmonics is introduced,

$$\mathbf{N}_{\omega_0 \beta m}(\rho_j) = \mathbf{e}_r \frac{i\beta}{\kappa_0} H'_m(\kappa_0\rho_j) - \mathbf{e}_\phi \frac{m\beta}{\kappa_0^2\rho_j} H_m(\kappa_0\rho_j) + \mathbf{e}_z H_m(\kappa_0\rho_j), \quad (6)$$

$$\mathbf{M}_{\omega_0 \beta m}(\rho_j) = \mathbf{e}_r \frac{m\omega_0}{\kappa_0^2\rho_j} H_m(\kappa_0\rho_j) + \mathbf{e}_\phi \frac{i\omega_0}{\kappa_0} H'_m(\kappa_0\rho_j), \quad (7)$$

where $H_m(\kappa_0\rho_j)$ is the Hankel function of the first kind, $\omega_0 = n_0\omega$, and $\kappa_0 = \sqrt{n_0^2\omega^2 - \beta^2}$. Thus, the contribution of the j th waveguide into the guided mode field outside of the array is determined by the frequency ω , the propagation constant β , and the partial amplitudes a_{jm} , b_{jm} . Note that for $\beta = 0$, expansions (1) and (5) transform into the corresponding expressions in [22]; however, different notations are used there. Below, the factor $e^{-i\omega t + i\beta z}$ is omitted.

Below, for the purpose of illustration of the effect, we confine ourselves to the zero-harmonic approximation. This means that only the terms with $m = 0$ are taken into account

in (1) and (5). It is easy to convince ourselves that in this case, the guided modes are either transverse-magnetic (TM) modes or transverse-electric (TE) ones. For the TM mode, $b_{j0} = d_{j0} = 0$, while for the TE mode, $a_{j0} = c_{j0} = 0$. As an example, let us consider the TM modes. Then, Eqs. (1) and (5) take the form

$$\tilde{\mathbf{E}}_j(\mathbf{r}) = c_j \tilde{\mathbf{N}}_{\omega'\beta 0}(\rho_j), \quad \tilde{\mathbf{H}}_j(\mathbf{r}) = c_j n \tilde{\mathbf{M}}_{\omega'\beta 0}(\rho_j), \quad \rho_j < R, \quad (8)$$

$$\mathbf{E}_j(\mathbf{r}) = a_j \mathbf{N}_{\omega_0\beta 0}(\rho_j), \quad \mathbf{H}_j(\mathbf{r}) = a_j n_0 \mathbf{M}_{\omega_0\beta 0}(\rho_j), \quad \rho_j > R. \quad (9)$$

Here the notations a_j, c_j are used instead of a_{j0}, c_{j0} .

Let \mathbf{R}_j be the radius vector of a point on the surface of the j th waveguide. Then the fields $\tilde{\mathbf{E}}_j(\mathbf{R}_j), \tilde{\mathbf{H}}_j(\mathbf{R}_j)$ in Eq. (1) and the fields $\mathbf{E}(\mathbf{R}_j), \mathbf{H}(\mathbf{R}_j)$ in Eq. (4) obey the boundary conditions on the surface of this waveguide. Generally, there are six boundary conditions. However, for the TM modes, only two of them are required,

$$[\mathbf{E}(\mathbf{R}_j)]_z = [\tilde{\mathbf{E}}_j(\mathbf{R}_j)]_z, \quad [\mathbf{H}(\mathbf{R}_j)]_\phi = [\tilde{\mathbf{H}}_j(\mathbf{R}_j)]_\phi. \quad (10)$$

These equations determine the partial amplitude a_j and c_j for given ω and β . To represent Eqs. (10) in a convenient form, one should express the fields $\mathbf{E}_l(\mathbf{R}_j)$ for $l \neq j$ entering in Eq. (4) in terms of the functions $\tilde{\mathbf{N}}$ using the Graph formula,

$$\begin{bmatrix} \mathbf{N}_{\omega_0\beta n}(\rho_l) \\ \mathbf{M}_{\omega_0\beta n}(\rho_l) \end{bmatrix} e^{in\phi_l} = \sum_{m=-\infty}^{+\infty} U_{nm}^{lj}(\omega, \beta) \begin{bmatrix} \tilde{\mathbf{N}}_{\omega_0\beta m}(\rho_j) \\ \tilde{\mathbf{M}}_{\omega_0\beta m}(\rho_j) \end{bmatrix} e^{im\phi_j}, \quad (11)$$

where

$$U_{nm}^{lj}(\omega, \beta) = H_{n-m}(\kappa_0 a |l - j|) [\text{sign}(j - l)]^{n-m}. \quad (12)$$

