

Quantum-mechanical inclusion of the source in the Aharonov-Bohm effectsPhilip Pearle^{1,*} and Anthony Rizzi^{2,†}¹*Department of Physics, Hamilton College, Clinton, New York 13323, USA*²*Institute for Advanced Physics, P.O. Box 15030, Baton Rouge, Louisiana 70895, USA*

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Following semiclassical arguments by Vaidman [*Phys. Rev. A* **86**, 040101(R) (2012)], we show that the phase shifts arising in the Aharonov-Bohm (AB) magnetic or electric effects can be treated as due to the electric force of a classical electron, respectively acting on quantized solenoid particles or quantized capacitor plates. This is in contrast to the usual approach that treats both effects as arising from non-field-producing potentials acting on the quantized electron. Moreover, we consider the problems of an interacting quantized electron and a quantized solenoid or a quantized capacitor to see what phase shift their joint wave function acquires. We show, in both cases, that the net phase shift is indeed the AB shift (for one might have expected twice the AB shift, given the above two mechanisms for each effect). The solution to the exact Schrödinger equation may be treated (approximately for the magnetic AB effect, which we show using a variational approach, exactly for the electric AB effect) as the product of two solutions of separate Schrödinger equations for each of the two quantized entities, but with an extra phase. The extra phase provides the negative of the AB shift, while the two separate Schrödinger equations without the extra phase each provide the AB phase shift so that the product wave function produces the net AB phase shift.

DOI: [10.1103/PhysRevA.95.052123](https://doi.org/10.1103/PhysRevA.95.052123)**I. INTRODUCTION**

The Aharonov-Bohm (AB) magnetic effect [1–3] predicts relative phase shifts of two electron wave packets moving in alternative paths around the outside of a long cylindrical solenoid. Classically, it appears impossible that there should be a measurable effect in such circumstances, where the electron moves in a region of non-field-producing potentials, i.e., in a region where there is a vector and/or scalar potential, but no electric or magnetic fields [4]. That such potentials can produce physical effects is considered by many to be a prime example of the marvelous novelty quantum theory reveals vis-à-vis classical theory.

Recently, Vaidman [5] argued, using a semiclassical analysis of a model of the solenoid, that, under the influence of the electric-field-producing vector potential of the electron treated classically, the solenoid produces a relative phase shift (depending upon the direction of traverse of the electron wave packet) exactly equal to the standard magnetic AB phase shift.

The importance of the AB effect suggests that Vaidman's result should be carefully considered in the context of a fully quantum mechanical treatment of the solenoid particles. We do this in this paper and obtain Vaidman's result.

However, given the radical interconnectedness implied by quantum mechanics, it is natural to consider the problem where *both* the electron and solenoid particles are quantized [6]. Does one then get twice the AB phase shift, the sum of the usual and Vaidman results? The answer is no: We consider that problem here and find that, indeed, one gets the usual AB phase shift. This occurs through a consistent treatment of the interaction between the particles of the solenoid and the electron.

We model the solenoid as consisting of N particles. They and the electron are considered to be described by well-defined wave packets and the joint wave function is approximated as

the product of such wave functions. Putting this approximate wave function into the variational principle for the Schrödinger equation results in $N + 1$ Schrödinger equations, where each particle evolves under the vector potential of the other particle's mean positions and momenta, which are the positions and momenta of the same classical problem.

However, there is an extra phase term in each Schrödinger equation. By a phase transformation that does not change the overall phase of the wave function, the extra phase term may be removed from all Schrödinger equations but one. We choose to put that extra phase term into the electron's Schrödinger equation. That extra phase turns out to be the time integral of the interaction term in the Hamiltonian, when operators are replaced by their classical counterparts. We then suppose that the $N + 1$ particles are involved in an interference experiment and show that the extra phase is the negative of the AB phase shift. The Schrödinger equation for the electron (evolving in the classical field of the solenoid), without the extra phase term, gives the usual AB phase shift. Therefore, the Schrödinger equation for the electron with the extra phase term gives zero phase shift. The Schrödinger equation for the N solenoid particles gives the AB phase shift. Thus, the net result for the product wave function is the AB phase shift.

Although we do not bother to consider it here, the extra phase might just as well have been eliminated from the electron's Schrödinger equation and distributed in any way among the N solenoid particles. The result then is that the joint wave function of the solenoid particles, including the extra phases, produces zero phase shift in the interference experiment. Then the AB phase shift is attributable solely to the electron's wave function, as in the usual description.

In the following paper [7], we point out that consideration of *all* relevant quantum objects in this problem requires quantizing the vector potential as well. A similar analysis of the joint wave function of electron, solenoid, and vector potential, approximated as a product wave function and governed by the separate Schrödinger equations that arise from the variational

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principle, gives the usual AB phase shift. As in this paper, depending upon which Schrödinger equation(s) get the extra phase, the AB shift may be attributed solely to the wave function of the electron, solely to the solenoid, solely to the vector potential, or to a combination thereof.

Our analysis also shows why the usual phase shift due to the electron and the phase shift due to the solenoid have to be the same. To complete the story we spell out the details of the model of the solenoid, show that the amplitude of the interference term is essentially 1 for reasonable values of the model parameters, and do the direct calculation of the AB phase shift arising from the solenoid.

Next we turn to show that similar considerations apply in the case of the electric AB effect. Vaidman proposed a simplified physical setup wherein the electric potential of two classical charges produces a zero-field environment for the electron and thereby an electric AB phase shift. Then he argued for an alternative treatment, where the electron is treated classically and the charges are quantized, and showed that the electric field of the electron produces the same electric AB phase shift for the two charges.

We will consider here, not Vaidman's simplified model, but the standard AB electric phase-shift situation of an electron passing over or under a charged capacitor. The usual calculation attributes the phase shift to the electron packets moving in regions of different constant scalar potential created by a classical capacitor. We will show, as Vaidman did in his quasiclassical analysis for his model, that this same AB phase shift is obtained when the electron is treated classically and the capacitor plates (more precisely, the plate's center of mass coordinates) are treated quantum mechanically, thereby acquiring the AB phase shift in the electric field of the electron. Again, these are two alternative, mathematically but not conceptually equivalent, ways to calculate the same thing. So we consider the situation where *both* the electron and the capacitor plates have quantum dynamics. Similarly to the magnetic AB effect, we show how the joint phase shift may be attributed solely to the electron's motion or may be attributed solely to the capacitor plate motion simply by shifting a term in the Hamiltonian to either the electron part of the Hamiltonian or the capacitor plate part of the Hamiltonian or that other splits may be made.

Thus, for both these classic examples of the AB effects, there is support for an alternative (espoused by Vaidman) to the usual views of these effects as due to the electron moving in a non-field-producing potential. It is that the effects may, with equal justification be viewed as due to motion of charged objects due to forces exerted by the electron. However, considerations other than ours may lead one to prefer one point of view over the other.

The plan of this paper is as follows. In Sec. II we calculate the amplitude and phase of a wave packet describing a charged particle with well-localized position and momentum on a one-dimensional predetermined path (e.g., moving in a tube with high potential walls), under arbitrary time-varying (but not space-varying) externally applied electric-field-producing vector and scalar potentials, for a time interval T . We show that the phase can be completely expressed in terms of the classical motion of the particle (which is the mean motion of the wave packet).

In Sec. III we generalize the result of the previous section to calculate the amplitude and phase associated with a wave packet in the more general case where the external electromagnetic forces experienced by the particle depend not only upon arbitrarily upon time, but also arbitrarily upon the location of the particle as well. The approximate solution we develop is for the case in which the particle's mean position follows the classical path.

In Sec. IV we first consider two quantized particles interacting via their mutual vector and scalar potentials. As described above, we approximate the wave function as the product of localized wave packets for each particle. Using the variational principle for the exact Schrödinger equation, we derive Schrödinger equations for each particle. We show that the Schrödinger equations describe each particle as moving under the potentials of the other particle treated classically. This allows us to use the results of Sec. III to calculate the amplitude and phase of each particle. However, in addition, each Schrödinger equation contains an extra phase term. By adding a phase to one particle and subtracting it from the other, so the overall phase of the wave function is not affected, the extra phase is removed from one particle and belongs solely the other.

It is shown that the extra phase is the time integral of the interaction term in the Hamiltonian, where all operators have been replaced by the corresponding classical variables; call this $-\Phi$. It is shown that the Schrödinger equation for the particle without the extra phase and the other particle's Schrödinger equation with the extra phase also removed each produces the same phase Φ . So the total phase is Φ . Further, in an interference experiment involving the two particles moving on two different paths, it is the value of Φ on one path minus the value of Φ on the other path that provides the total phase shift.

We then extend these results to the case of N particles, each interacting with one special particle we call the electron. Again, the exact wave function is approximated as a product of all $N + 1$ particle wave functions. Schrödinger equations are obtained for each particle and each equation has an extra phase term that may be eliminated from the N particles, leaving the electron wave function with all the extra phase.

The resulting phase situation is as follows. The electron's extra phase term is $-\Phi$, so the electron Schrödinger equation produces zero phase. Then the net phase is that of the N particles, which is Φ . Again, in a considered interference experiment where there are two different paths for all particles, the total net phase shift is the difference of this phase Φ for the two paths.

Additionally, for this interference experiment, we obtain an expression for the amplitude of the interference term, expressed as a function of the N -particle classical positions and momenta.

In Sec. V we present a fully-quantum-mechanical approach to Vaidman's semiclassical considerations. In particular, we apply the argument given above to a detailed model of the solenoid as a collection of N well-localized circulating quantized particles, interacting with the electron. The results of the preceding section immediately apply. The total net phase shift, the difference of the net phase of the joint electron-solenoid wave function for two different paths taken by the electron, is the AB phase shift.

Also, it is explained why the Vaidman scenario calculation *must* produce the same phase shift as the usual calculation. For further insight, we give an intuitive time-averaged derivation of our result.

Furthermore, we consider the magnitude of the interference term in this AB experiment, which experimentally [8] is 1. In Vaidman's semiclassical calculation of the magnetic AB effect, it is necessary for the electron at the end of its traverse to decelerate back to zero speed and thus cancel the initial impulse that started the solenoid cylinders moving. This is in order that each cylinder's final motion be unchanged from its initial (preacceleration) motion and its entanglement with the electron packets be removed, else there would be no interference [5]. In our case, there is no need for such a caveat. We show in our model, despite the two different displacements of each solenoid particle associated with the two different electron paths, that the magnitude of the interference term is close to 1 because the displacements are very small (since each contributes only a very tiny amount to the phase shift). This completes our presentation of the AB magnetic effect, showing how the AB phase shift with interference amplitude 1 is obtained when considering both the quantized electron and quantized solenoid.

Section VI applies the result of Sec. III to the electric AB effect. We show that the electric case is exactly parallel to the magnetic AB case. We consider the exact problem wherein we have a quantized electron and quantized capacitor plates (modeled as sheets of glued-together charges, so their centers of mass are the plate dynamical variables). We explain how the phase-shift expression can be written either as if it were due to the electron moving in the non-field-producing potential external to the capacitor or, alternatively, as if it were due to the capacitor plates moving in the electric field of the electron, or in some other equivalent way.

Section VII contains concluding remarks.

We set $\hbar = 1$. In order to avoid superfluous 4π factors, we employ cgs units in all but Sec. VI and Appendix D, where we employ rationalized mks units.

II. PARTICLE MOVING UNDER TIME-DEPENDENT NON-SPATIALLY-VARYING FORCES

In this and the next section, we consider the general problem of a particle well localized in position and momentum and moving in one dimension under scalar and vector potentials and calculate the magnitude and phase of its wave function. The problem of a particle moving in one dimension is to be considered as a limiting case of the three-dimensional problem of a particle moving in a three-dimensional pipe. This is relevant here, because we will model the solenoid as a collection of particles moving in stacked tubes, i.e., the pipe is a tube of toroidal shape. A bonus is that we can treat the orbiting electron the same way, taking its motion to be in a circular tube of its own, concentric with and outside the solenoid.

In cylindrical coordinates, the kinetic energy in the Hamiltonian is separable and so a tube may be modeled as providing, say, a square-well potential in the radial and z directions and supposing that the particle under consideration is in the ground state in these directions. Thus, it has free motion in the azimuthal direction. If the sides of the tube are allowed to

be as small as desired, the separation in energy between the ground and lowest excited states gets as large as desired, so any external field has negligible probability of exciting the radial or z motion and just consists in a distortion of the ground state. Thus, the only interesting motion is the free one-dimensional particle motion.

Of course, this one-dimensional motion approximation to describe a linearly constrained three-dimensional motion has greater applicability than to just circular motion in cylindrical coordinates. Immediately, one notes that this argument is just as readily applied to motion along a constant parameter curve in any of the 13 coordinate systems in which the Laplacian is separable. It should also be applicable to motion along a larger class of curves, along which one can choose local coordinates so that the Laplacian is effectively separable, at least into the one-dimensional linear motion and two-dimensional orthogonal motion; we will not pursue here an analysis of the class of curves for which the one-dimensional motion approximation to constrained three-dimensional motion has validity.

