

Hidden long-range order in a spin-orbit-coupled two-dimensional Bose gasShih-Wei Su,¹ I-Kang Liu,¹ Shih-Chuan Gou,^{1,2,*} Renyuan Liao,³ Oleksandr Fialko,⁴ and Joachim Brand^{5,†}¹*Department of Physics and Graduate Institute of Photonics, National Changhua University of Education, Changhua 50058, Taiwan*²*Physics Division, National Center for Theoretical Sciences, Hsinchu 30013, Taiwan*³*College of Physics and Energy, Fujian Normal University, Fuzhou 350108, China*⁴*Dodd-Walls Centre for Photonics and Quantum Technology and Centre for Theoretical Chemistry and Physics, Institute for Natural and Mathematical Sciences, Massey University, Auckland, New Zealand*⁵*Dodd-Walls Centre for Photonics and Quantum Technology and Centre for Theoretical Chemistry and Physics, New Zealand Institute for Advanced Study, Massey University, Auckland, New Zealand*

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A spin-orbit-coupled two-dimensional Bose gas is shown to simultaneously possess quasi- and true long-range order in the total and relative phase sectors, respectively. The total phase undergoes a Berezinskii-Kosterlitz-Thouless transition to a low-temperature phase with quasi-long-range order, as expected for a two-dimensional quantum gas. Additionally, the relative phase undergoes an Ising-type transition building up true long-range order, which is induced by the anisotropic spin-orbit coupling in combination with spin-dependent particle-particle interactions. Based on the Bogoliubov approach, expressions for the total- and relative-phase fluctuations are derived analytically for the low-temperature regime. Numerical simulations of the stochastic projected Gross-Pitaevskii equation give a good agreement with the analytical predictions.

DOI: [10.1103/PhysRevA.95.053629](https://doi.org/10.1103/PhysRevA.95.053629)**I. INTRODUCTION**

Spatial dimensionality and interactions play crucial roles in the physics of phase transitions. The governing paradigm is the Hohenberg-Mermin-Wagner theorem [1,2], which asserts that a uniform infinite system with short-range interaction possessing continuous symmetries cannot exhibit long-range order (LRO) at finite temperatures in $d \leq 2$ dimensions. In the context of Bose gases, this implies the nonexistence of Bose-Einstein condensation (BEC) in dimension $d \leq 2$ in the thermodynamic limit. Instead, a two-dimensional (2D) Bose gas can develop a quasi-LRO in the low-temperature phase, characterized by an algebraically decaying correlation function, and undergoes a phase transition to the high-temperature phase, where the correlation between particles decays exponentially. This mechanism is known as the Berezinskii-Kosterlitz-Thouless (BKT) transition [3–6].

Recent advances in the manipulation of ultracold atoms have made it possible to study uniform 2D quantum degenerate gases [7,8] and thus it is timely to probe the unexplored aspects of two-dimensional phase transitions. To this end, we are particularly interested in the condensation of spin-orbit-coupled pseudospin-1/2 Bose gases [9], which have attracted a great deal of attention in recent years [10–23]. The spin-orbit coupling (SOC) here refers to a synthetic gauge field originating from the laser-assisted coupling between the atomic center-of-mass motion and the internal degrees of freedom [9,10,24]. Synthetic SOC in ultracold gases has so far been realized in one-dimensional (1D) [9] and 2D form [25,26] and has become a tunable resource [27], with more exotic realizations proposed [28,29]. For an ideal two-component Bose gas, the presence of SOC can enhance the density of states at low energies, making the system more susceptible to

both quantum and thermal fluctuations and thus preventing the atoms from condensing [14–16]. On the other hand, the interatomic interactions can stabilize the condensate, and enhanced condensation due to SOC was found in superfluid Fermi gases [30–32]. It is thus anticipated that the competition between fluctuations and interactions in the presence of SOC can drastically affect the mechanism of the BEC phase transition. Recently the thermal properties of spin-orbit-coupled 2D Bose gases have been investigated and extended scenarios of BKT physics reaching from relative suppression of superfluidity to fractionalized vortex phases have been predicted [17,20]. In these studies, the corresponding effective theories were derived in terms of the total-phase degree of freedom by integrating out the relative-phase counterpart. Since the variables representing respectively the total- and relative-phase sectors are entwined via SOC, a more complete picture of the nature of the superfluid phase transition can be obtained by considering all degrees of freedom. The aim of the current work is to address this issue.

In this paper we study the low-temperature properties of a spin-orbit-coupled two-dimensional Bose gas in the plane-wave phase with Bogoliubov theory and simulations with the stochastic projected Gross-Pitaevskii equation (SPGPE) [33–36]. We find that quasi-long-range order in the total phase of the pseudospin- $\frac{1}{2}$ superfluid coexists with true long-range order of the relative phase between the two spin components.

The organization of this paper is as follows. In Sec. II, the exact solutions to Bogoliubov–de Gennes equations pertinent to the low-lying excitations of a 2D Bose gas with anisotropic SOC are presented, which reveal the low-temperature properties of the gas. In Sec. III, two-point correlation functions of the total- and relative phases are calculated both analytically and numerically. Based on the analytical results in Sec. II, close-form expressions of the phase correlation functions are derived. Meanwhile, to explore the essence of BEC phase transition in the current system on an *ab initio* basis, we perform SPGPE simulations to evaluate the correlation

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functions over a wide range of temperatures. The attributes of phase transitions in the total- and relative phases are verified according to the behavior of the correlation functions and the underlying physics is addressed. Concluding remarks are given in Sec. IV, including a discussion on the experimental implementation for measuring the hidden LRO of the system. Finally, auxiliary calculations and derivations are placed in the Appendix.

II. FORMULAS

The system under study is described by the Hamiltonian

$$\hat{H} = \int d^2r \left[\hat{\Psi}^\dagger \hat{H}_{\text{sp}} \hat{\Psi} + \frac{g_{11}}{2} (\hat{\Psi}_1^\dagger \hat{\Psi}_1)^2 + \frac{g_{22}}{2} (\hat{\Psi}_2^\dagger \hat{\Psi}_2)^2 + g_{12} \hat{\Psi}_1^\dagger \hat{\Psi}_1 \hat{\Psi}_2^\dagger \hat{\Psi}_2 \right], \quad (1)$$

where $\hat{\Psi} = (\hat{\Psi}_1, \hat{\Psi}_2)^T$ is the two-component spinor field operator and $\hat{H}_{\text{sp}} = -\hbar^2 \nabla^2 / 2m + \kappa_x \hat{p}_x \hat{\sigma}_x + \kappa_y \hat{p}_y \hat{\sigma}_y$ is the single-particle Hamiltonian with $\kappa_{x,y}$ the spin-orbit coupling strengths along different directions and $\hat{\sigma}_{x,y}$ are the Pauli matrices. The inter- and intraspecies atomic interaction strengths are characterized by g_{12} and g_{ii} ($i = 1, 2$), respectively. For simplicity, we will assume that the intra-species interactions are identical, i.e., $g_{11} = g_{22} \equiv g$, and note that $g_{12} \leq g$ is a necessary condition to obtain a miscible ground state. It is worth mentioning that for the fully anisotropic SOC ($\kappa_x = 0$ or $\kappa_y = 0$) the SOC term in Eq. (1) can be gauged away when all coupling constants are equal ($g_{12} = g$) leading to trivial results. In the following we will therefore assume $g_{12} < g$ (strictly smaller), and the spin-dependence of the coupling constants will be relevant for the physical outcomes. The assumption $g_{12} \leq g_{11} = g_{22}$ is a good approximation to the situation in experiments with ^{87}Rb [9].

Diagonalizing the single-particle Hamiltonian yields two dispersion branches, $\epsilon_{\pm} = p^2/2m \pm (\kappa_x^2 p_x^2 + \kappa_y^2 p_y^2)^{1/2}$, and the corresponding eigenvectors, $\phi_{\mathbf{k}}^{\pm} = (1, \pm e^{i\varphi_{\mathbf{k}}})^T e^{i\mathbf{p}\cdot\mathbf{x}/\hbar} / \sqrt{2}$, where $\varphi_{\mathbf{k}} = \arg(\kappa_x p_x + i\kappa_y p_y)$ [37]. For anisotropic SOC ($\kappa_x \neq \kappa_y$) the single-particle ground state lies in the lower branch, and is twofold degenerate at $\mathbf{k} =$

$\pm m\kappa_x \mathbf{e}_x$ ($\pm m\kappa_y \mathbf{e}_y$) for $|\kappa_x| > |\kappa_y|$ ($|\kappa_x| < |\kappa_y|$). On the other hand the single-particle ground state is infinitely degenerate on the Rashba ring of radius $|\mathbf{p}| = m\kappa$ in momentum space for isotropic SOC ($|\kappa_x| = |\kappa_y| \equiv \kappa$).