For the $m = 0$ approximation, one has

$$\begin{aligned} \mathbf{N}_{\omega_0\beta 0}(\rho_l) &\approx U_{l-j}(\omega, \beta) \tilde{\mathbf{N}}_{\omega_0\beta 0}(\rho_j), \\ \mathbf{M}_{\omega_0\beta 0}(\rho_l) &\approx U_{l-j}(\omega, \beta) \tilde{\mathbf{M}}_{\omega_0\beta 0}(\rho_j), \end{aligned} \quad (13)$$

where $U_{l-j}(\omega, \beta) = U_{00}^{lj}(\omega, \beta)$. Then, it follows from Eq. (9) that

$$\begin{aligned} \mathbf{E}_l(\mathbf{r}) &= a_l U_{l-j}(\omega, \beta) \tilde{\mathbf{N}}_{\omega_0\beta 0}(\rho_j), \\ \mathbf{H}_l(\mathbf{r}) &= a_l n_0 U_{l-j}(\omega, \beta) \tilde{\mathbf{M}}_{\omega_0\beta 0}(\rho_j). \end{aligned} \quad (14)$$

Thus,

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= a_j \mathbf{N}_{\omega_0\beta 0}(\rho_j) + \sum_{l \neq j} a_l U_{l-j}(\omega, \beta) \tilde{\mathbf{N}}_{\omega_0\beta 0}(\rho_j), \\ \mathbf{H}(\mathbf{r}) &= a_j n_0 \mathbf{M}_{\omega_0\beta 0}(\rho_j) + \sum_{l \neq j} a_l n_0 U_{l-j}(\omega, \beta) \tilde{\mathbf{M}}_{\omega_0\beta 0}(\rho_j). \end{aligned} \quad (15)$$

Substituting (15) and (8) into (10), one obtains

$$\begin{aligned} a_j H_0(\kappa_0 R) + \sum_{l \neq j} a_l U_{l-j}(\omega, \beta) J_0(\kappa_0 R) &= c_j J_0(\kappa R), \\ a_j n_0 \frac{i\omega_0}{\kappa_0} H'_0(\kappa_0 R) + \sum_{l \neq j} a_l n_0 U_{l-j}(\omega, \beta) \frac{i\omega_0}{\kappa_0} J'_0(\kappa_0 R) \\ &= c_j n \frac{i\omega'}{\kappa} J'_0(\kappa R). \end{aligned} \quad (16)$$

Then the system of equations (16) is reduced to the form

$$\frac{a_j}{\bar{a}(\omega, \beta)} - \sum_{l \neq j} U_{l-j}(\omega, \beta) a_l = 0, \quad (17)$$

$$c_j = \bar{c}(\omega, \beta) a_j, \quad (18)$$

where

$$\bar{a}(\omega, \beta) = \frac{n^2 \kappa_0 J'_0(\kappa R) J_0(\kappa_0 R) - n_0^2 \kappa J_0(\kappa R) J'_0(\kappa_0 R)}{n_0^2 \kappa J_0(\kappa R) H'_0(\kappa_0 R) - n^2 \kappa_0 J'_0(\kappa R) H_0(\kappa_0 R)}, \quad (19)$$

$$\bar{c}(\omega, \beta) = \frac{n_0^2 \kappa \{H_0(\kappa_0 R) J'_0(\kappa_0 R) - H'_0(\kappa_0 R) J_0(\kappa_0 R)\}}{n_0^2 \kappa J'_0(\kappa_0 R) J_0(\kappa R) - n^2 \kappa J_0(\kappa_0 R) J'_0(\kappa R)}. \quad (20)$$

The terms $U_{l-j}(\omega, \beta)$ in Eq. (17) describe the interaction between the waveguides. If the interaction $U_{l-j}(\omega, \beta)$ in Eq. (17) is neglected, the poles of $\bar{a}(\omega, \beta)$ or $\bar{c}(\omega, \beta)$ determine the guided modes for the isolated waveguide.

The system of equations (17) possesses nontrivial solutions if

$$\det \left\| \frac{\delta_{jl}}{\bar{a}(\omega, \beta)} - U_{l-j}(\omega, \beta) \right\| = 0. \quad (21)$$

This equation relates the frequency of the guided mode ω and its propagation constant β implicitly.