Obviously, this is just one possible configuration for the magnetic AB effect. In the only actually rigorously performed experiment [8], the solenoid was replaced by a small magnetized torus and, after one packet goes through the torus hole and the other passes outside, the electron wave functions were allowed to spread and overlap. While the results obtained here really only apply to the configuration of our model, they suggest that there should be counterparts when other configurations are considered.

In this section we consider that the motion of the particle in one dimension is under scalar and vector potentials associated with forces that are solely time dependent (i.e., not space independent). This is an exactly solvable problem and we find the expression for the amplitude and phase acquired by the particle's wave function in this case. We do this before tackling in the next section the more general but not exactly solvable problem where the force depends upon the particle's position as well as upon time.

The result from this section, limited to solely time-dependent (non-spatially-varying) forces, is however applicable in Sec. VI, to the case of the quantized capacitor plates moving under the electron's classical Coulomb field. The plates are assumed to be so massive that they do not displace significantly during the electron's traverse, which simplifies the calculation.

A. Classical motion

First, consider the classical motion of a particle, of charge q and mass m , under solely time-dependent electric forces due to a vector and scalar potential. If the vector potential is $A(t)$, then its associated electric force $-q \frac{d}{cdt} A(t)$ has just time dependence. Expand the potential energy $qV(x,t)$ for a particle that moves on the classical trajectory $x = x_{cl}(t)$ about the particle's position: $qV(x,t) = qV(x_{cl}(t),t) + [x - x_{cl}(t)]qV'(x_{cl}(t),t) + \dots$, where x is the distance along the path. Since the force experienced by the particle is to have no spatial dependence, we assume no higher power of $x - x_{cl}(t)$ than the first. Moreover, since all the problems of concern to us are of first order in q and since $x_{cl}(t) = x_0 + v_0 t + o(q)$ (x_0 is the particle's initial position, v_0 its initial velocity), we

replace $x_{cl}(t)$ by $x_0 + v_0 t$. Thus, we write the scalar potential energy we will consider as $qV(t) + (x - x_0 - v_0 t)qV'(t) \equiv qg(t) + qxV'(t)$ [where $g(t) \equiv V(t) - (x_0 + v_0 t)V'(t)$], i.e., we define $V(t) \equiv V(x_{cl}(t), t)$ and $V'(t) \equiv V'(x_{cl}(t), t)$. The classical Hamiltonian we consider is therefore

$$H = \frac{1}{2m} \left[p - \frac{q}{c} A(t) \right]^2 + qxV'(t) + qg(t), \quad (1)$$

where x and p are the canonical coordinates. The equations of motion follow from Hamilton's Poisson bracket equations

$$\begin{aligned} \frac{d}{dt} x &= \frac{1}{m} \left[p - \frac{q}{c} A(t) \right], \\ \frac{d}{dt} \left[p - \frac{q}{c} A(t) \right] &= -qV'(t) - \frac{q}{c} \frac{d}{dt} A(t) \end{aligned} \quad (2)$$

and solution

$$v_{cl}(t) = v_0 - \frac{q}{mc} [A(t) - A(0)] - \frac{q}{m} U(t), \quad (3a)$$

$$x_{cl}(t) = x_0 + v_0 t - \frac{q}{mc} [A - A(0)t] - \frac{q}{m} [tU(t) - W(t)], \quad (3b)$$

$$p_{cl}(t) = mv_{cl}(t) + \frac{q}{c} A(t) = p_0 - qU(t), \quad (3c)$$

where $\mathcal{A} \equiv \int_0^t dt' A(t')$,

$$\int_0^t dt' \int_0^{t'} dt'' V'(t'') = \int_0^t dt' [t - t'] V'(t') = tU(t) - W(t), \quad (3d)$$

$$p_0 \equiv mv_0 + \frac{q}{c} A(0),$$

$$U(t) \equiv \int_0^t dt' V'(t'), \quad (3e)$$

$$W(t) \equiv \int_0^t dt' t' V'(t').$$

(The subscript cl is to distinguish these classical variables from the quantum variables to appear later.) Note that the electric force only appears in these equations: A magnetic force does not appear for motion constrained to one dimension since that force is in the direction perpendicular to the velocity.

B. Wave function

The wave function in the momentum representation satisfies

$$\begin{aligned} i \frac{\partial}{\partial t} \psi(p, t) \\ = \left(\frac{1}{2m} \left[p - \frac{q}{c} A(t) \right]^2 + qg(t) + qV'(t) i \frac{\partial}{\partial p} \right) \psi(p, t) \end{aligned} \quad (4)$$

(we set $\hbar = 1$). It is initially taken to describe an object of momentum p_0 located at time 0 at x_0 :

$$\psi(p, 0) = N e^{-(p-p_0)^2 \sigma^2} e^{-ipx_0}. \quad (5)$$

To solve Eq. (4) with the initial condition (5), we make the ansatz

$$\psi(p, t) = N e^{-(\sigma^2 + it/2m)p^2 + \beta(t)p + i\gamma(t) - \beta_R^2(t)/4\sigma^2}, \quad (6)$$

where $\beta = \beta_R + i\beta_I$ and γ is real. The last factor in the exponent of (6) ensures $\int dp |\psi(p, t)|^2 = N^2$. From (5) and (6), the initial conditions are $\gamma(0) = 0$ and $\beta(0) = 2p_0\sigma^2 - ix_0$.

In Appendix A we put (6) into (4) and equate coefficients of p and 1, with the results

$$\begin{aligned} \beta_R &= 2\sigma^2 p_{cl}(t), \quad \beta_I = -x_{cl}(t) + \frac{p_{cl}(t)t}{m}, \\ \gamma &= -q \int_0^t dt' V(t') + v_0 W(t). \end{aligned} \quad (7)$$

We then go to the position representation by taking the Fourier transform of (6). The result (A12) is

$$\begin{aligned} \psi(x, t) &= N e^{-\{[x-x_{cl}(t)]^2/4\sigma^2\}} e^{ip_{cl}(t)[x-x_{cl}(t)]} e^{ip_{cl}^2(t)t/2m} \\ &\quad \times e^{-iq \int_0^t dt' V(t') + iv_0 W(t)}, \end{aligned} \quad (8)$$

under the assumptions that the mass of the object is large enough so that there is negligible spreading of the wave packet over time t and that the packet width σ is much larger than the wavelength associated with the momentum, $\sigma \gg h/mv$, so the momentum is well defined. Thus we see that it is a consequence of quantum theory that the mean particle motion follows the classical trajectory and that the phase shift is expressed in terms of the classical variables as well.

III. PARTICLE MOVING UNDER GENERAL FORCES

We now consider the motion of a charged particle in one dimension under externally applied potentials with arbitrary space and time dependence. An exact solution of Schrödinger's equation cannot be obtained for this more general case. Instead, we resort to an approximate method (which, as we will see, gives the same result for the phase as above when the force is restricted to be solely time dependent). We set the stage for the approximation in Secs. III A and III B. The approximation, consisting of expanding the exact Hamiltonian about the classical trajectory of the particle, is presented in Sec. III C. For a wave packet whose mean follows the classical trajectory, this is an exactly soluble problem and the phase of the wave function is obtained in Sec. III D.

A. Classical motion

The motion is governed by the classical Hamiltonian

$$H = \frac{1}{2m} \left(p_{cl} - \frac{q}{c} A(x_{cl}, t) \right)^2 + qV(x_{cl}, t), \quad (9)$$

where x_{cl} and p_{cl} are the canonical coordinates. The equations of motion follow from Hamilton's Poisson bracket equations

$$\begin{aligned} \frac{d}{dt} x_{cl}(t) &= \frac{1}{m} \left(p_{cl}(t) - \frac{q}{c} A(x_{cl}(t), t) \right), \\ \frac{d}{dt} \left(p_{cl}(t) - \frac{q}{c} A(x_{cl}(t), t) \right) &= -qV'(x_{cl}(t), t) - \frac{q}{c} \dot{A}(x_{cl}(t), t), \end{aligned} \quad (10)$$

where the overdot means ∂_t , the prime means ∂_x , and we have used $dF(x_{cl}, p_{cl}, t)/dt = [F, H]_{PB} + \partial F(x_{cl}, p_{cl}, t)/\partial t$. Defining $\mathcal{E}(x_{cl}(t), t) \equiv -V'(x_{cl}(t), t) - \frac{1}{c}\dot{A}(x_{cl}(t), t)$, the solution may formally be written

$$\begin{aligned} v_{cl}(t) &\equiv \frac{1}{m} \left(p_{cl}(t) - \frac{q}{c} A(x_{cl}(t), t) \right) \\ &= v_0 + \frac{q}{m} \int_0^t dt_1 \mathcal{E}(x_{cl}(t_1), t_1), \\ x_{cl}(t) &= x_0 + v_0 t + \frac{q}{m} \int_0^t dt_1 \int_0^{t_1} dt_2 \mathcal{E}(x_{cl}(t_2), t_2). \end{aligned} \quad (11)$$

B. Quantum phase

For the comparable quantum problem, the Hamiltonian is

$$H = \frac{1}{2m} \left[P - \frac{q}{c} A(X, t) \right]^2 + qV(X, t), \quad (12)$$

where X and P are the conjugate operators. In this case we will assume that the wave packet follows the classical trajectory, supposing that it spreads negligibly over the time interval of interest (i.e., $t/2m \ll \sigma^2$),

$$\psi(x, t) = e^{-[x-x_{cl}(t)]^2/4\sigma^2} e^{i\theta(x, t)}, \quad (13)$$

and see whether the phase $\theta(x, t)$ of the wave function can be obtained from Schrödinger's equation in some reasonable approximation.

As is well known from de Broglie–Bohm theory, putting (13) into Schrödinger's equation results in two equations, the imaginary part yielding conservation of probability, in the form $\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} [\rho(x, t)v(x, t)]$, the real part yielding the Hamilton-Jacobi equation, modified by a quantum potential

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} e^{-[x-x_{cl}(t)]^2/2\sigma^2} \\ &\quad + \frac{\partial}{\partial x} \left(e^{-[x-x_{cl}(t)]^2/2\sigma^2} \frac{1}{m} \left[\theta'(x, t) - \frac{q}{c} A(x, t) \right] \right), \\ -\dot{\theta}(x, t) &= \frac{1}{2m} \left[\theta'(x, t) - \frac{q}{c} A(x, t) \right]^2 + qV(x, t) \\ &\quad - \frac{1}{2m} e^{[x-x_{cl}(t)]^2/4\sigma^2} \frac{\partial^2}{\partial x^2} e^{-[x-x_{cl}(t)]^2/4\sigma^2}. \end{aligned} \quad (14)$$

The conservation equation, the first of (14), becomes

$$\begin{aligned} 0 &= -\frac{1}{\sigma^2} [x - x_{cl}(t)] \left(-v_{cl}(t) + \frac{1}{m} \left[\theta'(x, t) - \frac{q}{c} A(x, t) \right] \right) \\ &\quad + \frac{\partial}{\partial x} \frac{1}{m} \left[\theta'(x, t) - \frac{q}{c} A(x, t) \right], \end{aligned} \quad (15)$$

with solution

$$v_{cl}(t) - \frac{1}{m} \left[\theta'(x, t) - \frac{q}{c} A(x, t) \right] = C e^{[x-x_{cl}(t)]^2/2\sigma^2}. \quad (16)$$

The expectation value of the momentum $\int dx |\psi(x, t)|^2 \theta'(x, t)$ will diverge unless $C = 0$ (since the integrand goes $\sim C$ for large x), so the conservation of probability provides the condition

$$\theta'(x, t) = m v_{cl}(t) + \frac{q}{c} A(x, t). \quad (17)$$

The quantum potential part of the second equation of (14) is

$$\begin{aligned} &-\frac{1}{2m} e^{[x-x_{cl}(t)]^2/4\sigma^2} \frac{\partial^2}{\partial x^2} e^{-[x-x_{cl}(t)]^2/4\sigma^2} \\ &= -\frac{1}{8m\sigma^4} \{ [x - x_{cl}(t)]^2 - 2\sigma^2 \}. \end{aligned} \quad (18)$$

Because the magnitude of the wave function keeps $[x - x_{cl}(t)]^2$ of the order of σ^2 , (18) is of magnitude $1/m\sigma^2$. This may be neglected compared to the kinetic energy term in Eq. (14), $m v_{cl}^2/2$, since we assume that there are many wavelengths within σ , so the momentum of the packet is well defined.