For an interacting gas, depending on the interatomic interaction strengths, the ground state phases are characterized by the plane waves corresponding to the minima of the single-particle dispersion. For $g > g_{12}$, the ground state is a single plane-wave (PW) state while for $g < g_{12}$ the ground state is a standing wave created by the superposition of two plane waves carrying opposite momenta [37]. In the following calculation, we shall work in the dimensionless units where the length, time, and energy are scaled by $a_h = \sqrt{\hbar/m\omega_0}$, $1/\omega_0$, and $\hbar\omega_0$, respectively, with m the atomic mass and ω_0 the transverse trapping frequency. In the following, the dimensionless interatomic interaction strengths and SOC strengths are denoted by \tilde{g}_{ij} and $\tilde{\kappa}_{x,y}$, respectively.

Within the framework of mean-field theory, the dynamics of Bose gases is determined by the Gross-Pitaevskii energy functional $\mathcal{E}[\Psi^*, \Psi] = \langle \hat{H} \rangle$, where the Bose fields in Eq. (1) are replaced by the complex classical-field wave functions, $\Psi_j = \langle \hat{\Psi}_j \rangle$. The Gross-Pitaevskii equation, $i\hbar \partial_t \Psi_j = \mathcal{L}_j \Psi_j$, can be derived via the Hartree variational principle (see the Appendix). For definiteness and to assure the validity of the mean-field approach, we will consider anisotropic SOC and focus on the PW state in what follows, which avoids the degeneracies and ambiguities of scenarios with higher symmetry [15]. At zero temperature, the PW state wave function is $\Psi^0 = (\Psi_1^0, \Psi_2^0)^T = \sqrt{n} e^{-i\tilde{\kappa}_x x} (1, 1)^T$ where we assume that the condensation occurs at $\mathbf{p} = (-|\tilde{\kappa}_x|, 0)$ and n is the total particle density. Furthermore, the PW state is characterized by a nonvanishing pseudospin density, $\mathbf{S} = \sum_{\alpha, \beta} \Psi_{\alpha}^* \boldsymbol{\sigma}_{\alpha\beta} \Psi_{\beta}$, along x direction, $\mathbf{S}^0 = n \mathbf{e}_x$. To investigate the low-lying excitations, we adopt the Bogoliubov formulation where the total wave function is decomposed as $\Psi_j = e^{-i\mu t} e^{-i\tilde{\kappa}_x x} (\Psi_j^0 + \delta\Psi_j)$ with μ the chemical potential and $\delta\Psi_j$ the low-lying excitation. Inserting $\delta\Psi_j = \sum_{\mathbf{q}} (u_{\mathbf{q}}^j e^{i(\mathbf{q}\cdot\mathbf{r} - \omega t)} - v_{\mathbf{q}}^{j*} e^{-i(\mathbf{q}\cdot\mathbf{r} - \omega t)}) / \sqrt{A}$, where A is the area of system and ω is the excitation energy of the mode with momentum \mathbf{q} , into the Gross-Pitaevskii equation yields the Bogoliubov-de Gennes equation (also see the Appendix)

$$\begin{pmatrix} \mathcal{L}_0 - \tilde{\kappa}_x q_x & -\tilde{g}n & \tilde{g}_{12}n + h_{\text{soc}} - \tilde{\kappa}_x^2 & -\tilde{g}_{12}n \\ \tilde{g}n & -\mathcal{L}_0 - \tilde{\kappa}_x q_x & \tilde{g}_{12}n & h_{\text{soc}}^* + \tilde{\kappa}_x^2 - \tilde{g}_{12}n \\ \tilde{g}_{12}n + h_{\text{soc}}^* - \tilde{\kappa}_x^2 & -\tilde{g}_{12}n & \mathcal{L}_0 - \tilde{\kappa}_x q_x & -\tilde{g}n \\ \tilde{g}_{12}n & h_{\text{soc}} + \tilde{\kappa}_x^2 - \tilde{g}_{12}n & \tilde{g}n & -\mathcal{L}_0 - \tilde{\kappa}_x q_x \end{pmatrix} \begin{pmatrix} u_1^{\mathbf{q}} \\ v_1^{\mathbf{q}} \\ u_2^{\mathbf{q}} \\ v_2^{\mathbf{q}} \end{pmatrix} = \omega \begin{pmatrix} u_1^{\mathbf{q}} \\ v_1^{\mathbf{q}} \\ u_2^{\mathbf{q}} \\ v_2^{\mathbf{q}} \end{pmatrix}, \quad (2)$$

where $\mathcal{L}_0 = q^2/2 + \tilde{g}n + \tilde{\kappa}_x^2$, $h_{\text{soc}} = \tilde{\kappa}_x q_x - i\tilde{\kappa}_y q_y$, and $u_j^{\mathbf{q}}, v_j^{\mathbf{q}}$ satisfy the normalization condition $\sum_j |u_j^{\mathbf{q}}|^2 - |v_j^{\mathbf{q}}|^2 = 1$. For the fully anisotropic SOC ($\tilde{\kappa}_y = 0$), Eq. (2) is solved with the two distinct energy dispersion relations of the excitation:

$$\omega_{\pm}^{\mathbf{q}} = \sqrt{(\xi_{\pm}^{\mathbf{q}})^2 - (\tilde{g} + \tilde{g}_{12})^2 n^2}, \quad (3)$$

with $\xi_{\pm}^{\mathbf{q}} = q^2/2 + (\tilde{g} + \tilde{g}_{12})n$ and the eigenvector $\delta\Psi_{\pm}^{\mathbf{q}} \sim (u_{\pm}^{\mathbf{q}}, v_{\pm}^{\mathbf{q}}, u_{\pm}^{\mathbf{q}}, v_{\pm}^{\mathbf{q}})^T$;

$$\omega_{\pm}^{\mathbf{q}} = -2q_x \tilde{\kappa}_x + \sqrt{(\xi_{\pm}^{\mathbf{q}})^2 - (\tilde{g} - \tilde{g}_{12})^2 n^2}, \quad (4)$$

with $\xi_{\pm}^{\mathbf{q}} = q^2/2 + (\tilde{g} - \tilde{g}_{12})n + 2\tilde{\kappa}_x^2$ and the eigenvector $\delta\Psi_{\pm}^{\mathbf{q}} \sim (u_{\pm}^{\mathbf{q}}, v_{\pm}^{\mathbf{q}}, -u_{\pm}^{\mathbf{q}}, -v_{\pm}^{\mathbf{q}})^T$.

Equation (3) represents a gapless mode corresponding to the total-phase excitation that is immune to SOC. On the other hand, Eq. (4) indicates a mode corresponding to the

relative-phase spin excitation where the effect of SOC acts to open a gap but also shift the minimum of the dispersion. For nonvanishing κ_y the eigenenergies and eigenvectors can be calculated numerically and the above conclusion remains valid.