For the infinite periodical array of identical waveguides, the solution of Eq. (17) reads

$$a_j = a_0 e^{ikaj}, \quad -\pi/a < k \leq \pi/a. \quad (22)$$

In this case, the nontrivial solution exists if

$$\frac{1}{\bar{a}(\omega, \beta)} - U(\omega, \beta, k) = 0, \quad (23)$$

where

$$U(\omega, \beta, k) = \sum_{l \neq 0} U_l(\omega, \beta) e^{ikal}. \quad (24)$$

Equation (23) determines the dispersion law $\omega(\beta, k)$. If the propagation constant β is real, the corresponding mode frequencies may be either real or complex. If the frequency is real, the mode possesses an infinite Q factor. Otherwise, the mode has a finite lifetime and the imaginary part of the frequency determines the mode decay rate. However, if the corresponding quality factor is large, the mode may be considered as a guided one.

III. NUMERICAL SIMULATION FOR THE INFINITE AND THE FINITE ARRAYS

Let us consider the infinite array of the waveguides with the geometric parameters and the refractive indices which are chosen to be close to those in Ref. [14]. The specific values of the parameters are taken so that the illustration of the results looks quite representative. For this reason, one takes the waveguide radius $R = 1.975 \mu\text{m}$, the refractive index of the waveguides $n = 1.554$, and the refractive index of the environment $n_0 = 0.99n = 1.538$. Using (19), one finds the cutoff frequency for the isolated waveguide, $\omega_* = 5.57 \mu\text{m}^{-1}$. This value corresponds to the cutoff propagation constant $\beta_* = 8.57 \mu\text{m}^{-1}$. (Let us remind the reader that the speed

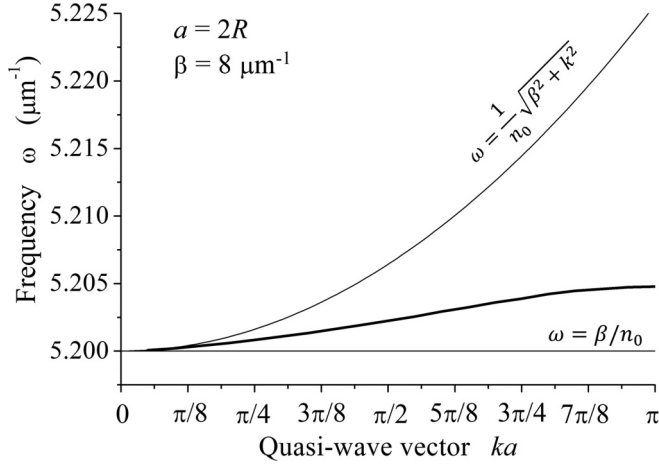


FIG. 2. The dispersion curve for the infinite array.

of light $c = 1$ and, therefore, the frequency has a dimension of the inverse length.)

The modes we are interested in appear due to the interaction between the waveguides. For this reason, we assume that the neighbor waveguides touch each other, since in this case the interaction reveals itself the most strongly. As an example, let us choose the propagation constant $\beta = 8 \mu\text{m}^{-1} < \beta_*$. The dispersion curve $\omega(\beta, k)$, which is a numerical solution to Eq. (23), is presented in Fig. 2 by the thick solid line.

One sees that the curve is completely located within the domain $\beta < n_0\omega(k) < \sqrt{\beta^2 + k^2}$. A physical explanation of this fact is given below. These numerical results completely support the kinematic criterion for the mode to be one without loss by radiation. Thus the infinite periodical array of the waveguides may possess the guided modes with the frequencies below the cutoff frequency of the single waveguide.

The guided modes that are inherent in the periodical array found above (see Fig. 2) possess the *infinite* quality factor $Q = 2\text{Re}\omega/|\text{Im}\omega|$. This is due to the fact that the array is infinite. However, actually one deals with the arrays composed of a finite number of the waveguides N . On the other hand, it is evident that for $N \gg 1$, the array should manifest the features similar to the infinite array. In particular, the guided modes should possess a high quality. Let us investigate how the quality factor Q depends on the number of the waveguides N .