Thus, we have the classical Hamilton-Jacobi equation

$$-\dot{\theta}(x, t) = \frac{1}{2m} \left[\theta'(x, t) - \frac{q}{c} A(x, t) \right]^2 + qV(x, t). \quad (19)$$

In order that these two equations (17) and (19) be integrable, it must be that $\partial_t \partial_x \theta(x, t) = \partial_x \partial_t \theta(x, t)$. From (17) we obtain

$$\begin{aligned} \partial_t \partial_x \theta(x, t) &= \partial_t \left[m v_{cl}(t) + \frac{q}{c} A(x, t) \right] = \frac{d}{dt} m v_{cl}(t) + \frac{q}{c} \dot{A}(x, t) \\ &= -qV'(x_{cl}(t), t) - \frac{q}{c} [\dot{A}(x_{cl}(t), t) - \dot{A}(x, t)], \end{aligned} \quad (20)$$

where, in going from the first line to the second line in Eq. (20), we have used the second of Eqs. (10). From (19), with use of (17), we obtain

$$\partial_x \partial_t \theta(x, t) = -\partial_x \frac{m}{2} v_{cl}^2(t) - qV'(x, t) = -qV'(x, t). \quad (21)$$

We see that the right-hand sides of (20) and (21) are not, in general, identical, so these equations are not, in general, integrable. The reason is that we have assumed the wave packet to have a very specific trajectory, its center at $x = x_{cl}(t)$, with specific initial conditions x_0 and v_0 , whereas the Hamiltonian dynamics is general, not specific to this trajectory.

While not integrable for this general case of the electric force depending upon position and time, they *are* integrable in the special case earlier dealt with, where the electric field only depends upon time. That is, if $V'(x, t) = V'(t)$ and $A(x, t) = A(t)$, the right-hand sides of (20) and (21) are identical.

C. Classical motion: Approximate Hamiltonian

Our procedure will be to construct an *approximate* Hamiltonian for which the problem is *exactly* soluble (i.e., for which the phase angle is integrable). The Hamiltonian will be designed so that the particular classical trajectory $x_{cl}(t)$, with specified initial values of x_0 and v_0 , which satisfies the exact Hamiltonian's equation of motion, now also satisfies the approximate Hamiltonian's equation of motion and that neighboring classical trajectories stay close (to be made precise at the end of this section) to $x_{cl}(t)$. The Hamiltonian we

propose is

$$H' = \frac{1}{2m} \left[P_{cl} - \frac{q}{c} A(x_{cl}(t), t) \right]^2 + qV(x_{cl}(t), t) + [X_{cl} - x_{cl}(t)]qV'(x_{cl}(t), t) - \frac{q}{c} v_{cl}(t)[X_{cl} - x_{cl}(t)]A'(x_{cl}(t), t), \quad (22)$$

where now X_{cl} and P_{cl} are the canonical coordinates. Thus, one particular $x_{cl}(t)$ plays three roles: It is a solution of the exact Hamiltonian dynamics, it is a crucial element in the approximate Hamiltonian (22), and, as we will see, it is a solution of the approximate Hamiltonian dynamics.

The Poisson bracket equations of motion corresponding to the Hamiltonian (22) are then

$$\begin{aligned} \frac{d}{dt} X_{cl}(t) &= \frac{1}{m} \left[P_{cl}(t) - \frac{q}{c} A(x_{cl}(t), t) \right], \\ \frac{d}{dt} \left[P_{cl}(t) - \frac{q}{c} A(x_{cl}(t), t) \right] &= -qV'(x_{cl}(t), t) + \frac{q}{c} v_{cl}(t)A'(x_{cl}(t), t) - \frac{d}{dt} \frac{q}{c} A(x_{cl}(t), t). \end{aligned} \quad (23)$$

Upon combining these two equations, we obtain

$$\frac{d^2}{dt^2} X_{cl}(t) = -qV'(x_{cl}(t), t) - \frac{q}{c} \dot{A}(x_{cl}(t), t). \quad (24)$$

Here $X_{cl}(t)$ is the general solution with arbitrary initial conditions $X_{cl}(0)$ and $V_{cl}(0)$. We see from (10) that $X_{cl}(t) = x_{cl}(t)$ is a solution of (24) [the solution with initial conditions $X_{cl}(0) = x_{cl}(0)$ and $V_{cl}(0) = v_{cl}(0)$]. Moreover, $d^2[X_{cl}(t) - x_{cl}(t)]/dt^2 = 0$, so for an arbitrary solution $X_{cl}(t)$,

$$X_{cl}(t) - x_{cl}(t) = [X_{cl}(0) + V_{cl}(0)t] - [x_{cl}(0) + v_{cl}(0)t] + ot^3.$$

Thus, two trajectories with the same initial speed and different initial positions only diverge to order t^3 under this Hamiltonian. This property, of neighboring classical trajectories staying close, might be expected of a soluble Hamiltonian because a quantum wave packet that holds together as time increases might be expected to have its classical counterparts hold together, for at least small times.

D. Quantum phase: Approximate Hamiltonian

For the comparable quantum problem, the Hamiltonian as usual is obtained by replacing in Eq. (22) the position and momentum variables X_{cl} and P_{cl} with operators X and P :

$$H' = \frac{1}{2m} \left[P - \frac{q}{c} A(x_{cl}(t), t) \right]^2 + qV(x_{cl}(t), t) + [X - x_{cl}(t)]qV'(x_{cl}(t), t) - \frac{q}{c} v_{cl}(t)[X - x_{cl}(t)]A'(x_{cl}(t), t). \quad (25)$$

Assuming again the wave function form (13),

$$\psi(x, t) = e^{-[x-x_{cl}(t)]^2/4\sigma^2} e^{i\theta(x, t)},$$

we obtain the two equations (with neglect of the quantum potential as before)

$$\begin{aligned} \theta'(x, t) &= mv_{cl}(t) + \frac{q}{c} A(x_{cl}(t), t), \\ -\dot{\theta}(x, t) &= \frac{m}{2} v_{cl}^2(t) + qV(x_{cl}(t), t) + q[x - x_{cl}(t)]V'(x_{cl}(t), t) - \frac{q}{c} v_{cl}(t)[x - x_{cl}(t)]A'(x_{cl}(t), t). \end{aligned} \quad (26)$$

From the first of Eqs. (26), using the second of Eqs. (10) again, we have

$$\begin{aligned} \partial_t \partial_x \theta(x, t) &= \partial_t \left[mv_{cl}(t) + \frac{q}{c} A(x_{cl}(t), t) \right] \\ &= -qV'(x_{cl}(t), t) - \frac{q}{c} \dot{A}(x_{cl}(t), t) + \frac{q}{c} [\dot{A}(x_{cl}(t), t) + v_{cl}(t)A'(x_{cl}(t), t)] \\ &= -qV'(x_{cl}(t), t) + \frac{q}{c} v_{cl}(t)A'(x_{cl}(t), t). \end{aligned} \quad (27)$$

We see from the second of (26) that this is equal to $\partial_x \partial_t \theta(x, t)$, so $\theta(x, t)$ can be found.

By integrating the first of (26), we find that $\theta(x, t)$ has the form

$$\theta(x, t) = x \left[mv_{cl}(t) + \frac{q}{c} A(x_{cl}(t), t) \right] + \alpha(t), \quad (28)$$

where $\alpha(t)$ is an arbitrary function of t . Taking the time derivative of this and comparing it with the second of (22), we find

$$\begin{aligned} \dot{\alpha}(t) &= -\frac{m}{2}v_{cl}^2(t) + x_{cl}(t)\left[qV'(x_{cl}(t),t) - \frac{q}{c}v_{cl}(t)A'(x_{cl}(t),t)\right] - qV(x_{cl}(t),t), \\ &= -\frac{1}{2m}\left[p_{cl}(t) - \frac{q}{c}A(x_{cl}(t),t)\right]^2 - x_{cl}(t)\frac{d}{dt}p_{cl}(t) - qV(x_{cl}(t),t), \\ &= -\frac{1}{2m}p_{cl}^2(t) + \frac{q}{mc}p_{cl}(t)A(x_{cl}(t),t) - \left[\frac{d}{dt}[x_{cl}(t)p_{cl}(t)] - v_{cl}(t)p_{cl}(t)\right] - qV(x_{cl}(t),t), \\ &= -\frac{1}{2m}p_{cl}^2(t) + \frac{q}{mc}p_{cl}(t)A(x_{cl}(t),t) - \frac{d}{dt}[x_{cl}(t)p_{cl}(t)] + \frac{1}{m}\left[p_{cl}(t) - \frac{q}{c}A(x_{cl}(t),t)\right]p_{cl}(t) - qV(x_{cl}(t),t), \\ &= \frac{1}{2m}p_{cl}^2(t) - \frac{d}{dt}[x_{cl}(t)p_{cl}(t)] - qV(x_{cl}(t),t). \end{aligned} \tag{29}$$

Putting the integral of (29) into (28), we obtain the expression for the phase [up to an additive constant, in which $x_{cl}(0)p_{cl}(0)$ has been absorbed]:

$$\theta(x,t) = p_{cl}(t)[x - x_{cl}(t)] + \int_0^t dt' \frac{1}{2m}p_{cl}^2(t') - q \int_0^t dt' V(x_{cl}(t'),t'). \tag{30}$$

[As a check, it is shown in Appendix B that when the electric force only depends upon time, this expression for the phase is equal to the phase in Eq. (8).]

IV. INTERFERENCE: EXTRA PHASE IN TWO-PARTICLE AND (N+1)-PARTICLE CASES

We wish to consider an interference situation governed by a Hamiltonian describing many particles interacting with a single particle. The situation is such that each particle’s wave function is expected to be well approximated as a spatially localized packet with well-defined momentum and the total wave function is expected to be well approximated as the product of these wave packets or the linear combination of such products. We turn now to show how to analyze this situation in a consistent fashion. Our result (13) for the amplitude and (30) for the phase associated with the motion of a single particle in known space-time-dependent potentials will be utilized in this endeavor.

In the next section we will consider the simplest problem, an interference situation involving just two particles, which however displays all the features of the more complicated problem.

A. Two-particle interaction: Wave function

The Schrödinger equation for two particles moving under their mutual vector and scalar potentials has the form

$$\begin{aligned} i \frac{d}{dt} |\Psi, t\rangle &= \left[\frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(r_{12}) \right. \\ &\quad \left. - \frac{q_1q_2}{m_1m_2c^2} \mathcal{S}[\mathbf{p}_1 \cdot \mathbf{D}^{\leftrightarrow}(\mathbf{r}_{12}) \cdot \mathbf{p}_2] \right] |\Psi, t\rangle. \end{aligned} \tag{31}$$

Here $r_{12} \equiv |\mathbf{x}_1 - \mathbf{x}_2|$, $\mathbf{D}^{\leftrightarrow}(\mathbf{r}_{12})$ is a dyad, and $\mathcal{S}[\]$ is an operation that makes the argument suitably Hermitian. In the Lorenz gauge¹ and the Coulomb gauge, $V(r_{12}) = q_1q_2/r_{12}$,

while the dyads for these two gauges (the Darwin Hamiltonian [9] in the case of the Coulomb gauge) are, respectively,

$$D^{\leftrightarrow}(\mathbf{r}_{12}) = \frac{\mathbf{1}^{\leftrightarrow}}{r_{12}}, \quad D^{\leftrightarrow}(\mathbf{r}_{12}) = \frac{1}{2} \left[\frac{\mathbf{1}^{\leftrightarrow}}{r_{12}} + \frac{\mathbf{r}_{12}\mathbf{r}_{12}}{r_{12}^3} \right],$$

so the vector potential, e.g., due to particle 2, is

$$\mathbf{A}_2 = \frac{q_2}{m_2c} \mathcal{S} \left[\frac{\mathbf{p}_2}{r_{12}} \right], \quad \mathbf{A}_2 = \frac{q_2}{2m_2c} \mathcal{S} \left[\frac{\mathbf{p}_2}{r_{12}} + \frac{\mathbf{r}_{12}\mathbf{r}_{12} \cdot \mathbf{p}_2}{r_{12}^3} \right], \tag{32}$$

the difference arising from the source of the vector potential being respectively the current and the transverse component of the current.

A general solution of (31) will be an entangled state for the two particles. However, for the physical situations we contemplate, we suppose the state vector is well approximated as a direct product of state vectors for each particle. The associated wave functions are to be those we have been considering, well-localized wave packets moving under the vector and scalar potentials due to the other particle, where the other particle’s operators are replaced by their classical counterparts. How do we make a consistent approximation?

Consider the variational principle for the Schrödinger equation

$$\delta \int_0^T dt \langle \Psi, t | \left[i \frac{d}{dt} - H \right] | \Psi, t \rangle = 0.$$

Inserting the approximate solution $|\Psi, t\rangle = |\psi_1, t\rangle |\psi_2, t\rangle$ and varying separately for each particle state vector, we obtain for

gauge condition $\nabla \cdot \mathbf{A} + c^{-1}\dot{V} = 0$ just uses the zeroth-order term in the scalar potential, the extra term is not needed for that, nor does it affect anything in our calculation, so it may be omitted.