III. RESULTS AND DISCUSSIONS

To study the phase fluctuations in the spin-orbit-coupled Bose gas, the Bose field can be expressed as [38]

$$\hat{\Psi} = \begin{pmatrix} \hat{\Psi}_1(\mathbf{r}') \\ \hat{\Psi}_2(\mathbf{r}') \end{pmatrix} = \sqrt{n} e^{i\hat{\phi}_t(\mathbf{r}')} \begin{pmatrix} e^{i\hat{\phi}_r(\mathbf{r}')} \\ e^{-i\hat{\phi}_r(\mathbf{r}')} \end{pmatrix}, \quad (5)$$

where $\hat{\phi}_{t,r}$ denote the total- and relative-phase operators, respectively, and we have neglected the density fluctuations. For small fluctuations, Eq. (5) can be expanded to the first order which gives $\hat{\phi}_{t,r} = \sum_{\mathbf{q}} [(\mathcal{U}_{t,r}^{\mathbf{q}} + \mathcal{V}_{t,r}^{\mathbf{q}}) \hat{\alpha}_{t,r}^{\mathbf{q}} - \text{H.c.}] / 2i\sqrt{n}$, where $\hat{\alpha}_{t,r}^{\mathbf{q}}$ ($\hat{\alpha}_{t,r}^{\mathbf{q}\dagger}$) is the annihilation (creation) operator that destroys (creates) the excitation in the corresponding branch $\omega_{t,r}^{\mathbf{q}}$ and $(\mathcal{U}_{t,r}^{\mathbf{q}}, \mathcal{V}_{t,r}^{\mathbf{q}}) = (u_{t,r}^{\mathbf{q}}, v_{t,r}^{\mathbf{q}}) e^{i\mathbf{q}\cdot\mathbf{r}} / \sqrt{A}$ is the amplitude of Bogoliubov excitation. In the linear approximation, the total- and relative-phase operators are decoupled and can be expressed in terms of the excitations $\delta\Psi_t^{\mathbf{q}}$ and $\delta\Psi_r^{\mathbf{q}}$, respectively. The two-point phase correlation functions are given by

$$G_{t,r}(\mathbf{r}', \mathbf{r}'') = \langle e^{i\hat{\phi}_{t,r}(\mathbf{r}') - i\hat{\phi}_{t,r}(\mathbf{r}'')} \rangle = e^{-\langle(\Delta\phi_{t,r})^2\rangle/2}, \quad (6)$$

where $\langle \dots \rangle$ denotes the ensemble average and $\Delta\phi_{t,r} = \hat{\phi}_{t,r}(\mathbf{r}') - \hat{\phi}_{t,r}(\mathbf{r}'')$. The thermal average can be expressed in terms of the Bogoliubov amplitudes

$$\langle(\Delta\phi_{t,r})^2\rangle = \int \frac{d^2q}{\pi n} \left(N_{t,r}^{\mathbf{q}} + \frac{1}{2} \right) (u_{t,r}^{\mathbf{q}} + v_{t,r}^{\mathbf{q}})^2 \sin^2 \frac{\mathbf{q}\cdot\mathbf{r}}{2}, \quad (7)$$

where $N_{t,r}^{\mathbf{q}} = 1/[\exp(\omega_{t,r}^{\mathbf{q}}/T) - 1]$ is the Bose-Einstein distribution function with T the temperature measured in units of $\hbar\omega_0/k_B$. Due to translational invariance the averaged phase fluctuations and the correlation function only depend on the separation $|\mathbf{r}| = |\mathbf{r}' - \mathbf{r}''|$. The Bogoliubov amplitudes in the integrand are $(u_t^{\mathbf{q}} + v_t^{\mathbf{q}})^2 = [\xi_t^{\mathbf{q}} + (\tilde{g} + \tilde{g}_{12})n]/2\omega_t^{\mathbf{q}}$ and $(u_r^{\mathbf{q}} + v_r^{\mathbf{q}})^2 = [\xi_r^{\mathbf{q}} + (\tilde{g} - \tilde{g}_{12})n]/2(\omega_r^{\mathbf{q}} + 2\tilde{\kappa}_x q_x)$. The total-phase fluctuation shown in Eq. (7) exhibits an infrared divergence similar to that of a 2D scalar Bose gas. Accordingly, the total-phase correlation function is shown in Fig. 4 in the Appendix. In the thermodynamic limit it is expected that the long-range correlation $\lim_{|\mathbf{r}|\rightarrow\infty} e^{-\langle(\Delta\phi_t)^2\rangle/2}$ would be destroyed by the total-phase fluctuations, leading to the BKT-type physics which is characterized by the quasi-LRO as discussed in Ref. [17]. The BKT transition temperature for the 2D scalar Bose gas is given by $T_{\text{BKT, scalar}}^{\infty} = 2\pi\hbar^2 n / \{mk_B \ln[(380 \pm 3)/\tilde{g}_0]\}$ with \tilde{g}_0 the dimensionless interaction strength [39,40]. Comparing the excitation spectrum of the 2D scalar Bose gas with the in-phase excitation energy $\omega_t^{\mathbf{q}}$, the BKT transition temperature T_{BKT}^{∞} for the total-phase degree of freedom can be estimated by replacing \tilde{g}_0 with $\tilde{g} + \tilde{g}_{12}$. On the contrary the fluctuation $\langle(\Delta\phi_t)^2\rangle$ is suppressed due to the gapped and anisotropic excitation energy, leading to the existence of true LRO in the relative-phase correlation. The relative-phase fluctuations evaluated from Eq. (7) are shown in Fig. 1.

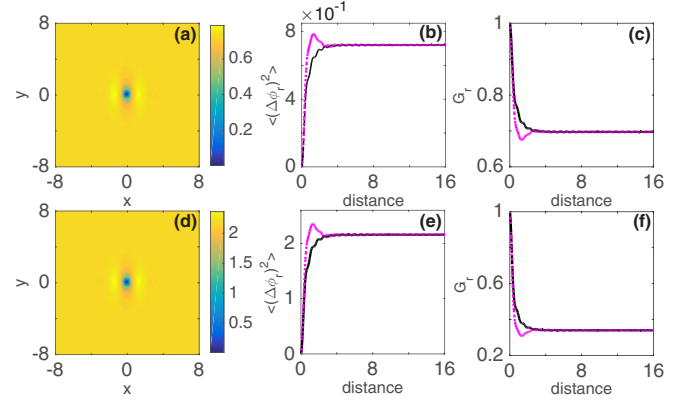


FIG. 1. Relative-phase fluctuations from Bogoliubov theory at two different temperatures. Panels (a) and (d) show the relative-phase fluctuations $\langle(\Delta\phi_t)^2\rangle$ from Eq. (6) where the axes denote the separations $x = x' - x''$ and $y = y' - y''$. A plateau is seen to develop at large separation. The fluctuations $\langle(\Delta\phi_t)^2\rangle$ are also shown in panels (b) and (e) while panels (c) and (f) depict the correlation function G_r from Eq. (7). The magenta (grey) lines are plotted along the x -axis and black lines along the y -axis for panels (b), (c), (e), and (f). The temperature is set to $T/T_{\text{BKT}}^{\infty} \approx 0.44$ for panels (a), (b), (c) and to $T/T_{\text{BKT}}^{\infty} \approx 1.33$ for panels (d), (e), (f), and $\mu = 13$, $\tilde{g}_{12}/\tilde{g} = 0.9$ and $(\tilde{\kappa}_x, \tilde{\kappa}_y) = (1, 0)$.

Plateaus of constant fluctuation and correlation are visible at a separation $|\mathbf{r}|$ larger than $\approx 4 = 4\tilde{\kappa}_x^{-1} \approx 20\xi$, where $\xi = 1/\sqrt{2\mu}$ is the zero-temperature healing length in scaled units. It is remarkable that the length scale for plateau formation is independent of temperature while the magnitude decreases with increasing temperature. Additionally, the effect of anisotropic SOC appears in the spatial variation at short length scales as clearly seen in Fig. 1.

To verify the analytical prediction, we numerically calculate the first-order correlation functions by evolving the stochastic projected Gross-Pitaevskii equation [33–36]

$$d\Psi_j = \mathcal{P}\{-i\mathcal{L}_j\Psi_j dt + \Gamma(\mu - \mathcal{L}_j)\Psi_j dt + dW_j\}, \quad (8)$$

where \mathcal{P} is the projection operator restricting the evolution to the region of $E < \epsilon_{\text{cut}}$, μ is the chemical potential, Γ is the growth rate, and dW_j is the complex white noise satisfying the fluctuation-dissipation relation $\langle dW_j^*(\mathbf{r}', t) dW_k(\mathbf{r}'', t) \rangle = 2\Gamma T \delta(\mathbf{r}', \mathbf{r}'') \delta_{jk} dt$. The phase correlation function of Eq. (6) can be numerically computed via the expression $G_{t,r}(\mathbf{r}', \mathbf{r}'') = \frac{1}{N_s} \sum_{j=1}^{N_s} \exp[i\phi_{t,r}(\mathbf{r}', t_j) - i\phi_{t,r}(\mathbf{r}'', t_j)]$, where t_j is a set of N_s times at which the field is sampled after the system reaches equilibrium [6,33]. In the numerical simulation, we consider the parameters $\mu = 13$, $\epsilon_{\text{cut}} \approx 42$, $\tilde{g}_{12}/\tilde{g} = 0.9$, and $(\tilde{\kappa}_x, \tilde{\kappa}_y) = (1, 0)$ at various temperatures. To obtain an equilibrated sample for calculating the correlation function, we let the system evolve for a sufficiently long time ($\gg 1/\Gamma$) and then take 10^3 samples to implement the averaging.