Using Eq. (21), one can obtain numerically that for a finite $N \gg 1$, the highest-quality factor is reached for the

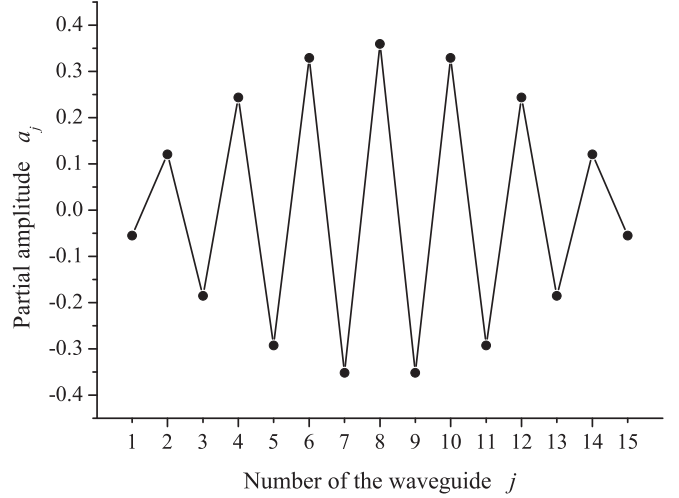
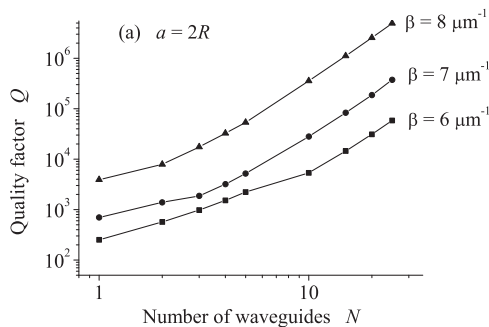


FIG. 4. The partial amplitude for the mode with the highest-quality factor.

modes whose frequency is close to the upper edge of the Brillouin zone, $k \approx \pi/a$ (such a feature is also inherent for the array of spherical particles [17,19,21]). The dependence of the quality factor on the number of the waveguides N just for modes with the highest Q factor is illustrated by two examples: the waveguides are touching, $a = 2R$, and the waveguides are spatially separated, $a = 3R$. The three values for the propagation constant smaller than β_* are taken: $\beta_1 = 6 \mu\text{m}^{-1}$, $\beta_2 = 7 \mu\text{m}^{-1}$, $\beta_3 = 8 \mu\text{m}^{-1} < \beta_*$. The results of the numerical simulation are presented in Fig. 3. The analysis of the dependencies in this figure reveals a remarkable feature: for $N > 10$, the dependence of the quality factor $Q(N) \sim N^3$.

Let us retrieve, using Eq. (17), the relation between the partial amplitudes a_j . A typical dependence, obtained numerically, is presented as an example in Fig. 4 for the case $N = 15$.

IV. THE INTERPRETATION OF THE NUMERICAL RESULTS

Knowing the dependence $\omega(\beta, k)$ allows us to determine the features of the guided modes. Let us pay attention to the fact that the reason for the mode to possess a finite lifetime is a conversion of it into a free photon. That mode is a radiative one. First, let us consider a single waveguide. Then the mode is

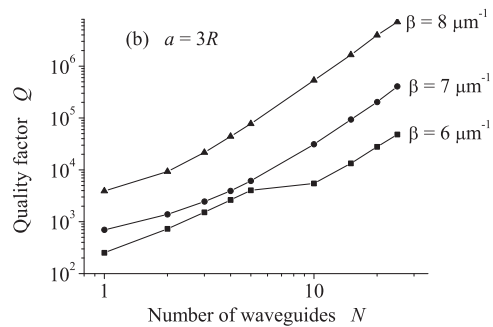


FIG. 3. The dependence of the quality factor on the number of the waveguides N for the plane array of waveguides.

described by a frequency ω and a propagation constant β . For the conversion into a free photon to take place, the photon wave vector \mathbf{q} should satisfy the following two conditions: $|\mathbf{q}| = n_0\omega$ and $q_z = \beta$. Since $q_z < |\mathbf{q}|$, the photon can be emitted *only* if $\beta < n_0\omega$. In the opposite case $\beta > n_0\omega$, the mode is one without loss by radiation and it is a guided one with the infinite quality factor. Let us turn to the infinite plane periodic array. In this case, a mode is determined by a quasiwave vector k , in addition to the frequency ω and the propagation constant β . Thus, the wave vector \mathbf{q} of the emitted free photon satisfies three conditions: $|\mathbf{q}| = n_0\omega$, $q_z = \beta$, and $q_x = k$. It is obvious that $\sqrt{q_x^2 + q_z^2} < |\mathbf{q}|$. So, the mode can be converted into a free photon only if $\sqrt{k^2 + \beta^2} < n_0\omega$. Thus, the infinite periodical array possesses guided modes within the frequency domain, which obeys the kinematic criterion

$$\beta < n_0\omega < \sqrt{\beta^2 + k^2}, \quad (25)$$

where the single waveguide allows only the radiating modes. Note that since $n\omega(\beta, k) < \sqrt{\beta^2 + k^2}$, a guided mode may not exist for small k at all.