¹Since the vector potential term is of order c^{-1} , one might add the term in the scalar potential to that order. However, since the Lorenz

particle 1 (and similarly for particle 2)

$$\begin{aligned}
& i \frac{d}{dt} |\psi_1, t\rangle + \left(\langle \psi_2, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_2^2}{2m_2} \right] | \psi_2, t \rangle \right) |\psi_1, t\rangle \\
&= \left[\frac{\mathbf{p}_1^2}{2m_1} + \langle \psi_2, t | V(r_{12}) | \psi_2, t \rangle - \frac{q_1 q_2}{m_1 m_2 c^2} \mathcal{S}[\mathbf{p}_1 \cdot \langle \psi_2, t | \mathbf{D}^{\leftrightarrow}(\mathbf{r}_{12}) \cdot \mathbf{p}_2 | \psi_2, t \rangle] \right] |\psi_1, t\rangle \\
&\approx \left[\frac{\mathbf{p}_1^2}{2m_1} + V(|\mathbf{x}_1 - \mathbf{x}_{2cl}(t)|) - \frac{q_1 q_2}{m_1 m_2 c^2} \mathcal{S}[\mathbf{p}_1 \cdot \mathbf{D}^{\leftrightarrow}(\mathbf{x}_1 - \mathbf{x}_{2cl}(t)) \cdot \mathbf{p}_{2cl}(t)] \right] |\psi_1, t\rangle,
\end{aligned} \tag{33}$$

where the approximation in the last step uses the localized nature of packet 2 to replace the operators \mathbf{x}_2 and \mathbf{p}_2 by their mean values $\mathbf{x}_{2cl}(t)$ and $\mathbf{p}_{2cl}(t)$.

Equation (33) is just the equation we have solved for the magnitude and phase of a localized packet moving in an external vector and scalar potential *except* that there is an extra phase term on the left-hand side, which we define

$$\dot{\phi}_j(t) \equiv \langle \psi_j, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_j^2}{2m_j} \right] | \psi_j, t \rangle, \quad j = 1, 2. \tag{34}$$

(That this is a real number may be seen by taking its complex conjugate and utilizing $d\langle \psi_j, t | \psi_j, t \rangle / dt = 0$.)

Now the equations for $|\psi_1, t\rangle, |\psi_2, t\rangle$ are form invariant under the replacements $|\psi_1, t\rangle = |\psi_1, t'\rangle e^{i\beta(t)}$ and $|\psi_2, t\rangle = |\psi_2, t'\rangle e^{-i\beta(t)}$: This is to be expected, since the product wave function is unchanged by this phase transformation. This gives the freedom to choose $\beta(t)$ to remove the extra phase term from, say, the equation for $|\psi_1, t\rangle$. Given a solution for $|\psi_1, t\rangle$ and $|\psi_2, t\rangle$, choose $\beta(t) = \phi_2(t)$, so the transformed equations become

$$i \frac{d}{dt} |\psi_1, t'\rangle = \left[\frac{\mathbf{p}_1^2}{2m_1} + V(|\mathbf{x}_1 - \mathbf{x}_{2cl}(t)|) - \frac{q_1 q_2}{m_1 m_2 c^2} \mathcal{S}[\mathbf{p}_1 \cdot \mathbf{D}^{\leftrightarrow}(\mathbf{x}_1 - \mathbf{x}_{2cl}(t)) \cdot \mathbf{p}_{2cl}(t)] \right] |\psi_1, t'\rangle, \tag{35a}$$

$$\begin{aligned}
& i \frac{d}{dt} |\psi_2, t'\rangle + \left(\langle \psi_1, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_1^2}{2m_1} \right] | \psi_1, t' \rangle \right) |\psi_2, t'\rangle \\
&= \left[\frac{\mathbf{p}_2^2}{2m_2} + V(|\mathbf{x}_2 - \mathbf{x}_{1cl}(t)|) - \frac{q_1 q_2}{m_1 m_2 c^2} \mathcal{S}[\mathbf{p}_2 \cdot \mathbf{D}^{\leftrightarrow}(\mathbf{x}_2 - \mathbf{x}_{1cl}(t)) \cdot \mathbf{p}_{1cl}(t)] \right] |\psi_2, t'\rangle.
\end{aligned} \tag{35b}$$

By subtracting $\frac{\mathbf{p}_1^2}{2m_1} |\psi_1, t'\rangle$ from (35a) and taking the scalar product with $\langle \psi_1, t |$, we see that the extra phase in Eq. (35b) is given by

$$\dot{\phi}'_1(t) \equiv \langle \psi_1, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_1^2}{2m_1} \right] | \psi_1, t' \rangle = V(|\mathbf{x}_{1cl}(t) - \mathbf{x}_{2cl}(t)|) - \frac{q_1 q_2}{m_1 m_2 c^2} \mathbf{p}_{1cl}(t) \cdot \mathbf{D}^{\leftrightarrow}(\mathbf{x}_{1cl}(t) - \mathbf{x}_{2cl}(t)) \cdot \mathbf{p}_{2cl}(t). \tag{36}$$

That is, this phase is the time integral of the interaction energy term of the Hamiltonian, where all operators are replaced by their classical values. We also note from (35b), by subtracting $\frac{\mathbf{p}_2^2}{2m_2}$ and taking the scalar product with $\langle \psi_2, t |$, that

$$\langle \psi_2, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_2^2}{2m_2} \right] | \psi_2, t' \rangle + \langle \psi_1, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_1^2}{2m_1} \right] | \psi_1, t' \rangle = \dot{\phi}'_1(t), \tag{37}$$

that is,

$$\langle \psi_2, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_2^2}{2m_2} \right] | \psi_2, t' \rangle = 0.$$

If we write $|\psi_2, t'\rangle \equiv e^{i\phi'_1(t)} |\psi_2, t''\rangle$, then $|\psi_2, t''\rangle$ satisfies (35b) without the extra phase term. Then it and the solution of (35a) are both the solutions we have found for a localized packet moving in external potentials, where those potentials now are internal potentials so to speak, potentials due to the other particle with operator values replaced by classical ones. So the wave function that solves (31) approximately is the product of these two solutions, multiplied by the extra phase factor $e^{i\phi'_1(t)}$.

B. Two-particle interaction: Interference

We will now show, in an interference experiment involving the two particles interacting, that each particle makes the same contribution to the phase shift, each the negative of the contribution of (36), so the net phase shift is the negative of the time integral of (36). Thus we see the germ here of what will be fleshed out later, that the motion of the electron (think particle 1) in the vector potential of the solenoid and the motion of the solenoid (think particle 2) in the vector potential of the electron

each make identical contributions to the phase shift. That would give twice the expected phase shift, except that the proper approximation generates an additional phase shift that is the negative of these and so one ends up with the expected phase shift, the shift due to either mechanism.

Consider an interference experiment where, at time zero, packets of the two particles split in two, with equal amplitude both particles going on trajectories labeled A or on trajectories labeled B . The wave function at some time $t < T$ may be written as

$$\Psi(x_1, x_2, t) = \frac{1}{\sqrt{2}} [\psi_1'^A(x_1, t) \psi_2''^A(x_2, t) e^{i\phi_1^A(t)} + \psi_1'^B(x_1, t) \psi_2''^B(x_2, t) e^{i\phi_1^B(t)}]. \quad (38)$$

Here $\psi_1'(x_1, t) = \langle x_1 | \psi_1, t \rangle'$ and $\psi_2''(x_2, t) = \langle x_2 | \psi_2, t \rangle''$ satisfy their respective Schrödinger equations, motion under potentials where the position and momentum operators of the other particle have been replaced by their time-dependent classical values, with no extra phase terms; $\phi_1^{A,B}(t)$ is the extra phase factor for trajectory A or B .

The packets finally come together at time T . We suppose that particle 2's packets precisely overlap then and thereafter. We suppose that particle 1's packets meet at (the particle analog of) a half-silvered mirror, coming from opposite directions. If particle 1 had been on path A , it splits into two packets that go to the right or left with equal amplitude $1/\sqrt{2}$. If particle 1 had been on path B , one packet goes to the right with amplitude $1/\sqrt{2}$ and the other to the left with amplitude $-1/\sqrt{2}$. The probability of particle 1 being detected either at the left or the right is what is measured. The wave function at time T (trivially extended thereafter) is therefore $\Psi(x_1, x_2, T) = \frac{1}{\sqrt{2}} [\Psi_+(x_1, x_2, T) + \Psi_-(x_1, x_2, T)]$, with

$$\Psi_{\pm}(x_1, x_2, T) = \frac{1}{\sqrt{2}} [\psi_1'^A(x_1, T) \psi_2''^A(x_2, T) e^{i\phi_1^A(T)} \pm \psi_1'^B(x_1, T) \psi_2''^B(x_2, T) e^{i\phi_1^B(T)}], \quad (39)$$

with the subscript $+$ referring to what is measured to the right and $-$ referring to what is measured to the left and the superscripts A and B denoting the path. The individual particle wave functions in Eq. (39) are of the form (13), with phase (30), e.g.,

$$\psi_1'^A(x_1, T) = N e^{-[x_1 - x_{1,cl}^A(T)]^2 / 4\sigma^2} \exp \left[i \left(p_{1,cl}^A(T) [x_1 - x_{1,cl}^A(T)] + \int_0^T dt \frac{1}{2m} [p_{1,cl}^A(t)]^2 - q_1 \int_0^T dt V_1(x_{1,cl}^A(t), t) \right) \right]. \quad (40)$$

The probabilities of the two outcomes are

$$P_{\pm} \equiv \int dx_1 dx_2 |\Psi_{\pm}(x_1, x_2, T)|^2 = \frac{1}{4} \left[2 \pm e^{i[\phi_1^A(T) - \phi_1^B(T)]} \int dx_1 \psi_1'^A(x_1, T) \psi_1'^B(x_1, T) \int dx_2 \psi_2''^A(x_2, T) \psi_2''^B(x_2, T) + \text{c.c.} \right]. \quad (41)$$

Using the expressions (40), the integrals in Eq. (41) are readily performed and give 1, since the packets completely overlap at time T so $x_{cl}^A(T) = x_{cl}^B(T)$ and $p_{cl}^A(T) = p_{cl}^B(T)$. We are left then with the phase factors. From (40), the factor associated with interference of each particle is (leaving off the primes)

$$\begin{aligned} e^{i(\Phi_i^A - \Phi_i^B)} &\equiv \int_{-\infty}^{\infty} dx_i \psi_i^A(x_i, T) \psi_i^{*B}(x_i, T) \\ &= \exp \left(i \int_0^T dt \frac{1}{2m_i} [p_{i,cl}^A(t)]^2 - i \int_0^T dt \frac{1}{2m_i} [p_{i,cl}^B(t)]^2 \right) \exp \left[-i q_i \left(\int_0^T dt (V(x_{i,cl}^A(t), t) - V(x_{i,cl}^B(t), t)) \right) \right]. \end{aligned} \quad (42)$$

Then the probabilities are

$$P_{\pm} = \frac{1}{2} \{ 1 \pm \cos [(\Phi_1^A + \Phi_2^A + \phi_1^A) - (\Phi_1^B + \Phi_2^B + \phi_1^B)] \}. \quad (43)$$

In order to evaluate the phase in Eq. (42), we utilize Eq. (11). We find, for trajectory A or B ,

$$p_{i,cl}(t) = m_i v_{i,0} + \frac{q_i}{c} A(x_{i,cl}(t), t) + q_i \int_0^t dt \mathcal{E}(x_{i,cl}(t), t), \quad (44a)$$

$$x_{i,cl}(t) = x_{i,0} + v_{i,0} t + \frac{q_i}{m_i} \int_0^t dt' \int_0^{t'} dt'' \mathcal{E}(x_{i,cl}(t''), t''). \quad (44b)$$

Since $x_{i,cl}^A(T) = x_{i,cl}^B(T)$, it follows from (44b) that

$$\int_0^T dt \int_0^t dt' [\mathcal{E}(x_{i,cl}^A(t'), t') - \mathcal{E}(x_{i,cl}^B(t'), t')] = 0. \quad (45)$$

We may write the momentum-dependent phase for each particle that appears in Eq. (42), using (44a) and (45), as

$$\begin{aligned}
 & \int_0^T dt \frac{1}{2m_i} [p_{i,cl}^A(t)]^2 - \int_0^T dt \frac{1}{2m_i} [p_{i,cl}^B(t)]^2 \\
 &= \int_0^T dt \frac{1}{2m_i} [p_{i,cl}^A(t) + p_{i,cl}^B(t)][p_{i,cl}^A(t) - p_{i,cl}^B(t)] \\
 &\approx \int_0^T dt v_i(0) \left(\frac{q_i}{c} [A(x_{i,cl}^A(t), t) - A(x_{i,cl}^B(t), t)] + q_i \int_0^t dt' [\mathcal{E}(x_{i,cl}^A(t'), t') - \mathcal{E}(x_{i,cl}^B(t'), t')] \right) \\
 &= \int_0^T dt v_i(0) \frac{q_i}{c} [A(x_{i,cl}^A(t), t) - A(x_{i,cl}^B(t), t)] \approx \int_0^T dt \left(v_{i,cl}^A(t) \frac{q_i}{c} A(x_{i,cl}^A(t), t) - v_{i,cl}^B(t) \frac{q_i}{c} A(x_{i,cl}^B(t), t) \right) \\
 &= \int_0^T dt \left(\mathbf{v}_{i,cl}^A(t) \cdot \frac{q_i}{c} \mathbf{A}(x_{i,cl}^A(t), t) - \mathbf{v}_{i,cl}^B(t) \cdot \frac{q_i}{c} \mathbf{A}(x_{i,cl}^B(t), t) \right). \tag{46}
 \end{aligned}$$

In the approximations made in the second and third lines of (46), we have used $v_i(0) = v_{i,cl}(t) + 0(q)$ from (11) and dropped terms in the square of the charges, as has been done throughout this paper. In the last line, we have reintroduced vector notation, since the vector potential in our equations has always been the component parallel to the velocity of the particle.