Figure 2 depicts the total-phase profile and correlation at various temperatures. At low temperatures the total-phase exhibits the periodic structure shown in Fig. 2(a), a consequence of the PW state entailing the phase factor $e^{-2i\tilde{\kappa}_x x}$. At high temperatures, the increasing thermal fluctuations smear out the quasiperiodic structure in Fig. 2(a) and results in a fluctuating

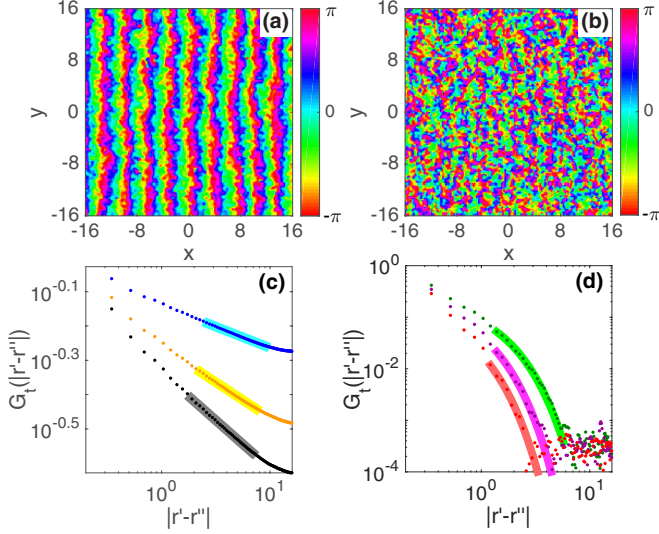


FIG. 2. Total phase from stochastic simulations of Eq. (8). Panels (a) and (b) depict the snapshot of the total-phase profile $\arg(\Psi_1) + \arg(\Psi_2)$ at $T/T_{\text{BKT}}^\infty \approx 0.44$ and 1.33 respectively. The correlation function $G_t(|\mathbf{r}' - \mathbf{r}''|)$ is shown on a doubly-logarithmic scale in panels (c) and (d). Dots represent numerical data and solid lines are algebraic fits for the lower temperatures in panel (c) and exponential fits in panel (d). The temperatures are $T/T_{\text{BKT}}^\infty \approx 0.44$ (blue, top), 0.67 (orange, middle), 0.78 (black, bottom) in panel (c), and 1.33 (green, top), 1.56 (magenta, middle) and 1.78 (red, bottom) in panel (d).

total-phase profile as shown in Fig. 2(b). Further analyses of the total-phase correlation are shown in Figs. 2(c) and 2(d). For $T < T_{\text{BKT}}^\infty$, the results are consistent with algebraic decay of the correlation function while for $T > T_{\text{BKT}}^\infty$ the correlation function decays exponentially, a defining feature of the BKT transition.

The relative-phase profiles and the correlation functions are shown in Fig. 3. Unlike the total-phase case, thermal fluctuations in relative-phase sector are suppressed in the low-temperature regime, as shown in Fig. 3(a), and the corresponding correlation function shown in Fig. 3(c) develops a plateau structure at large separation, implying an established LRO. On the other hand, the strong thermal fluctuations in the high-temperature regime completely randomize the phase distribution, leading to an exponentially decaying correlation function, as shown in Fig. 3(d). The value of phase correlation decreases with increasing temperature and eventually vanishes for $T > T_{\text{BKT}}^\infty$, as shown in Figs. 3(d) and 3(e). We note that in Fig. 3(c) the correlation function exhibits oscillations at small separation along the x direction. This qualitatively agrees with the oscillations in Figs. 1(c) and 1(f), which can be attributed to the anisotropic SOC. We note that the analytical and numerical calculations for the LRO are in close agreement at low temperatures, but inconsistent at high temperatures where Bogoliubov theory is expected to be inapplicable. In Figs. 1(f) and 3(d), the analytical calculation predicts a nonzero value whereas the numerical one gives a zero value. This discrepancy is attributed to the fact that Bogoliubov theory is poorly justified outside the perturbative low-temperature regime.

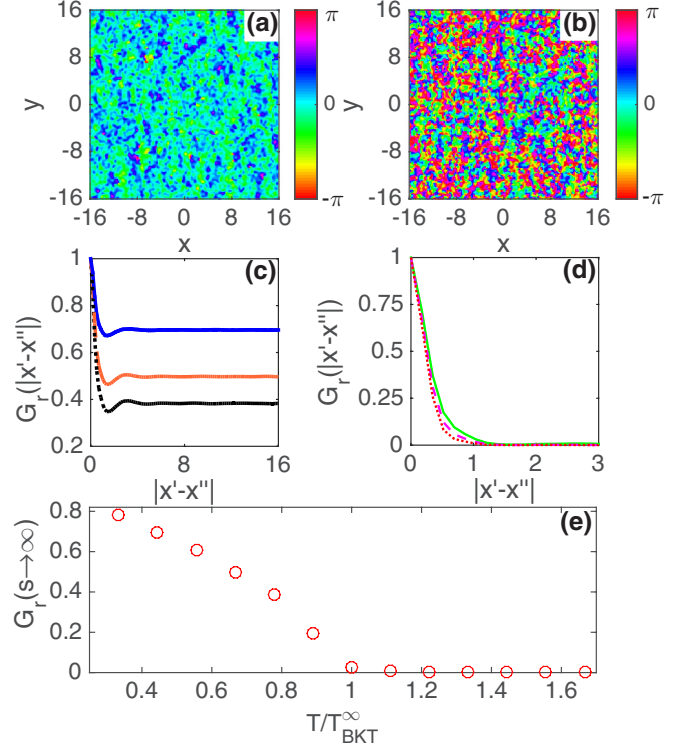


FIG. 3. Relative phase from stochastic simulations. Panels (a) and (b) depict the snapshots of the relative-phase profile $\arg(\Psi_1) - \arg(\Psi_2)$ at $T/T_{\text{BKT}}^\infty \approx 0.44$ and 1.33 respectively. The correlation function G_r is plotted in panels (c) and (d) along the SOC direction. While plateaus are reached in (c) for $T/T_{\text{BKT}}^\infty \approx 0.44$ (blue, top), 0.67 (orange, middle), 0.78 (black, bottom), the correlation function quickly decays to zero for the higher temperatures $T/T_{\text{BKT}}^\infty \approx 1.33$ (green, solid line), 1.56 (magenta, dashed line) and 1.78 (red, dotted line) in panel (d). Panel (e) shows the plateau values for the phase correlation $G_r(|x' - x''| \rightarrow \infty)$ versus temperature.

We have shown that LRO does exist in the relative-phase sector. But would it imply the existence of an otherwise different form of BEC? To address this problem, we inspect the single-particle density matrix (SPDM) for the two-component system defined in analogy with the scalar BEC (see the Appendix). Retaining the phase fluctuations, the matrix elements of the generalized SPDM can be presented as a 2-by-2 matrix:

$$\rho(\mathbf{r}', \mathbf{r}'') = n \begin{bmatrix} e^{-\frac{(\Delta\phi_1)^2}{2}} - \frac{(\Delta\phi_2)^2}{2} & e^{-\frac{(\Delta\phi_1)^2}{2}} - \frac{(\Delta_+\phi_r)^2}{2} \\ e^{-\frac{(\Delta\phi_1)^2}{2}} - \frac{(\Delta_+\phi_r)^2}{2} & e^{-\frac{(\Delta\phi_1)^2}{2}} - \frac{(\Delta\phi_2)^2}{2} \end{bmatrix}, \quad (9)$$

where $\Delta_+\phi_r = \hat{\phi}_r(\mathbf{r}') + \hat{\phi}_r(\mathbf{r}'')$ (see the Appendix). The matrix elements of Eq. (9) represent various correlations between atomic fields at different locations, where the diagonal elements denote the prototypal SPDMs corresponding to components 1 and 2 respectively. Note that all matrix elements in Eq. (9) contain the prefactor $e^{-\frac{(\Delta\phi_1)^2}{2}}$, which vanishes at large distances. As a result, the off-diagonal long-range order does not extend to the matrix elements of the SPDM implying that there is no macroscopic eigenvalue and hence the 2D spin-orbit-coupled Bose gas does not exhibit BEC, according to a well-known criterion for BEC [41].