Then, let us explain qualitatively the cubic dependence for the Q factor found in the previous section. First, let us consider the infinite array of the waveguides. Let $A_j(t)$ be the effective time-dependent partial amplitude for the j th waveguide, which characterizes the waveguide as a whole. The time evolution of $A_j(t)$ may be approximately described by the equation which is similar to a Schrödinger one,

$$i \frac{dA_j}{dt}(t) + V[A_{j-1}(t) + A_{j+1}(t)] = 0. \quad (26)$$

Here, V is the effective coupling between the nearest waveguides. Let us find the solution for this equation in the form

$$A_j(t) = A_0 e^{-i\omega t + ikj}, \quad -\pi < k \leq \pi. \quad (27)$$

Substituting (27) into (26), one obtains the dispersion law

$$\omega(k) = -2V \cos k. \quad (28)$$

Now let us turn to the finite array. As found above, the infinite array possesses the infinite Q , while the finite array possesses a large but finite Q (see Fig. 3). For this reason, it is natural to assume that this is connected with the availability of the *edge* waveguides in the array, which are responsible for the radiation of the photon. Based on this fact, one can write for the finite array the equation similar to Eq. (26),

$$i \frac{dA_j}{dt}(t) + V(1 - \delta_{j1})A_{j-1}(t) + V(1 - \delta_{jN})A_{j+1}(t) + i\gamma(\delta_{j1} + \delta_{jN})A_j(t) = 0. \quad (29)$$

The parameter $\gamma \ll V$ is responsible for the free photon emission. Using (27), one obtains, from (29),

$$\omega A_j + V(1 - \delta_{j1})A_{j-1} + V(1 - \delta_{jN})A_{j+1} + i\gamma(\delta_{j1} + \delta_{jN})A_j = 0. \quad (30)$$

Note that $\omega = \omega' + i\gamma$ may be complex. For the particular $j = N$, this equation takes the form

$$(\omega' + i\gamma)A_N + V A_{N-1} = 0. \quad (31)$$

So,

$$\omega' + i\gamma = -V \frac{A_{N-1}}{A_N}. \quad (32)$$

The dependence for a_j in Fig. 4 approaches zero at the edges of the array and resembles a cosine one. For this reason, let us seek the solution to Eq. (30) in the form

$$A_j \sim \cos k(j - N/2), \quad (33)$$

with k being close to $\pi - \pi/N$. Substituting (28) and (33) into (32), one gets

$$-2V \cos k + i\gamma = -V \frac{\cos k(N/2 - 1)}{\cos kN/2}. \quad (34)$$

Let

$$k = \pi - \pi/N + x, \quad (35)$$

where x is complex and $|x| \ll \pi/N$. For the sake of simplicity, let us assume N to be even. Then, substituting (35) into (34), one gets

$$2V \cos\left(\frac{\pi}{N} - x\right) + i\gamma = V \frac{\sin(\pi/N + Nx/2)}{\sin Nx/2}. \quad (36)$$

Taking into account a smallness of the arguments in the trigonometric functions in (36), one obtains

$$2V + i\gamma \approx V \left(\frac{2\pi}{N^2 x} + 1 \right). \quad (37)$$

Then, since $\gamma \ll V$, one has

$$x \approx \frac{2\pi}{N^2} \left(1 - i \frac{\gamma}{V} \right). \quad (38)$$

Substituting (38) and (35) into (28), one gets

$$\omega \approx 2V - i \frac{4\pi^2 \gamma}{N^3}. \quad (39)$$

Then, the quality factor

$$Q = \frac{2\text{Re } \omega}{|\text{Im } \omega|} = \frac{V N^3}{\pi^2 \gamma} \quad (40)$$

reveals the sought-for cubic dependence.