Thus, from (42) and (46) we have

$$\Phi_i^A - \Phi_i^B = \int_0^T dt \left(\mathbf{v}_{i,cl}^A(t) \cdot \frac{q_i}{c} \mathbf{A}(x_{i,cl}^A(t), t) - q_i V(x_{i,cl}^A(t), t) \right) - \int_0^T dt \left(\mathbf{v}_{i,cl}^B(t) \cdot \frac{q_i}{c} \mathbf{A}(x_{i,cl}^B(t), t) - q_i V(x_{i,cl}^B(t), t) \right). \tag{47}$$

That is, Φ_i^A and Φ_i^B are each the time integral of the negative of the Hamiltonian interaction energy with operators replaced by classical variables for that trajectory (A or B). Due to the symmetry of the interaction under exchange of particles 1 and 2, these are the same, independent of i . Since this is the negative of the extra phase $\phi_i^A(T)$ and $\phi_i^B(T)$, we have for the probabilities (43)

$$P_{\pm} = \frac{1}{2} [1 \pm \cos(\Phi^A - \Phi^B)], \tag{48a}$$

where

$$\Phi^{A,B} = \int_0^T dt \left(\mathbf{v}_{i,cl}(t)^{A,B} \cdot \frac{q_i}{c} \mathbf{A}(x_{i,cl}^{A,B}(t), t) - q_i V(x_{i,cl}^{A,B}(t), t) \right) = - \int_0^T dt H_{\text{int}}(x_{1,cl}^{A,B}(t), x_{2,cl}^{A,B}(t), p_{1,cl}^{A,B}(t), p_{2,cl}^{A,B}(t)). \tag{48b}$$

Thus, we have confirmed the assertion made in the beginning of this section, that the phase shift is the difference for each path of the time integral of the negative of the interaction Hamiltonian with operators replaced by time-dependent classical variables.

C. N particles interacting with a single particle: Wave function

We will extend the result of the preceding section since we eventually wish to consider an N -particle solenoid interacting with an electron. We will repeat as closely as possible the steps taken in the discussion of the interaction of two particles.

The Schrödinger equation for N identical particles of mass m , charge q , interacting with a single particle of mass m_e , charge e , under mutual vector and scalar potentials has the form

$$i \frac{d}{dt} |\Psi, t\rangle = \left[\frac{\mathbf{p}_e^2}{2m_e} + \sum_{n=1}^N \frac{\mathbf{p}_n^2}{2m} + \sum_{n=1}^N V(r_{en}) - \sum_{n=1}^N \frac{eq}{m_e m c^2} \mathcal{S}[\mathbf{p}_e \cdot \mathbf{D}^{\leftrightarrow}(\mathbf{r}_{en}) \cdot \mathbf{p}_n] \right] |\Psi, t\rangle. \tag{49}$$

(We will call the single particle the electron.) Upon considering the variational principle for the Schrödinger equation with the approximate solution $|\Psi, t\rangle = |\psi_e, t\rangle \prod_{n=1}^N |\psi_n, t\rangle$ and varying separately for each particle state vector, we obtain

$$\begin{aligned}
 & i \frac{d}{dt} |\psi_e, t\rangle + \left(\sum_{n=1}^N \langle \psi_n, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_n^2}{2m} \right] | \psi_n, t \rangle \right) |\psi_e, t\rangle \\
 &= \left(\frac{\mathbf{p}_e^2}{2m_e} + \sum_{n=1}^N \langle \psi_n, t | V(r_{en}) | \psi_n, t \rangle - \frac{eq}{m_e m c^2} \mathcal{S} \left[\mathbf{p}_e \cdot \sum_{n=1}^N \langle \psi_n, t | \mathcal{S}[\mathbf{D}^{\leftrightarrow}(\mathbf{r}_{en}) \cdot \mathbf{p}_n] | \psi_n, t \rangle \right] \right) |\psi_e, t\rangle \\
 &\approx \left(\frac{\mathbf{p}_e^2}{2m_e} + \sum_{n=1}^N V(|\mathbf{x}_e - \mathbf{x}_{ncl}(t)|) - \frac{eq}{m_e m c^2} \mathcal{S} \left[\mathbf{p}_e \cdot \sum_{n=1}^N \mathbf{D}^{\leftrightarrow}(\mathbf{x}_e - \mathbf{x}_{ncl}(t)) \cdot \mathbf{p}_{ncl}(t) \right] \right) |\psi_e, t\rangle, \tag{50a}
 \end{aligned}$$

$$\begin{aligned}
 & i \frac{d}{dt} |\psi_n, t\rangle + \left(\sum_{n'=1, n' \neq n}^N \langle \psi_{n'}, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_{n'}^2}{2m} \right] | \psi_{n'}, t \rangle + \langle \psi_e, t | \left[i \frac{d}{dt} - \frac{\mathbf{p}_e^2}{2m_e} \right] | \psi_e, t \rangle \right. \\
 & \quad \left. - \sum_{n'=1, n' \neq n}^N V(|\mathbf{x}_{ecl}(t) - \mathbf{x}_{n'cl}(t)|) + \frac{eq}{m_e m c^2} \mathbf{p}_{ecl}(t) \cdot \sum_{n'=1, n' \neq n}^N \mathbf{D}^{\leftrightarrow}(\mathbf{x}_{ecl}(t) - \mathbf{x}_{n'cl}(t)) \cdot \mathbf{p}_{n'cl}(t) \right) |\psi_n, t\rangle \\
 & = \left[\frac{\mathbf{p}_n^2}{2m} + \langle \psi_e, t | V(\mathbf{r}_{en}) | \psi_e, t \rangle - \frac{eq}{m_e m c^2} \mathcal{S}[\mathbf{p}_n \cdot \langle \psi_e, t | \mathcal{S}[\mathbf{D}^{\leftrightarrow}(\mathbf{r}_{en}) \cdot \mathbf{p}_e] | \psi_e, t \rangle] \right] |\psi_n, t\rangle \\
 & \approx \left[\frac{\mathbf{p}_n^2}{2m} + V(|\mathbf{x}_n - \mathbf{x}_{ecl}(t)|) - \frac{eq}{m_e m c^2} \mathcal{S}[\mathbf{p}_n \cdot [\mathbf{D}^{\leftrightarrow}(\mathbf{x}_n - \mathbf{x}_{ecl}(t)) \cdot \mathbf{p}_{ecl}(t)]] \right] |\psi_n, t\rangle. \tag{50b}
 \end{aligned}$$

(Note the new terms in the scalar and vector potentials on the second line of (50b) that do not appear in the two particle case.)

We can remove the extra phase terms from the N particles by the phase transformation $|\psi_n, t\rangle = |\psi_n, t'\rangle e^{i\alpha_n(t)}$ and $|\psi_e, t\rangle = |\psi_e, t'\rangle \exp[-i \sum_{n=1}^N \alpha_n(t)]$. Equations (50a) and (50b) then become

$$\begin{aligned}
 & i \frac{d}{dt} |\psi_e, t'\rangle + \left(\sum_{n=1}^N \langle \psi_n, t' | \left[i \frac{d}{dt} - \frac{\mathbf{p}_n^2}{2m} \right] | \psi_n, t' \rangle \right) |\psi_e, t'\rangle \\
 & \approx \left(\frac{\mathbf{p}_e^2}{2m_e} + \sum_{n=1}^N V(|\mathbf{x}_e - \mathbf{x}_{ncl}(t)|) - \frac{eq}{m_e m c^2} \mathcal{S} \left[\mathbf{p}_e \cdot \sum_{n=1}^N \mathbf{D}^{\leftrightarrow}(\mathbf{x}_e - \mathbf{x}_{ncl}(t)) \cdot \mathbf{p}_{ncl}(t) \right] \right) |\psi_e, t'\rangle, \tag{51a}
 \end{aligned}$$

$$i \frac{d}{dt} |\psi_n, t'\rangle \approx \left[\frac{\mathbf{p}_n^2}{2m_n} + V(|\mathbf{x}_n - \mathbf{x}_{ecl}(t)|) - \frac{eq}{m_e m c^2} \mathcal{S}[\mathbf{p}_n \cdot [\mathbf{D}^{\leftrightarrow}(\mathbf{x}_n - \mathbf{x}_{ecl}(t)) \cdot \mathbf{p}_{ecl}(t)]] \right] |\psi_n, t'\rangle. \tag{51b}$$

By subtracting $\frac{\mathbf{p}_e^2}{2m} |\psi_n, t'\rangle$ from (51b) and taking the scalar product with $\langle \psi_n, t' |$, we see that the extra phase in Eq. (51a) is given by

$$\dot{\phi}'_e(t) \equiv \sum_{n=1}^N \langle \psi_n, t' | \left[i \frac{d}{dt} - \frac{\mathbf{p}_n^2}{2m} \right] | \psi_n, t' \rangle = \sum_{n=1}^N V(|\mathbf{x}_{ncl}(t) - \mathbf{x}_{ecl}(t)|) - \frac{eq}{m_e m c^2} \mathbf{p}_{ecl}(t) \cdot \sum_{n=1}^N \mathbf{D}^{\leftrightarrow}(\mathbf{x}_{ecl} - \mathbf{x}_{ncl}(t)) \cdot \mathbf{p}_{ncl}(t). \tag{52}$$

Again, we find that the phase $\phi'_e(t)$ is the time integral of the interaction energy term of the Hamiltonian, where all operators are replaced by their classical values.

If we write $|\psi_e, t'\rangle \equiv e^{i\phi'_e(t)} |\psi_j, t''\rangle$, then $|\psi_j, t''\rangle$ satisfies (51a) without the extra phase term. So the wave function that solves (49) approximately is the product of $N + 1$ wave functions of the type we have considered, for the N particles and the electron, multiplied by the extra phase factor $e^{i\phi'_e(t)}$.

D. N particles interacting with a single particle: Interference

We will now consider a situation where the electron goes on either path A or path B and so the N interacting particles likewise move on trajectories A or B determined by their interaction with the electron. The electron packets that traveled by route A and B are brought, at time T , to exactly overlap at a half-silvered mirror. The electron is measured as either going right or left, just as was the case for particle 1 in the two-particle example; however, for each of the N particles, the wave packet on trajectory A does not precisely overlap with the wave packet on trajectory B at time T (the overlap is presumed the same thereafter). So we will have to consider the effect of this on the measured interference.

The wave function at time $t < T$ may be written as

$$\Psi(x_e, x_1, \dots, x_N, t) = \frac{1}{\sqrt{2}} \left(e^{i\phi'_e{}^A(t)} \psi_e''^A(x_e, t) \prod_{n=1}^N \psi_n'^A(x_n, t) + e^{i\phi'_e{}^B(t)} \psi_e''^B(x_e, t) \prod_{n=1}^N \psi_n'^B(x_n, t) \right), \tag{53}$$

where $\psi_e''(x_e, t) = \langle x_e | \psi_e, t \rangle''$ and $\psi_n'(x_n, t) = \langle x_n | \psi_n, t \rangle'$ satisfy their respective Schrödinger equations; motion under potentials where the position and momentum operators of the other interacting particle have been replaced by their time-dependent classical values. The wave function at time T , just after the electron has passed through the half-silvered mirror, is $\Psi(x_e, x_1, \dots, x_N, T) = \frac{1}{\sqrt{2}} [\Psi_+(x_e, x_1, \dots, x_N, T) + \Psi_-(x_e, x_1, \dots, x_N, T)]$, with

$$\Psi_{\pm}(x_e, x_1, \dots, x_N, T) = \frac{1}{\sqrt{2}} \left(e^{i\phi'_e{}^A(t)} \psi_e''^A(x_e, T) \prod_{n=1}^N \psi_n'^A(x_n, T) \pm e^{i\phi'_e{}^B(t)} \psi_e''^B(x_e, T) \prod_{n=1}^N \psi_n'^B(x_n, T) \right). \tag{54}$$

The individual particle wave functions in Eq. (54) are of the form (40).

The probabilities of the two outcomes are

$$\begin{aligned}
 P_{\pm} &\equiv \int dx_e dx_1 |\Psi_{\pm}(x_1, x_2, T)|^2 \\
 &= \frac{1}{4} \left[2 \pm e^{i[\phi_e^A(T) - \phi_e^B(T)]} \int dx_e \psi_e^{\prime\prime A}(x_e, T) \psi_e^{\prime\prime B}(x_e, T) \prod_{n=1}^N \int dx_n \psi_n^{\prime A}(x_n, T) \psi_n^{\prime B}(x_n, T) + \text{c.c.} \right]. \quad (55)
 \end{aligned}$$

We now must perform the integrals in Eq. (55) using the expressions (40). The integral over x_e is readily performed, since the electron wave functions completely overlap at time T . The integrals over x_n are more complicated, since the wave functions do not overlap.