As the orientation of local spin density \mathbf{S} is determined by the relative phase between the components of $\hat{\Psi}$, the LRO discussed above manifests a “spin-spin” correlation. As far as the PW phase is concerned, an anisotropic SOC is bound to result in two degenerate lowest energy states characterized by two counteroriented planar spins, $\pm\mathbf{S}^0$, respectively. This configuration features a 2D Ising-type ground state in the relative-phase sector and is protected by the energy gap in the dispersion, ω_r^q . It exhibits LRO by spontaneously breaking Z_2 symmetry at finite temperatures. Our numerical simulations suggests that the Ising-type and BKT transitions occur at the same temperature T_{BKT}^∞ , and the system simultaneously builds up the quasi- and true LROs in the total- and relative-phase sectors, correspondingly, when $T < T_{\text{BKT}}^\infty$. It is interesting to point out that a similar Ising-type phase transition was predicted to arise in the 2D polar spin-1 condensate subject to finite quadratic Zeeman energy [42]. For isotropic SOC, the 2D spin- $\frac{1}{2}$ Bose gas was shown to undergo the BKT transition at $T = 0$ [17,20], suggesting that at any nonzero temperatures the LRO in the relative-phase sector would be destroyed by the substantially intensified fluctuations due to the infinitely degenerate ground state.

IV. CONCLUSIONS

We theoretically approach the problem of BEC phase transition in a 2D Bose gas subjected to anisotropic SOC. By pinpointing the behavior of correlation functions, we verify that the total phase undergoes a conventional BKT transition, characterized by quasi-LRO, whereas the relative phase undergoes an Ising-type transition establishing true LRO.

It should be noted that we have used a generic SOC Hamiltonian in our model rather than the experimentally realized one which has different orientation to the spin-quantization axis and contains extra Rabi and Zeeman terms [9]. Nonetheless, 2D generalization of the spin-orbit-coupled Bose gas in Ref. [9] is shown to possess excitation spectra similar to those in our case [43], and this promises to observe the described phenomena in our study.

Finally, we briefly account for the idea of probing the hidden LRO in the relative phase by means of atomic interferometry. After the optical pumping by a $\pi/2$ pulse [44,45], the resultant density of each component becomes

$$n_\pm = \frac{1}{2}(\Psi_1 \pm \Psi_2)^*(\Psi_1 \pm \Psi_2). \quad (10)$$

The relative phase ϕ_r can be extracted from the density fringes f which is expressed in terms of n_\pm ,

$$j = \frac{n_+ - n_-}{2(n_+ + n_-)} \approx \frac{1}{2} \cos \phi_r. \quad (11)$$

The information of the relative phase can be measured through j and the relative-phase correlation can be evaluated.

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APPENDIX

Bogoliubov-de Gennes equation. In the the mean-field approximation, the energy functional of the spin-orbit-coupled 2D Bose gas is

$$E[\Psi^*, \Psi] = \int \left[\Psi^* \left(-\frac{\hbar^2}{2m} \nabla^2 + \kappa_x \hat{\rho}_x \hat{\sigma}_x + \kappa_y \hat{\rho}_y \hat{\sigma}_y \right) \Psi + \frac{g_{11}}{2} |\Psi_1|^4 + \frac{g_{22}}{2} |\Psi_2|^4 + g_{12} |\Psi_1|^2 |\Psi_2|^2 \right] d^2r, \quad (A1)$$

where $\Psi = (\Psi_1, \Psi_2)^T$, $\kappa_{x,y}$ are the strengths of SOC, $\hat{\sigma}_{x,y}$ are the Pauli matrices, and g_{ij} are the nonlinear interaction strengths. In the following, we consider the case $g_{11} = g_{22} \equiv g$. The dynamics is described by the GP equation which can be derived via the Hartree variational principle $i\hbar \partial_t \Psi_j = \delta E / \delta \Psi_j^* = \mathcal{L}_j \Psi_j$ with \mathcal{L}_j the GP evolution operator which takes the form

$$\begin{aligned} i\hbar \partial_t \Psi_1 &= \left(-\frac{\hbar^2}{2m} \nabla^2 + g\rho_1 + g_{12}\rho_2 \right) \Psi_1 \\ &\quad + \left(\frac{\hbar}{i} \kappa_x \partial_x - \hbar \kappa_y \partial_y \right) \Psi_2, \\ i\hbar \partial_t \Psi_2 &= \left(-\frac{\hbar^2}{2m} \nabla^2 + g\rho_2 + g_{12}\rho_1 \right) \Psi_2 \\ &\quad + \left(\frac{\hbar}{i} \kappa_x \partial_x + \hbar \kappa_y \partial_y \right) \Psi_1, \end{aligned} \quad (A2)$$

where $\rho_j = |\Psi_j|^2$ is the density of j th component. In the following calculation, we shall work in the dimensionless units that the length, time, and energy are scaled by $a_h = \sqrt{\hbar/m\omega_0}$, $1/\omega_0$, and $\hbar\omega_0$, respectively, with m the atomic mass and ω the transverse trapping frequency. In the following, the dimensionless interatomic interaction strengths and SOC strengths are denoted by \tilde{g}_{ij} and $\tilde{\kappa}_{x,y}$, respectively.

For $g > g_{12}$, the ground state is a single plane-wave (PW) state while for $g < g_{12}$ the ground state is the standing-wave state which is the superposition of two plane waves carrying two opposite momenta [37]. Here we focus on the PW state only that the ground-state wave function is $\Psi^0 = (\Psi_1^0, \Psi_2^0)^T = \sqrt{n} e^{-i\tilde{\kappa}_x x} (1, 1)^T$ where we assume the condensation at $\mathbf{p} = (-|\tilde{\kappa}_x|, 0)$. To investigate the low-lying excitations, we adopt the Bogoliubov formulation that the total wave function is decomposed as $\Psi_j = e^{-i\mu t} e^{-i\tilde{\kappa}_x x} (\Psi_j^0 + \delta\Psi_j)$ with μ the chemical potential and $\delta\Psi_j$ the low-lying excitation. We substitute the Bogoliubov decomposition into Eq. (A2) and retain the correction up to the first order. As a result, the chemical potential is determined by the zeroth-order equation

$$\mu = (\tilde{g} + \tilde{g}_{12})n - \frac{\tilde{\kappa}_x^2}{2}, \quad (A3)$$

and the first-order equation takes the form

$$\begin{aligned} i\partial_t\delta\Psi_1 &= \left(\frac{-\nabla^2}{2} + i\tilde{\kappa}_x\partial_x + 2\tilde{g}n + \tilde{g}_{12}n + \frac{\tilde{\kappa}_x^2}{2} - \mu\right)\delta\Psi_1 + \tilde{g}n\delta\Psi_1^* + \tilde{g}_{12}n\delta\Psi_2 + \tilde{g}_{12}n\delta\Psi_2^* + \left(\frac{\tilde{\kappa}_x}{i}\partial_x - \tilde{\kappa}_y\partial_y\right)\delta\Psi_2 - \tilde{\kappa}_x^2\delta\Psi_2, \\ i\partial_t\delta\Psi_2 &= \left(\frac{-\nabla^2}{2} + i\tilde{\kappa}_x\partial_x + 2\tilde{g}n + \tilde{g}_{12}n + \frac{\tilde{\kappa}_x^2}{2} - \mu\right)\delta\Psi_2 + \tilde{g}n\delta\Psi_2^* + \tilde{g}_{12}n\delta\Psi_1 + \tilde{g}_{12}n\delta\Psi_1^* + \left(\frac{\tilde{\kappa}_x}{i}\partial_x + \tilde{\kappa}_y\partial_y\right)\delta\Psi_1 - \tilde{\kappa}_x^2\delta\Psi_1. \end{aligned} \quad (\text{A4})$$