V. CONCLUSION

In this paper, on the simple example of $m = 0$ modes, we investigated the guided modes in the array of coupled waveguides below the cutoff frequency of a single waveguide, i.e., in the frequency domain where the single waveguide possesses only the radiating modes. It is shown that the infinite periodic array possesses a band of the guided modes with the infinite Q factor. This is due to the inevitable interaction between the waveguides. In the case of the finite array, the modes below the cutoff frequency are weakly radiating ones. Their quality factor increases with the number of waveguides as $Q(N) \sim N^3$. These results are obtained numerically using the multiple scattering formalism. A clear physical interpretation of the numerical results is given. Note that the effect is also similar to

the interacting micro- and nanospheres, as mentioned in Sec. I. The increasing of the lifetime of the excited identical nuclei which resonantly interact by the virtual x-radiation emitted by the nuclei was observed in [23].

ACKNOWLEDGMENT

The study is partially supported by the Russian Fund for Basic Research (Grant No. 16-02-00660).

-
- [1] J.-M. Lourtioz, H. Benisty, V. Berger, J.-M. Gerard, D. Maystre, and A. Tchebnokov, *Photonic Crystals: Towards Nanoscale Photonic Devices* (Springer, New York, 2008).
- [2] J. Joannopoulos, P. R. Villeneuve, and S. Fan, *Nature (London)* **386**, 143 (1997).
- [3] *Photonic Crystals. Advances in Design, Fabrication, and Characterization*, edited by K. Busch, S. Lölkes, R. B. Wehrspohn, and H. Föll (Wiley-VCH, Berlin, 2004).
- [4] S. Longhi, *Laser Photon. Rev.* **3**, 243 (2009).
- [5] A. R. McGurn and A. A. Maradudin, *Phys. Rev. B* **48**, 17576 (1993).
- [6] V. Kuzmiak and A. A. Maradudin, *Phys. Rev. B* **55**, 7427 (1997).
- [7] O. L. Berman, Yu. E. Lozovik, S. L. Eiderman, and R. D. Coalson, *Phys. Rev. B* **74**, 092505 (2006).
- [8] N. A. Giannakis, J. E. Inglesfield, A. K. Jastrzebski, and P. R. Young, *J. Opt. Soc. Am. B* **30**, 1755 (2013).
- [9] S. Belan and S. Vergeles, *Opt. Mater. Express* **5**, 130 (2015).
- [10] M. J. Zheng, J. J. Xiao, and K. W. Yu, *Phys. Rev. A* **81**, 033829 (2010).
- [11] F. Lederer, G. I. Stegemanb, D. N. Christodoulides, G. Assanto, M. Segev, and Y. Silberberg, *Phys. Rep.* **463**, 1 (2008).
- [12] D. N. Christodoulides, F. Lederer, and Y. Silberberg, *Nature (London)* **424**, 817 (2003).
- [13] A. Szameit, T. Pertsch, S. Nolte, A. Tünnermann, and F. Lederer, *Phys. Rev. A* **77**, 043804 (2008).
- [14] F. Dreisow, A. Szameit, M. Heinrich, T. Pertsch, S. Nolte, A. Tünnermann, and S. Longhi, *Phys. Rev. Lett.* **102**, 076802 (2009).
- [15] M. I. Gozman, Yu. I. Polishchuk, I. Ya. Polishchuk, and E. A. Tsivkunova, *Solid State Commun.* **213-214**, 16 (2015).
- [16] D. Marcuse, *Light Transmission Optics* (Van Nostrand Reinhold, New York, 1972).
- [17] G. S. Blaustein, M. I. Gozman, O. M. Samoylova, I. Ya. Polishchuk, and A. L. Burin, *Opt. Express* **15**, 17380 (2007).
- [18] A. L. Burin, H. Cao, G. C. Schatz, and M. A. Ratner, *J. Opt. Soc. Am. B* **21**, 121 (2004).
- [19] A. L. Burin, *Phys. Rev. E* **73**, 066614 (2006).
- [20] L. I. Deych and O. Roslyak, *Phys. Rev. E* **73**, 036606 (2006).
- [21] I. Ya. Polishchuk, M. I. Gozman, G. S. Blaustein, and A. L. Burin, *Phys. Rev. E* **81**, 026601 (2010).
- [22] H. C. Van de Hulst, *Light Scattering by Small Particles* (Dover, New York, 1981).
- [23] Yu. Kagan and A. M. Afanas'ev, *Zh. Eksp. Teor. Phys.* **50**, 271 (1966) [*Sov. Phys. JETP* **23**, 178 (1966)].