The factor associated with interference of each particle in Eq. (55) (leaving off the subscript and the primes) is

$$\begin{aligned}
 \int_{-\infty}^{\infty} dx \psi^A(x, T) \psi^{B*}(x, T) &\sim \int_{-\infty}^{\infty} dx e^{-i[x-x_{cl}^A(T)]^2/4\sigma^2} e^{-i[x-x_{cl}^B(T)]^2/4\sigma^2} e^{ip_{cl}^A(T)[x-x_{cl}^A(T)]} e^{-ip_{cl}^B(T)[x-x_{cl}^B(T)]} \\
 &\times \exp\left(i \int_0^T dt \frac{1}{2m} [p_{cl}^A(t)]^2 - i \int_0^T dt \frac{1}{2m} [p_{cl}^B(t)]^2\right) \\
 &\times \exp\left[-iq \left(\int_0^T dt [V(x_{cl}^A(t), t) - V(x_{cl}^B(t), t)]\right)\right]. \quad (56)
 \end{aligned}$$

We may now use

$$\begin{aligned}
 \int_{-\infty}^{\infty} dx e^{-[(x-a)^2/4\sigma^2]} e^{-[(x-b)^2/4\sigma^2]} e^{ip^1(x-a)} e^{-ip^2(x-b)} &= \int_{-\infty}^{\infty} dx e^{-(x^2/2\sigma^2)} e^{x(a+b)/2\sigma^2} e^{-(a^2/4\sigma^2)} e^{-(b^2/4\sigma^2)} e^{i(p^1-p^2)x} e^{-ip^1a} e^{ip^2b} \\
 &\sim e^{(\sigma^2/2)[(a+b)/2\sigma^2 + i(p^1-p^2)]^2} e^{-(a^2/4\sigma^2)} e^{-(b^2/4\sigma^2)} e^{-ip^1a} e^{ip^2b} \\
 &= e^{-[(a-b)^2/8\sigma^2]} e^{-[\sigma^2(p^1-p^2)^2/2]} e^{i[(p^1+p^2)(b-a)/2]}. \quad (57)
 \end{aligned}$$

The factor in Eq. (56) is then

$$\begin{aligned}
 \langle \psi^B, T | \psi^A, T \rangle &\equiv |\langle \psi^B, T | \psi^A, T \rangle| e^{i\Phi} \\
 &= \exp\left(-\frac{[x_{cl}^A(T) - x_{cl}^B(T)]^2}{8\sigma^2}\right) \exp\left(-\frac{\sigma^2[p_{cl}^A(T) - p_{cl}^B(T)]^2}{2}\right) \exp\left(i\frac{1}{2}[p_{cl}^A(T) + p_{cl}^B(T)][x_{cl}^B(T) - x_{cl}^A(T)]\right) \\
 &\quad + i\frac{1}{2m} \int_0^T dt \{[p_{cl}^A(t)]^2 - [p_{cl}^B(t)]^2\} - iq \int_0^T dt [V(x_{cl}^A(t), t) - V(x_{cl}^B(t), t)] \quad (58)
 \end{aligned}$$

For the electron, since $x_{cl}^A(T) = x_{cl}^B(T)$ and $p_{cl}^A(T) = p_{cl}^B(T)$, the magnitude in Eq. (58) is 1. The phase analysis is precisely the same as for either particle in the case of two particles, resulting in the phase difference we will call $\Phi_e^A - \Phi_e^B$. Here $\Phi_e^{A,B}$ is the negative time integral of the interaction Hamiltonian with all particle operators replaced by time-dependent classical variables for that trajectory (A or B). This cancels the extra phase factor $\phi_e^A(T) - \phi_e^B(T)$. Therefore, the net phase is that of the N particles.

The phase shift contributed by each particle, according to (58), is

$$\Phi_n = \frac{1}{2} [p_{ncl}^A(T) + p_{ncl}^B(T)][x_{ncl}^B(T) - x_{ncl}^A(T)] + \frac{1}{2m} \int_0^T dt \{[p_{ncl}^A(t)]^2 - [p_{ncl}^B(t)]^2\} - q \int_0^T dt [V(x_{ncl}^A(t), t) - V(x_{ncl}^B(t), t)]. \quad (59)$$

For the electron, the first term vanishes, but here it does not. The analysis is the same as previously, using the expressions (11) for the classical position and momentum in terms of the potentials, recalling that terms proportional to the square of the charges are disregarded:

$$\begin{aligned}
 \frac{1}{2} [p_{cl}^A(T) + p_{cl}^B(T)][x_{cl}^B(T) - x_{cl}^A(T)] &\approx \frac{1}{2} [2mv_0] \frac{q}{m} \int_0^T dt \int_0^t dt' [\mathcal{E}(x_{cl}^B(t'), t') - \mathcal{E}(x_{cl}^A(t'), t')], \\
 \frac{1}{2m} \int_0^T dt \{[p_{cl}^A(t)]^2 - [p_{cl}^B(t)]^2\} &= \frac{1}{2m} \int_0^T dt [p_{cl}^A(t) + p_{cl}^B(t)][p_{cl}^A(t) - p_{cl}^B(t)] \quad (60a)
 \end{aligned}$$

$$\begin{aligned}
 &\approx \frac{1}{2m} \int_0^T dt [2mv_0] \left(\frac{q}{c} [A(x_{cl}^A(t), t) - A(x_{cl}^B(t), t)] \right. \\
 &\quad \left. + q \int_0^t dt' [\mathcal{E}(x_{cl}^A(t'), t') - \mathcal{E}(x_{cl}^B(t'), t')] \right). \quad (60b)
 \end{aligned}$$

Adding (60a) and (60b), the double integral of the electric field vanishes, and putting the sum into (59) gives the phase contribution of the n th particle,

$$\begin{aligned}\Phi_n &= \int_0^T dt \frac{qv_{n0}}{c} [A(x_{ncl}^A(t), t) - A(x_{ncl}^B(t), t)] - q \int_0^T dt [V(x_{ncl}^A(t), t) - V(x_{ncl}^B(t), t)] \\ &\approx \int_0^T dt \frac{qv_{ncl}(t)}{c} [A(x_{ncl}^A(t), t) - A(x_{ncl}^B(t), t)] - q \int_0^T dt [V(x_{ncl}^A(t), t) - V(x_{ncl}^B(t), t)] \\ &= \int_0^T dt \frac{q\mathbf{v}_{ncl}(t)}{c} \cdot [\mathbf{A}(x_{ncl}^A(t), t) - \mathbf{A}(x_{ncl}^B(t), t)] - q \int_0^T dt [V(x_{ncl}^A(t), t) - V(x_{ncl}^B(t), t)].\end{aligned}\quad (61)$$

Therefore, the phase contributed by all N particles is

$$\begin{aligned}\Phi_N &\equiv \Phi_N^A - \Phi_N^B = \sum_{n=1}^N \Phi_n \\ &= \int_0^T dt \sum_{n=1}^N \left[\frac{q\mathbf{v}_{ncl}(t)}{c} \cdot \mathbf{A}(x_{ncl}^A(t), t) - qV(x_{ncl}^A(t), t) \right] - \int_0^T dt \sum_{n=1}^N \left[\frac{q\mathbf{v}_{ncl}(t)}{c} \cdot \mathbf{A}(x_{ncl}^B(t), t) - qV(x_{ncl}^B(t), t) \right].\end{aligned}\quad (62)$$

Thus, we see that the phase $\Phi_N^{A,B} = \Phi_e^{A,B} \equiv \Phi^{A,B}$ is the negative of the interaction Hamiltonian with operators replaced by their classical counterparts for the associated trajectory. Again we see the crucial effect of the extra phase: Since $\Phi^A - \Phi^B = -[\phi_e'^A(T) - \phi_e'^B(T)]$, we have for the probabilities (55), using (58),

$$P_{\pm} = \frac{1}{2} \left[1 \pm \prod_{n=1}^N e^{-\{[x_{ncl}^A(T) - x_{ncl}^B(T)]^2 / 8\sigma^2\}} e^{-\{\sigma^2 [p_{ncl}^A(T) - p_{ncl}^B(T)]^2 / 2\}} \cos(\Phi^A - \Phi^B) \right].\quad (63)$$

When we apply Eq. (63) to the magnetic AB effect, where the N particles comprise the solenoid, we will evaluate the magnitude of the interference term.

A few remarks are in order before we close this section.

Of course, the wave function of a quantized solenoid under the influence of the classical field of an electron (as done by Vaidman and verified by us in the next section) has a phase, since all wave functions have a phase; however, why should that phase be precisely the same as that acquired by the quantized electron moving in the classical potentials of the solenoid? We have answered that question here. At heart, it is due to the symmetry of the potentials under exchange of the electron and solenoid variables.

A concise way to see the equality of these phases is to note that the current of the electron is $\mathbf{J}_e(\mathbf{x}, t) = e\mathbf{v}_e(t)\delta(\mathbf{x} - \mathbf{x}_e(t))$, so the phase contribution of the electron that moved in the classical field of the N particles until time T may be expressed as

$$\Phi_e = \frac{1}{c} \int_0^T dt \int d\mathbf{x} \mathbf{J}_e(\mathbf{x}, t) \cdot \mathbf{A}_N(\mathbf{x}, t),$$

where $\mathbf{A}_N(\mathbf{x}, t)$ is the vector potential due to the N particles.² In addition, the current of the N particles is $\mathbf{J}_N(\mathbf{x}, t) = q \sum_{n=1}^N \mathbf{v}_n(t)\delta(\mathbf{x} - \mathbf{x}_n(t))$, so their phase acquired while moving in the classical field of the electron is

$$\Phi_N = \frac{1}{c} \int_0^T dt \int d\mathbf{x} \mathbf{J}_N(\mathbf{x}, t) \cdot \mathbf{A}_e(\mathbf{x}, t).$$

However, using integration by parts,

$$\begin{aligned}\Phi_e &= \frac{1}{c} \int_0^T dt \int d\mathbf{x} [-c\nabla^2 \mathbf{A}_e(\mathbf{x}, t)] \cdot \mathbf{A}_N(\mathbf{x}, t) = \frac{1}{c} \int_0^T dt \int d\mathbf{x} \mathbf{A}_e(\mathbf{x}, t) [-c\nabla^2 \cdot \mathbf{A}_N(\mathbf{x}, t)] \\ &= \frac{1}{c} \int_0^T dt \int d\mathbf{x} \mathbf{A}_e(\mathbf{x}, t) \cdot \mathbf{J}_N(\mathbf{x}, t) = \Phi_N.\end{aligned}$$

²In the Coulomb gauge, either the current or the transverse current $\mathbf{J}_{eT}(\mathbf{x}, t)$ may be put into this equation: $\int d\mathbf{x} \mathbf{J}_e(\mathbf{x}, t) \cdot \mathbf{A}_N(\mathbf{x}, t) = \int d\mathbf{x} \mathbf{J}_{eT}(\mathbf{x}, t) \cdot \mathbf{A}_N(\mathbf{x}, t)$. This is because $\mathbf{J}_e(\mathbf{x}, t) - \mathbf{J}_{eT}(\mathbf{x}, t)$ is a gradient and its contribution to the integral vanishes, since $\nabla \cdot \mathbf{A}_N(\mathbf{x}, t) = 0$.

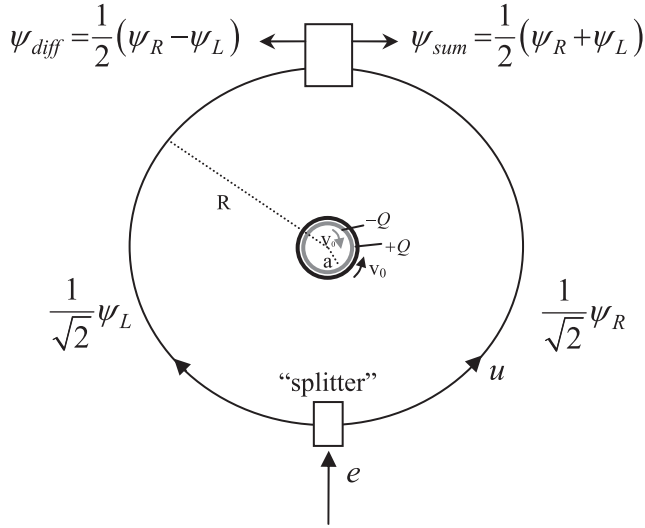


FIG. 1. Electron packet trajectories in the $z = 0$ plane and a cross section of the rotating cylindrical shells. The splitter boxes do not represent simply beam splitters but whatever optics is necessary to execute the electron behaviors explained in the text.

V. APPLICATION TO THE MAGNETIC AB EFFECT

We now apply these considerations to the Aharonov-Bohm magnetic effect, where we model the solenoid as a collection of N particles. As remarked in the Introduction, Vaidman demonstrated by a semiclassical calculation that the phase shift associated with the motion of the solenoid under the vector potential of the electric field of the electron is equal to the usual phase shift associated with the motion of the electron under the vector potential of the solenoid. Using our fully quantum calculation, we have confirmed this in the preceding section by a general argument. In this section we will show this by direct calculation. In addition, we will show that the magnitude of the interference term in Eq. (63) is essentially 1 for reasonable values of the parameters in our model.