Expanding the deviation as $\delta\Psi_j = \sum_{\mathbf{q}}(u_j^{\mathbf{q}}e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} - v_j^{\mathbf{q}*}e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)})/\sqrt{A}$ with A the area of the system and ω the excitation energy of the mode with momentum \mathbf{q} and substituting into Eq. (A4) yields the Bogoliubov–de Gennes equation

$$\begin{pmatrix} \mathcal{L}_0 - \tilde{\kappa}_x q_x & -\tilde{g}n & \tilde{g}_{12}n + h_{\text{soc}} - \tilde{\kappa}_x^2 & -\tilde{g}_{12}n \\ \tilde{g}n & -\mathcal{L}_0 - \tilde{\kappa}_x q_x & \tilde{g}_{12}n & h_{\text{soc}}^* + \tilde{\kappa}_x^2 - \tilde{g}_{12}n \\ \tilde{g}_{12}n + h_{\text{soc}}^* - \tilde{\kappa}_x^2 & -\tilde{g}_{12}n & \mathcal{L}_0 - \tilde{\kappa}_x q_x & -\tilde{g}n \\ \tilde{g}_{12}n & h_{\text{soc}} + \tilde{\kappa}_x^2 - \tilde{g}_{12}n & \tilde{g}n & -\mathcal{L}_0 - \tilde{\kappa}_x q_x \end{pmatrix} \begin{pmatrix} u_1^{\mathbf{q}} \\ v_1^{\mathbf{q}} \\ u_2^{\mathbf{q}} \\ v_2^{\mathbf{q}} \end{pmatrix} = \omega \begin{pmatrix} u_1^{\mathbf{q}} \\ v_1^{\mathbf{q}} \\ u_2^{\mathbf{q}} \\ v_2^{\mathbf{q}} \end{pmatrix}, \quad (\text{A5})$$

where $\mathcal{L}_0 = q^2/2 + \tilde{g}n + \tilde{\kappa}_x^2$, $h_{\text{soc}} = \tilde{\kappa}_x q_x - i\tilde{\kappa}_y q_y$, and $u_j^{\mathbf{q}}, v_j^{\mathbf{q}}$ satisfy the normalization condition $\sum_j |u_j^{\mathbf{q}}|^2 - |v_j^{\mathbf{q}}|^2 = 1$. For the fully anisotropic SOC ($\tilde{\kappa}_y = 0$), Eq. (A5) can be diagonalized analytically which yields two distinct dispersion relations for the excitation modes:

$$\omega_{\text{t}}^{\mathbf{q}} = \sqrt{(\xi_{\text{t}}^{\mathbf{q}})^2 - (\tilde{g} + \tilde{g}_{12})^2 n^2}, \quad (\text{A6})$$

$$\omega_{\text{r}}^{\mathbf{q}} = -2q_x \tilde{\kappa}_x + \sqrt{(\xi_{\text{r}}^{\mathbf{q}})^2 - (\tilde{g} - \tilde{g}_{12})^2 n^2} \quad (\text{A7})$$

with the corresponding eigenvectors

$$\delta\Psi_{\text{t}}^{\mathbf{q}} = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{\xi_{\text{t}}^{\mathbf{q}}}{\omega_{\text{t}}^{\mathbf{q}}} + 1} \\ \sqrt{\frac{\xi_{\text{t}}^{\mathbf{q}}}{\omega_{\text{t}}^{\mathbf{q}}} - 1} \\ \sqrt{\frac{\xi_{\text{t}}^{\mathbf{q}}}{\omega_{\text{t}}^{\mathbf{q}}} + 1} \\ \sqrt{\frac{\xi_{\text{t}}^{\mathbf{q}}}{\omega_{\text{t}}^{\mathbf{q}}} - 1} \end{pmatrix}, \quad \delta\Psi_{\text{r}}^{\mathbf{q}} = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{\xi_{\text{r}}^{\mathbf{q}} + \omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}{\omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}} \\ \sqrt{\frac{\xi_{\text{r}}^{\mathbf{q}} - \omega_{\text{r}}^{\mathbf{q}} - 2q_x \tilde{\kappa}_x}{\omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}} \\ -\sqrt{\frac{\xi_{\text{r}}^{\mathbf{q}} + \omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}{\omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}} \\ -\sqrt{\frac{\xi_{\text{r}}^{\mathbf{q}} - \omega_{\text{r}}^{\mathbf{q}} - 2q_x \tilde{\kappa}_x}{\omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}} \end{pmatrix}, \quad (\text{A8})$$

where $\xi_{\text{t}}^{\mathbf{q}} = q^2/2 + (\tilde{g} + \tilde{g}_{12})n$ and $\xi_{\text{r}}^{\mathbf{q}} = q^2/2 + (\tilde{g} - \tilde{g}_{12})n + 2\tilde{\kappa}_x^2$. The bosonic field can be expressed in the form [38]

$$\hat{\Psi}(\mathbf{r}) = \begin{pmatrix} \hat{\Psi}_1(\mathbf{r}) \\ \hat{\Psi}_2(\mathbf{r}) \end{pmatrix} = e^{i\hat{\phi}_{\text{t}}(\mathbf{r})} \begin{pmatrix} \sqrt{n + \delta n_1(\mathbf{r})} e^{i\hat{\phi}_{\text{t}}(\mathbf{r})} \\ \sqrt{n + \delta n_2(\mathbf{r})} e^{-i\hat{\phi}_{\text{t}}(\mathbf{r})} \end{pmatrix}, \quad (\text{A9})$$

where $\hat{\phi}_{\text{t}}(\mathbf{r})$ and $\hat{\phi}_{\text{r}}(\mathbf{r})$ are respectively the total and relative phase fluctuations. Therefore for small fluctuations we have [expanding Eq. (A9) to first order]

$$\begin{aligned} \hat{\phi}_{\text{t}} &\approx \frac{1}{4in^{1/2}} [(\delta\Psi_1 - \delta\Psi_2^\dagger) - \text{H.c.}], \\ \hat{\phi}_{\text{r}} &\approx \frac{1}{4in^{1/2}} [(\delta\Psi_1 - \delta\Psi_2) - \text{H.c.}], \end{aligned} \quad (\text{A10})$$

where $\hat{\phi}_{\text{t}}(\mathbf{r})$ and $\hat{\phi}_{\text{r}}(\mathbf{r})$ are Hermitian operators. In the linear approximation, the total and relative phase operators are decoupled and can be respectively expressed in terms of the excitations $\delta\Psi_{\text{t}}^{\mathbf{q}}$ and $\delta\Psi_{\text{r}}^{\mathbf{q}}$, by writing $\delta\hat{\Psi}_{\text{t}}^{\mathbf{q}} = (\delta\hat{\Psi}_{1,\text{t}}^{\mathbf{q}}, \delta\hat{\Psi}_{2,\text{t}}^{\mathbf{q}})^\text{T}$

and $\delta\hat{\Psi}_{\text{r}}^{\mathbf{q}} = (\delta\hat{\Psi}_{1,\text{r}}^{\mathbf{q}}, \delta\hat{\Psi}_{2,\text{r}}^{\mathbf{q}})^\text{T}$. Express the fluctuations as

$$\begin{aligned} \delta\hat{\Psi}_{1,\text{t}}(\mathbf{r}) &= \delta\hat{\Psi}_{2,\text{t}}(\mathbf{r}) = \frac{1}{\sqrt{A}} \sum_{\mathbf{q}} [u_{\text{t}}^{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \hat{\alpha}_{\text{t}}^{\mathbf{q}} - v_{\text{t}}^{\mathbf{q}*} e^{-i\mathbf{q}\cdot\mathbf{r}} \hat{\alpha}_{\text{t}}^{\mathbf{q}\dagger}] \\ &= \sum_{\mathbf{q}} [\mathcal{U}_{\text{t}}^{\mathbf{q}}(\mathbf{r}) \hat{\alpha}_{\text{t}}^{\mathbf{q}} - \mathcal{V}_{\text{t}}^{\mathbf{q}*}(\mathbf{r}) \hat{\alpha}_{\text{t}}^{\mathbf{q}\dagger}], \\ \delta\hat{\Psi}_{1,\text{r}}(\mathbf{r}) &= -\delta\hat{\Psi}_{2,\text{r}}(\mathbf{r}) = \frac{1}{\sqrt{A}} \sum_{\mathbf{q}} [u_{\text{r}}^{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \hat{\alpha}_{\text{r}}^{\mathbf{q}} - v_{\text{r}}^{\mathbf{q}*} e^{-i\mathbf{q}\cdot\mathbf{r}} \hat{\alpha}_{\text{r}}^{\mathbf{q}\dagger}] \\ &= \sum_{\mathbf{q}} [\mathcal{U}_{\text{r}}^{\mathbf{q}}(\mathbf{r}) \hat{\alpha}_{\text{r}}^{\mathbf{q}} - \mathcal{V}_{\text{r}}^{\mathbf{q}*}(\mathbf{r}) \hat{\alpha}_{\text{r}}^{\mathbf{q}\dagger}], \end{aligned} \quad (\text{A11})$$