A. Model

We suppose that the electron moves in from infinity to location $(x = 0, y = -R, z = 0) \equiv (R, \phi' = -\frac{\pi}{2}, z = 0)$ at time $t = 0$, where it is split by a beam splitter into two packets, each packet then circulating around the solenoid in a half circle (in the $z = 0$ plane) of radius R with speed u (Fig. 1). We choose the discontinuity in angle at $-\frac{\pi}{2} \leftrightarrow \frac{3\pi}{2}$. The right-side packet goes counterclockwise with angle $\phi'(t) = -\frac{\pi}{2} + \frac{ut}{R}$ and the left-side packet goes clockwise with angle $\phi'(t) = \frac{3\pi}{2} - \frac{ut}{R}$, with the packets meeting again after time $T = \frac{\pi R}{u}$ at $(R, \pi/2, z = 0)$ at a second beam splitter, from which the sum of packets emerges from one side and the difference from the other side.

We model the solenoid as comprised of two concentric cylindrical shells of nearly the same radius (which, for simplicity, we will consider as superimposed). Each cylinder consists of a stack of, say, square cross-sectioned hollow rings (see Fig. 2), the center of each ring at $(x = 0, y = 0, z)$, with the rings extending from $z = -L/2$ to $z = L/2$. Each ring is of radius a and height characterized by coordinate z . Each

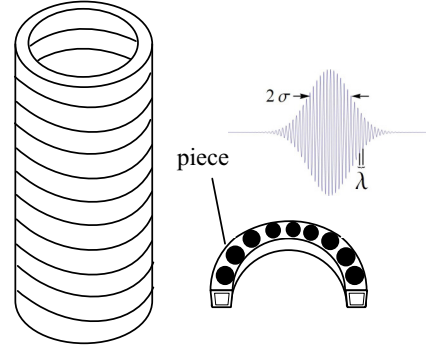


FIG. 2. Illustrating one of the two superimposed cylindrical shells comprising the solenoid, a hollow ring within the shell containing charged pieces, and the wave packet of a piece.

ring contains charged pieces, each of mass m and charge $\pm q$, moving with speed v_0 , each of localized extent and well-defined momentum so that our previous results apply. As the electron moves in, its Coulomb field alters the speed of these pieces as well as their density, but the current that is their product is unchanged.³

The charge may be written as

$$q = dz d\phi Q / 2\pi L, \quad (64)$$

where ϕ is the angular coordinate of a piece, Q (M) is the magnitude of the total charge (total mass) in one cylindrical shell, dz is the height of a ring, and $ad\phi$ is the length of each piece (see Fig. 3). The charge to mass ratio is $q/m = Q/M$ for each piece.

The positively charged pieces move counterclockwise, the negatively charged pieces move clockwise, with uniform surface charge density $\sigma_0 = Q/2\pi aL$, so there is linear current density $J_0 = Qv_0/2\pi aL$ in each ring. Thus, the magnetic field due to the two cylinders is

$$B_0 = 2 \frac{4\pi}{c} J_0 = 4 \frac{v_0}{c} \frac{Q}{aL}. \quad (65)$$

B. Magnitude of the interference term

One expects the magnitude to differ negligibly from 1. For the electron, in either the magnetic or the electric AB effect, when the packets recombine at time T , they are brought by the beam splitter to have the same position and momentum

³Suppose the electron wave packet moves in adiabatically from infinity to $(x = 0, y = -R, z = 0)$. Then one can neglect the vector potential of the electron and neglect \vec{E} . This leaves the electric field as the gradient of the scalar potential. It then follows from Maxwell's equation $\vec{B} = -c\nabla \times \vec{E} = 0$ that the magnetic field is unchanged. Thus, we see from $\vec{J} = (c/4\pi)\nabla \times \vec{B}$ that $\vec{J} = 0$, so the current is unchanged, $J = J_0$. Thus, the current associated with the n th piece $J_0 = \sigma_n v_n$ is unchanged, although both the surface charge density σ_n and velocity v_n have been changed from σ_0 and v_0 . One may further show that $v_n(\phi) \approx v_0 + \frac{qeaR \sin \phi}{mv_0 D^3}$. Thus, we may take the particle densities and velocities as having the constant values σ_0 and v_0 since the effect of the correction term $\sim q$ upon the phase shift is $\sim q^2$, which may be ignored.

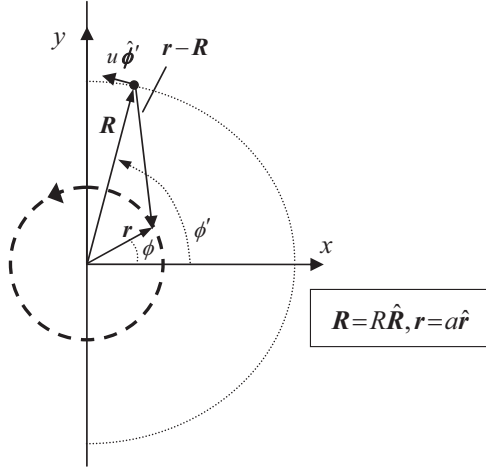


FIG. 3. Illustration of the notation.

$x_{ecl}^1(T) = x_{ecl}^2(T)$ and $p_{ecl}^1(T) = p_{ecl}^2(T)$, which we have seen implies that the electron contributes a factor 1 to the magnitude.

For the electric AB effect, as discussed in Sec. VII and Appendix D, the capacitor plate dynamics is that of a single particle (i.e., the plate wave function describes its center of mass behavior) and when the electron is brought to interfere, the two plates then overlap almost completely in position and momentum and so their contribution to the magnitude is ≈ 1 .

The solenoid in the magnetic AB effect requires much more discussion. We see from (63) that there is a dependence upon both position and momentum for each piece. Qualitatively, the AB shift is shared among a large number of pieces (as we will call the charged particles making up the solenoid). Each piece's wave function then has such a small relative shift (for the electron packet's two traverses) in position and momentum that the magnitude of overlap is ≈ 1 . In terms of de Broglie waves, this corresponds to the need for the de Broglie waves to overlap spatially almost completely, to shift very slightly, and to have nearly the same frequency to get maximal interference.

In our model, suppose each ring of radius a contains n_a pieces of wave packet size σ moving clockwise with speed v_0 , each with n_e electrons. Each ring also contains n_a similar sized pieces moving counterclockwise with speed v_0 , each with n_e "positive electrons." We note that $\sigma = 2\pi a/n_a$ and the mass of a piece is $m = n_e m_e$, where m_e is the mass of an electron. The solenoid is of length L and contains $n_L \equiv L/\sigma$ rings. The number of pieces is $N = 2n_L n_a$.

We will give here an example to illustrate how, for reasonable values of the parameters, the sum of all the exponents in Eq. (63) corresponding to all N pieces in the solenoid can be negligibly small. Say a is of order 1 cm, R is of order 10 cm, L is of order 100 cm, v_0 is of order 1 cm/s (a typical drift velocity of electrons in a conductor), and u is of order 1 m/s. In terms of the total number of positive or negative electron charges $N_e = n_e n_a n_L$, the AB shift is $\Phi_{AB} = 4\pi N_e \frac{e^2 v_0}{\hbar c} \frac{a}{L}$ [see Eq. (67), below]. Say the setup gives a shift $\Phi_{AB} = \pi$. Then $N_e \approx 10^{14}$ electrons.

The strongest constraint on n_a is the condition for our model that $\lambda \equiv \hbar/mv_0 \ll \sigma$, i.e., that a solenoid piece wave packet contain many wavelengths so that its momentum

is well defined. Using $m = n_e m_e = \frac{N_e}{n_a n_L} m_e = \frac{N_e}{n_a} \frac{L}{\sigma} m_e$ and $\sigma = 2\pi a/n_a$, this constraint becomes $n_a^3 \ll N_e \frac{m_e v_0 a}{\hbar} \frac{2\pi a}{L} \approx 6 \times 10^{12}$, which we satisfy by choosing $n_a \approx 1000$ pieces per ring.

This determines m and so makes $\lambda \approx 10^{-6}$ cm and determines $\sigma \approx 6 \times 10^{-3}$ cm, so there are ≈ 6000 wavelengths in a wave packet. Also, then there are $n_e \approx 6 \times 10^6$ electrons per piece, $n_L \approx 1.5 \times 10^4$ rings in the solenoid, and $n_p = 1.5 \times 10^7$ pieces in the solenoid.

For this estimate, we will treat each piece of the solenoid as participating equally in the phase shift, even though the pieces in the ring in the plane of the electron's motion, being closer to the electron, can receive a larger force and therefore displacement and also pieces in the same ring are displaced differently. Since the AB shift $\Phi_{AB} = \pi$ corresponds to a total shift of all pieces by a distance $\lambda/2 \approx 5 \times 10^{-7}$ cm, each individual piece shifts by a distance $\lambda/2n_p \approx 3 \times 10^{-14}$ cm. The squared fractional displacement appearing in the exponent of the first term in Eq. (63) for one piece is therefore $\frac{[x_{cl}^1(T) - x_{cl}^2(T)]^2}{8\sigma^2} \approx 3 \times 10^{-24}$ and for all $n_p = 1.5 \times 10^7$ pieces the spatial displacement total exponent is $\approx 10^{-16}$.

In order to achieve the displacement of $\approx 3 \times 10^{-14}$ cm in the time $T = \pi R/u \approx 0.3$ s it takes the electron packets to complete their traverse, the relative speed change of the pieces is $\Delta v \approx 10^{-13}$ cm/s. Therefore, since $\sigma[p_{cl}^1(T) - p_{cl}^2(T)]T/\hbar \approx \sigma m \Delta v/\hbar$, using $m \approx n_e m_e$, we have $\sigma[p_{cl}^1(T) - p_{cl}^2(T)]T/\hbar \approx \times 10^{-8}$. Thus, for one piece the momentum contribution to the exponent in Eq. (63) is therefore $\approx 0.5 \times 10^{-16}$ and for all n_p pieces the momentum-dependent total exponent is $\approx 10^{-9}$. We conclude that the magnitude in Eq. (63) is 1 to high accuracy.

C. Direct calculation of the phase shift

The phase shift of the electron moving in the vector potential of the solenoid, the well-known calculation of the AB phase shift, is

$$\begin{aligned} \Phi_{AB} &= \Phi_e^A - \Phi_e^B \\ &= \frac{1}{c} \int_0^T dt euA(R) - \frac{1}{c} \int_0^T dt e(-u)[-A(R)] \\ &= 2T \frac{e}{c} u \frac{a^2 B_0}{2R} = \frac{e}{c} \pi a^2 B_0, \end{aligned} \quad (66)$$

where $A(R) = \frac{a^2 B_0}{2R}$ is magnitude of the solenoid's vector potential, which is parallel to the electron velocity for trajectory A and antiparallel for trajectory B, and we have used $uT = \pi R$. In the preceding section's discussion of the magnitude of the interference term, we used the expression

$$\Phi_{AB} = \frac{e}{c} \pi a^2 B_0 = \frac{e}{c} \pi a^2 \left[4 \frac{v_0 N_e e}{c a L} \right] \frac{1}{\hbar}, \quad (67)$$

where B_0 comes from (65), $Q = N_e e$, and the correct factor of \hbar has been inserted.

Now we explicitly calculate the phase contribution of the pieces of the solenoid due to the field of the electron. We will do this in both the Lorenz gauge and the Coulomb gauge. At time t , the electron is at the location $(R, \phi'(t), 0)$, where $\phi'(t) = -\pi/2 + ut/R$, and the solenoid piece is at $(a, \phi_n(t), z_n)$, where $\phi_n(t) = \phi(0) + v_0 t/a + o(q)$. We identify each piece, labeled by the (discrete) index n , as we take the continuum limit, by the continuous variables $\phi(0)$ describing its initial angular value and its z value. According to (32), the vector potential caused by the electron at the location of the piece in the solenoid at time t is

$$\mathbf{A}(\phi_n(0), z_n, t) = \frac{e}{c} \frac{\mathbf{u}(t)}{|\mathbf{r} - \mathbf{R}|}, \quad (68a)$$

$$\mathbf{A}(\phi_n(0), z_n, t) = \frac{e}{2c} \frac{\mathbf{u}(t)}{|\mathbf{r} - \mathbf{R}|} + \frac{e}{2c} [\mathbf{r} - \mathbf{R}] \frac{\mathbf{u}(t) \cdot [\mathbf{r} - \mathbf{R}]}{|\mathbf{r} - \mathbf{R}|^3} \quad (68b)$$

in the Lorenz and Coulomb gauges, respectively. Here $\mathbf{r} = \mathbf{r}(a, \phi_n(t), z_n)$ and $\mathbf{R} = \mathbf{R}(R, \phi'(t), 0)$.