where $(u_{\text{t}}^{\mathbf{q}}, v_{\text{t}}^{\mathbf{q}}) = (\sqrt{\frac{\xi_{\text{t}}^{\mathbf{q}}}{\omega_{\text{t}}^{\mathbf{q}}} + 1}, \sqrt{\frac{\xi_{\text{t}}^{\mathbf{q}}}{\omega_{\text{t}}^{\mathbf{q}}} - 1})$, $(u_{\text{r}}^{\mathbf{q}}, v_{\text{r}}^{\mathbf{q}}) = (\sqrt{\frac{\xi_{\text{r}}^{\mathbf{q}} + \omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}{\omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}}, \sqrt{\frac{\xi_{\text{r}}^{\mathbf{q}} - \omega_{\text{r}}^{\mathbf{q}} - 2q_x \tilde{\kappa}_x}{\omega_{\text{r}}^{\mathbf{q}} + 2q_x \tilde{\kappa}_x}})$, $\hat{\alpha}_{\text{t},\text{r}}^{\mathbf{q}}$ ($\hat{\alpha}_{\text{t},\text{r}}^{\mathbf{q}\dagger}$) is the annihilation (creation) operator that destroys (creates) the excitation in the corresponding branches $\omega_{\text{t},\text{r}}^{\mathbf{q}}$, and $(\mathcal{U}_{\text{t},\text{r}}^{\mathbf{q}}, \mathcal{V}_{\text{t},\text{r}}^{\mathbf{q}}) = (u_{\text{t},\text{r}}^{\mathbf{q}}, v_{\text{t},\text{r}}^{\mathbf{q}}) e^{i\mathbf{q}\cdot\mathbf{r}}/\sqrt{A}$.

The two-point phase correlation functions are given by

$$G_{\text{t},\text{r}}(\mathbf{r}', \mathbf{r}'') = \langle e^{i\hat{\phi}_{\text{t},\text{r}}(\mathbf{r}') - i\hat{\phi}_{\text{t},\text{r}}(\mathbf{r}'')} \rangle = e^{-((\Delta\phi_{\text{t},\text{r}})^2)/2}, \quad (\text{A12})$$

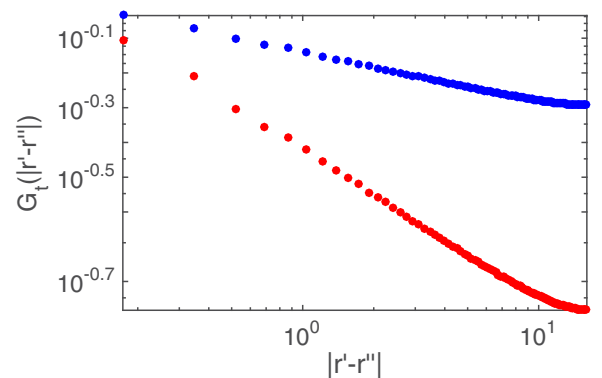


FIG. 4. Total-phase correlation functions based on Eq. (A12). Blue (top) and red (bottom) dots indicate the total-phase correlation functions evaluated at temperatures $T/T_{\text{BKT}}^\infty \approx 0.44$ and 1.33 respectively. The linearity of the curves implies a power-law behavior.

where $\langle \dots \rangle$ denotes the ensemble average and $\Delta\phi_{t,r} = \hat{\phi}_{t,r}(\mathbf{r}') - \hat{\phi}_{t,r}(\mathbf{r}'')$. The thermal average can be expressed in terms of the Bogoliubov amplitudes

$$\langle (\Delta\phi_{t,r})^2 \rangle = \int \frac{d^2q}{\pi n} \left(N_{t,r}^q + \frac{1}{2} \right) (u_{t,r}^q + v_{t,r}^q)^2 \sin^2 \frac{\mathbf{q} \cdot \mathbf{r}}{2}, \quad (\text{A13})$$

where $N_{t,r}^q = 1/(e^{\omega_{t,r}^q/T} - 1)$ is the Bose-Einstein distribution function, T is the temperature measured in units of $\hbar\omega_0/k_B$, $\mathbf{r} = \mathbf{r}' - \mathbf{r}''$, and $(u_{t,r}^q + v_{t,r}^q)^2 = [\xi_t^q + (\tilde{g} + \tilde{g}_{12}n)/2\omega_t^q]$ and $(u_{t,r}^q - v_{t,r}^q)^2 = [\xi_r^q + (\tilde{g} - \tilde{g}_{12}n)/2(\omega_r^q + 2\tilde{\kappa}_x q_x)]$. Evidently, the total-phase correlation functions so obtained are nothing but precisely the case of a 2D scalar Bose gas, which entails a power-law decay irrespective of

temperature [38], as shown in Fig. 4 where the functions are plotted on a doubly logarithmic scale.

Density matrix. For quasicondensates, the density fluctuation is negligible that the density matrix can be expressed as

$$\begin{aligned} \rho(\mathbf{r}', \mathbf{r}'') &= n \begin{bmatrix} \langle e^{i[\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}'')] } \rangle & \langle e^{i[\phi_1(\mathbf{r}') - \phi_2(\mathbf{r}'')] } \rangle \\ \langle e^{i[\phi_2(\mathbf{r}') - \phi_1(\mathbf{r}'')] } \rangle & \langle e^{i[\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}'')] } \rangle \end{bmatrix} \\ &= n \begin{bmatrix} e^{-\langle \Delta\phi_1^2 \rangle / 2} & e^{-\langle \Delta\phi^2 \rangle / 2} \\ e^{-\langle \Delta\phi^2 \rangle / 2} & e^{-\langle \Delta\phi_2^2 \rangle / 2} \end{bmatrix}, \end{aligned} \quad (\text{A14})$$

where $\hat{\phi}_j = \hat{\phi}_t + (-1)^{j-1} \hat{\phi}_r$, $\Delta\phi_j = \phi_j(\mathbf{r}') - \phi_j(\mathbf{r}'')$, and $\Delta\phi = \phi_1(\mathbf{r}') - \phi_2(\mathbf{r}'')$. Accordingly, the phase fluctuations are explicitly expressed as

$$\begin{aligned} (\Delta\phi_1)^2 &= (\Delta\phi_t)^2 + (\Delta\phi_r)^2 + 2[\hat{\phi}_t(\mathbf{r}')\hat{\phi}_r(\mathbf{r}') + \hat{\phi}_t(\mathbf{r}'')\hat{\phi}_r(\mathbf{r}'') - \hat{\phi}_t(\mathbf{r}')\hat{\phi}_r(\mathbf{r}'') - \hat{\phi}_t(\mathbf{r}'')\hat{\phi}_r(\mathbf{r}')], \\ (\Delta\phi_2)^2 &= (\Delta\phi_t)^2 + (\Delta\phi_r)^2 - 2[\hat{\phi}_t(\mathbf{r}')\hat{\phi}_r(\mathbf{r}') + \hat{\phi}_t(\mathbf{r}'')\hat{\phi}_r(\mathbf{r}'') - \hat{\phi}_t(\mathbf{r}')\hat{\phi}_r(\mathbf{r}'') - \hat{\phi}_t(\mathbf{r}'')\hat{\phi}_r(\mathbf{r}')], \\ (\Delta\phi)^2 &= (\Delta\phi_t)^2 + (\hat{\phi}_r(\mathbf{r}') + \hat{\phi}_r(\mathbf{r}''))^2 + 2[\hat{\phi}_t(\mathbf{r}')\hat{\phi}_r(\mathbf{r}') - \hat{\phi}_t(\mathbf{r}'')\hat{\phi}_r(\mathbf{r}'') - \hat{\phi}_t(\mathbf{r}')\hat{\phi}_r(\mathbf{r}'') + \hat{\phi}_t(\mathbf{r}'')\hat{\phi}_r(\mathbf{r}')]. \end{aligned} \quad (\text{A15})$$

Equation (A14) can be simplified as

$$\rho(\mathbf{r}', \mathbf{r}'') = n \begin{bmatrix} e^{-\langle \Delta\phi_t^2 \rangle / 2} e^{-\langle \Delta\phi_r^2 \rangle / 2} & e^{-\langle \Delta\phi_t^2 \rangle / 2} e^{-\langle \Delta\phi_r^2 \rangle / 2} \\ e^{-\langle \Delta\phi_t^2 \rangle / 2} e^{-\langle \Delta\phi_r^2 \rangle / 2} & e^{-\langle \Delta\phi_t^2 \rangle / 2} e^{-\langle \Delta\phi_r^2 \rangle / 2} \end{bmatrix}, \quad (\text{A16})$$

where $\Delta_+\phi_r = \hat{\phi}_r(\mathbf{r}') + \hat{\phi}_r(\mathbf{r}'')$ and the full density matrix consists of block matrices. The ensemble average of the cross terms in Eq. (A15) vanish since $\langle \hat{\alpha}_t^q \hat{\alpha}_r^q \rangle = \langle \hat{\alpha}_t^q \hat{\alpha}_r^q \rangle = \langle \hat{\alpha}_t^q \hat{\alpha}_r^q \rangle = \langle \hat{\alpha}_t^q \hat{\alpha}_r^q \rangle = 0$. Following the same procedure of deriving Eq. (A13), the expectation value of $(\Delta_+\phi_r)^2$ is

$$\langle (\Delta_+\phi_r)^2 \rangle = \int \frac{d^2q}{\pi n} \left(N_r^q + \frac{1}{2} \right) (u_r^q + v_r^q)^2 \cos^2 \frac{\mathbf{q} \cdot \mathbf{r}}{2}, \quad (\text{A17})$$

where $\langle (\Delta_+\phi_r)^2 \rangle$ is maximized at $\mathbf{r} = 0$. Therefore in the thermodynamics limit, the off-diagonal elements of the block matrices vanishes when $|\mathbf{r}' - \mathbf{r}''| \rightarrow \infty$ since all the block matrices contain the same prefactor $e^{-\langle \Delta\phi_t^2 \rangle / 2}$ which would destroy the off-diagonal LRO. The absence of off-diagonal LRO of the full density matrix Eq. (A14) indicates that the 2D SO-coupled Bose gas could not undergo the Bose-Einstein condensation even though the relative-phase sector could possess a true long-range order.

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- [1] N. D. Mermin and H. Wagner, *Phys. Rev. Lett.* **17**, 1133 (1966).
[2] P. C. Hohenberg, *Phys. Rev.* **158**, 383 (1967).
[3] V. L. Berezinskii, *Sov. Phys. JETP* **32**, 493 (1971).
[4] J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* **6**, 1181 (1973).
[5] Z. Hadzibabic, P. Krüger, M. Cheneau, B. Battelier, and J. Dalibard, *Nature (London)* **441**, 1118 (2006).
[6] C. J. Foster, P. B. Blakie, and M. J. Davis, *Phys. Rev. A* **81**, 023623 (2010).
[7] A. L. Gaunt, T. F. Schmidutz, I. Gotlibovych, R. P. Smith, and Z. Hadzibabic, *Phys. Rev. Lett.* **110**, 200406 (2013).
[8] L. Chomaz, L. Corman, T. Bienaimé, R. Desbuquois, C. Weitenberg, S. Nascimbène, J. Beugnon, and J. Dalibard, *Nat. Commun.* **6**, 6162 (2015).
[9] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, *Nature (London)* **471**, 83 (2011).
[10] D. L. Campbell, G. Juzeliūnas, and I. B. Spielman, *Phys. Rev. A* **84**, 025602 (2011).
[11] Z. F. Xu, R. Lü, and L. You, *Phys. Rev. A* **83**, 053602 (2011).
[12] S. Sinha, R. Nath, and L. Santos, *Phys. Rev. Lett.* **107**, 270401 (2011).
[13] S.-W. Su, I.-K. Liu, Y.-C. Tsai, W. M. Liu, and S.-C. Gou, *Phys. Rev. A* **86**, 023601 (2012).
[14] H. Hu and X.-J. Liu, *Phys. Rev. A* **85**, 013619 (2012).
[15] Q. Zhou and X. Cui, *Phys. Rev. Lett.* **110**, 140407 (2013).
[16] X. Cui and Q. Zhou, *Phys. Rev. A* **87**, 031604 (2013).
[17] C.-M. Jian and H. Zhai, *Phys. Rev. B* **84**, 060508 (2011).
[18] Y. Li, L. P. Pitaevskii, and S. Stringari, *Phys. Rev. Lett.* **108**, 225301 (2012).
[19] Y. Li, G. I. Martone, L. P. Pitaevskii, and S. Stringari, *Phys. Rev. Lett.* **110**, 235302 (2013).
[20] R. Liao, Z.-G. Huang, X.-M. Lin, and O. Fialko, *Phys. Rev. A* **89**, 063614 (2014).
[21] Q. Sun, L. Wen, W.-M. Liu, G. Juzeliūnas, and A.-C. Ji, *Phys. Rev. A* **91**, 033619 (2015).
[22] S.-W. Su, S.-C. Gou, I.-K. Liu, I. B. Spielman, L. Santos, A. Acus, A. Mekys, J. Ruseckas, and G. Juzeliūnas, *New J. Phys.* **17**, 033045 (2015).
[23] S.-W. Su, S.-C. Gou, Q. Sun, L. Wen, W.-M. Liu, A.-C. Ji, J. Ruseckas, and G. Juzeliūnas, *Phys. Rev. A* **93**, 053630 (2016).

- [24] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, *Rev. Mod. Phys.* **83**, 1523 (2011).
- [25] L. Huang, Z. Meng, P. Wang, P. Peng, S.-L. Zhang, L. Chen, D. Li, Q. Zhou, and J. Zhang, *Nat. Phys.* **12**, 540 (2016).
- [26] Z. Wu, L. Zhang, W. Sun, X.-T. Xu, B.-Z. Wang, S.-C. Ji, Y. Deng, S. Chen, X.-J. Liu, and J.-W. Pan, *Science* **354**, 83 (2016).
- [27] K. Jiménez-García, L. J. LeBlanc, R. A. Williams, M. C. Beeler, C. Qu, M. Gong, C. Zhang, and I. B. Spielman, *Phys. Rev. Lett.* **114**, 125301 (2015).
- [28] B. M. Anderson, G. Juzeliūnas, V. M. Galitski, and I. B. Spielman, *Phys. Rev. Lett.* **108**, 235301 (2012).
- [29] B. M. Anderson, I. B. Spielman, and G. Juzeliūnas, *Phys. Rev. Lett.* **111**, 125301 (2013).
- [30] K. Zhou and Z. Zhang, *Phys. Rev. Lett.* **108**, 025301 (2012).
- [31] L. He and X.-G. Huang, *Phys. Rev. Lett.* **108**, 145302 (2012).
- [32] Y. Xu and C. Zhang, *Phys. Rev. Lett.* **114**, 110401 (2015).
- [33] P. Blakie, A. Bradley, M. Davis, R. Ballagh, and C. Gardiner, *Adv. Phys.* **57**, 363 (2008).
- [34] S. J. Rooney, P. B. Blakie, and A. S. Bradley, *Phys. Rev. A* **86**, 053634 (2012).
- [35] S.-W. Su, S.-C. Gou, A. Bradley, O. Fialko, and J. Brand, *Phys. Rev. Lett.* **110**, 215302 (2013).
- [36] A. S. Bradley and P. B. Blakie, *Phys. Rev. A* **90**, 023631 (2014).
- [37] C. Wang, C. Gao, C. M. Jian, and H. Zhai, *Phys. Rev. Lett.* **105**, 160403 (2010).
- [38] H. Pethick, C. J. Smith, *Bose-Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, England, 2002).
- [39] N. Prokof'ev, O. Ruebenacker, and B. Svistunov, *Phys. Rev. Lett.* **87**, 270402 (2001).
- [40] N. Prokof'ev and B. Svistunov, *Phys. Rev. A* **66**, 043608 (2002).
- [41] A. J. Leggett, *Quantum Liquids* (Oxford University Press, Oxford, 2006).
- [42] A. J. A. James and A. Lamacraft, *Phys. Rev. Lett.* **106**, 140402 (2011).
- [43] G. I. Martone, Y. Li, L. P. Pitaevskii, and S. Stringari, *Phys. Rev. A* **86**, 063621 (2012).
- [44] M. Egorov, R. P. Anderson, V. Ivannikov, B. Opanchuk, P. Drummond, B. V. Hall, and A. I. Sidorov, *Phys. Rev. A* **84**, 021605(R) (2011).
- [45] B. Opanchuk, R. Polkinghorne, O. Fialko, J. Brand, and P. D. Drummond, *Ann. Phys. (Berlin, Ger.)* **525**, 866 (2013).