First we treat the Lorenz gauge. We consider the counterclockwise traverse of the electron (path A) and the phase it gives to the positively charged pieces of the solenoid. The contribution to the phase for the n th piece is

$$\begin{aligned} \Phi_n^A(z_n, \phi_n(0), T) &= \int_0^T dt \frac{q}{c} v_0 \hat{\phi}(t) \cdot \mathbf{A}(z, t) \\ &= \frac{v_0 e q u}{c^2} \int_0^T dt \frac{\cos(\phi_0 + \frac{v_0 t}{a} + \pi/2 - \frac{ut}{R})}{\sqrt{R^2 + z^2 + a^2 - 2aR \cos(\phi_0 + \frac{v_0 t}{a} + \pi/2 - \frac{ut}{R})}}, \end{aligned} \quad (69)$$

where we have used $\hat{\phi}(t) \cdot \mathbf{u}(t) = u \hat{\phi}(t) \cdot \hat{\phi}'(t) = u \cos[\phi(t) - \phi'(t)]$ and $\mathbf{r} \cdot \mathbf{R} = aR \cos[\phi(t) - \phi'(t)]$. We now use (64) to write $q = dz d\phi_0 Q/2\pi L$ and sum over all pieces, i.e., integrate over ϕ_0 and z . Since ϕ_0 is integrated over a 2π range, the argument of the cosine can be replaced by an angle θ integrated over that range. Then we assume that the solenoid length $L \gg a, R$, which simplifies the integral over z ,

$$\begin{aligned} \Phi_{\text{sol}}^A &= \frac{v_0 Q e u}{2\pi L c^2} \int_0^T dt \int_0^{2\pi} d\theta \int_{-L/2}^{L/2} dz \frac{\cos \theta}{\sqrt{R^2 + z^2 + a^2 - 2aR \cos \theta}} \\ &\approx \frac{v_0 Q e u T}{2\pi L c^2} \int_0^{2\pi} d\theta \cos \theta [2 \ln L - \ln(R^2 + a^2 - 2aR \cos \theta)] \\ &= \frac{v_0 Q e u T}{2\pi L c^2} \int_0^{2\pi} \sin \theta d \ln[R^2 + a^2 - 2aR \cos \theta] \\ &= \frac{v_0 Q e u T}{2\pi L c^2} 2aR \int_0^{2\pi} d\theta \frac{\sin^2 \theta}{R^2 + a^2 - 2aR \cos \theta} \\ &= \frac{v_0 Q e u T}{2\pi L c^2} 2aR \frac{1}{R^2} \int_0^{2\pi} d\theta \frac{\sin^2 \theta}{1 + (a/R)^2 - 2a/R \cos \theta} \\ &= \frac{v_0 Q e u T}{2\pi L c^2} 2aR \frac{1}{R^2} \pi = \frac{e}{4c} \pi a^2 B_0, \end{aligned} \quad (70)$$

where the third line is obtained by integrating by parts, the last integral over θ is π if $a/R < 1$, which it is, and we have used $uT = \pi R$ and the expression (65) for B_0 .

The result (70) is 1/4 the AB shift (66). The negatively charged rings give the same result ($Q \rightarrow -Q, v_0 \rightarrow -v_0$), so traverse A gives 1/2 the AB shift. The counterclockwise traverse also gives 1/2 the AB shift ($u \rightarrow -u$ and the B traverse phase is subtracted). Thus we get the full AB shift.

For the Coulomb gauge, the first term in the vector potential expression (68b) is 1/2 the Lorenz gauge potential (68a), so this term gives 1/8 the AB shift. Using $\hat{\phi}(t) \cdot [\mathbf{r} - \mathbf{R}] = -\hat{\phi}(t) \cdot \mathbf{R} = R \sin[\phi(t) - \phi'(t)]$ and $\mathbf{u}(t) \cdot [\mathbf{r} - \mathbf{R}] = \mathbf{u}(t) \cdot \mathbf{r} = ua \sin[\phi(t) - \phi'(t)]$, the second term in Eq. (68b) gives the shift, for large L ,

$$\begin{aligned} \Phi_{\text{sol}}^A &= \frac{1}{2} \frac{v_0 Q e u a R}{2\pi L c^2} \int_0^T dt \int_0^{2\pi} d\theta \int_{-L/2}^{L/2} dz \frac{\sin^2 \theta}{[R^2 + z^2 + a^2 - 2aR \cos \theta]^{3/2}} \\ &\approx \frac{v_0 Q e u T a R}{4\pi L c^2} \int_0^{2\pi} d\theta \frac{2 \sin^2 \theta}{R^2 + a^2 - 2aR \cos \theta}. \end{aligned} \quad (71)$$

This is 1/2 the result given by the last term on the third line in Eq. (70), i.e., it is 1/8 of the AB shift, so both vector potential terms combined in the Coulomb gauge give 1/4 the AB phase shift.

D. Time-average approach to understanding the contribution of a solenoid piece to the phase shift

Here is a simple, intuitively appealing, approach that gives the result (70). Suppose each piece of the solenoid moves slowly compared to the speed of the electron over the interval T . Then one might consider the approximation of replacing the vector potential created by the electron at the site of a piece, during the piece's traverse along its own path, by the time-averaged potential it experiences.

This averaging technique will also apply even if the speed of a solenoid pieces is not slow compared to the electron speed, as in the preceding section. Because one piece of the solenoid replaces another during their motion, a piece will always be present at a given place in the solenoid and thus that place can be considered to experience an average vector potential and we can add the contribution from all the places. This approach starts with the following Hamiltonian for a piece:

$$H = \frac{1}{2m} \left(\hat{p} - \frac{q}{c} \langle A \rangle_T \right)^2, \tag{72}$$

where $\langle A \rangle_T \equiv \frac{1}{T} \int_0^T dt A(t)$ and $A(t)$ is the component of the electron's vector potential along the path of the solenoid piece. Because $\langle A \rangle_T$ is a constant, it immediately follows (as is usual when treating the phase contributed by the electron's motion) that the phase \mathcal{P} contributed by the piece is the path integral of the vector potential:

$$\mathcal{P}(T) = \int_{\text{path}} ds \frac{q}{c} \langle A \rangle_T = v_0 T \frac{q}{c} \langle A \rangle_T. \tag{73}$$

However, this is precisely the expression given in Eq. (40) for the phase contributed by the motion of a piece that, when summed over all positively charged pieces as in the previous section, gives 1/4 the AB phase. Including, in a similar way, the contribution of the negatively charged pieces and the contribution of the left traverse, gives the full AB shift. Thus, the phase shift can be viewed as due to the action of the time-averaged electric-field-producing vector potential of the electron acting on the charged pieces of the solenoid as they travel their short paths, the small arcs of the ring in which they are constrained to move. A more detailed analysis from this point of view appears in Appendix C.

E. Electric force in the magnetic AB effect

We have shown that the standard AB calculation and the solenoid-involved calculation are two alternative ways of calculating the same thing. It is remarkable how conceptually dissimilar they are. In the usual case, there is the situation of a non-field-producing potential causing the phase shift. We would like to point out in this section, in the other case, although we have expressed the phase shift in terms of the vector potential, that it can be looked at as due to the electric force of the electron on the solenoid pieces, the view that Vaidman took in his semiclassical calculation [5].

It may be noted that Vaidman reasoned the existence of the electric force on the solenoid using the Faraday law. Faraday's law encompasses two distinct effects (see, e.g., Ref. [8]), one dealing with the electromotive force (EMF) generated in a conductor traveling through a magnetic field, the other dealing

with the EMF generated when there is a changing magnetic field inside a closed path. Vaidman's analysis uses the latter. He reasoned that, when the electron accelerates from rest and begins its traverse, it produces a changing magnetic flux in the solenoid. Thus, he obtained the electric field induced in the solenoid (since its integral over the solenoid circumference is the EMF), whose force changes the speed of the charged solenoid cylinders. This expression of the second Faraday law effect is equivalent to noting that the accelerating electron causes a time-changing vector potential, whose (negative) time derivative is this same electric field.

Consider, for example, the effect of the electron's vector potential, on the positively charged solenoid pieces, during its right traverse. We will suppose that the electron starts from rest so that its initial vector potential is zero (we ignore the solenoid's self-vector potential). We take the time dependence of the electron's vector potential at the site of each solenoid piece to be just due to the electron's motion and not the piece's motion, i.e., $\mathbf{A}(\mathbf{r}_n, t)$ at the site \mathbf{r}_n . This is because the replacement of solenoid pieces by successive pieces means we need only label the pieces' locations and ignore their translation. Then the phase associated with these solenoid pieces may be written as

$$\begin{aligned} \Phi^A &= \int_0^T dt \sum_n \frac{q}{c} \mathbf{v}_n \cdot \mathbf{A}(\mathbf{r}_n, t) \\ &= - \sum_n \int_0^T d\mathbf{x}_n(t) \cdot \int_0^t dt' \left[- \frac{\partial}{\partial t'} \frac{q}{c} \mathbf{A}(\mathbf{r}_n, t') \right] \\ &= - \sum_n \int_0^T d\mathbf{x}_n(t) \cdot \int_0^t dt' \mathbf{F}_n(\mathbf{r}_n, t'). \end{aligned} \tag{74}$$

We have written $\mathbf{v}_n dt = d\mathbf{x}_n$ as the displacement of the n th piece during dt . We have noted that the bracketed expression is the electric force $\mathbf{F}_n(\mathbf{r}_i, t')$ exerted by the electron on the n th piece in the interval dt' . Thus, this phase-shift contribution has been expressed in terms of $dt' \mathbf{F}_n(\mathbf{r}_n, t') = dp_n(t')$, the impulse exerted during dt' by the electron on the solenoid pieces.

If we consider that the electron rapidly accelerates to speed u at the beginning of its trajectory, since the vector potential is proportional to $u(t)$, there is a large initial impulse (followed, as the electron continues its traverse, by a force on each solenoid piece that averages out over all pieces to zero), causing a sudden initial change of the momentum $\Delta \mathbf{p}_n$ of the positively charged pieces (and the opposite change in momentum for the negatively charged pieces), so $\Phi^A = - \sum_n \int_0^T d\mathbf{x}_n(t) \cdot \Delta \mathbf{p}_n$.

This impulse-induced differential of momentum, occurring oppositely for the electron's left traverse, causes the phase shift to accumulate over the interval T , which was the point of view taken by Vaidman in his semiclassical calculation. We can see that here by writing $\Delta p_n = mv_n - mv_0$, which is constant and close to parallel to the piece displacement $\int_0^T d\mathbf{x}_n \equiv \Delta \sigma_n$ over time T . Tossing out the term mv_0 , which does not contribute to the AB shift as it is the same for left and right traverses of the electron, we have the effective $\Phi^A = - \sum_n \Delta \sigma_n mv_n = \sum_n \Delta \sigma_n / \lambda_n$, where λ_n is the de Broglie wavelength for the piece.

VI. APPLICATION TO THE ELECTRIC AB EFFECT

Finally, we turn to discuss the electric AB effect. This has precisely the same dichotomy as the magnetic AB effect, two different ways of viewing and calculating the AB phase shift. However, here it can be shown more simply (because the Coulomb interaction is a direct particle interaction, not an interaction mediated by an intermediate field as is the interaction through the vector potential) precisely how this involves two alternative views of the identically same calculation.

Using a simplified model, where the electron sits in a zero-field environment halfway between two identical point charges, again by a semiclassical argument, Vaidman [5] showed that, while the AB shift is obtainable as due to the electron’s motion in the field of the charges, it can also be obtained as due to the motion of the charges in the electric field of the electron.

Instead of Vaidman’s simplified model, we will apply our considerations to the usually discussed electric AB effect setup, where electron packets passing over and under a charged capacitor are brought together to interfere. According to the usual view, the phase shift $\Phi_{AB} = -e\sigma DT$ (e is the charge on the electron, σ is the surface charge on the plates, D is the plate separation, and T is the time of traverse of the electron; in this section we use rationalized mks units) is due to the quantized electron moving in the constant potential of the classical capacitor. We will show that the same phase shift is obtained by consideration of the quantized capacitor plates moving in the electric field of the classical electron.

However, we do more. We consider what happens when both electron and plates are quantized. The Hamiltonian is separable, the wave function is the product of wave functions for the electron and plates, and we show how the phase shift of this product wave function, the AB electric phase shift, can be viewed as either due to the electron motion or due to the capacitor plate motion, alternative ways of calculating the same thing.

A. Model

We consider two very large plates containing charges glued down, with uniform charge densities $\pm\sigma$ and area A , lying initially on top of one another in the x - y plane at $z = 0$ (see Fig. 4). There are two long hollow frictionless tubes parallel to the x axis, one at $(y = 0, z = d)$ and the other at $(y = 0, z = -d)$, with the entrance and exit of each tube at the plate edges. From the left, an electron approaches a beam splitter that creates two wave packets of equal amplitude., following which two mirrors direct the packets into the tubes.

After the packets have traveled well away from the capacitor edges, the two plates are separated, rapidly moved by an external field (over a time interval much shorter than T) so that the positive and negative plates are left at rest at $\pm D/2$ at time $t = 0$. The plates are free to move thereafter for a time interval T , but it is assumed that they are so massive that their displacement is relatively small compared to D over that interval.

At time T , while the electron is still far from the capacitor edge, the plates are rapidly returned to the plane $z \approx 0$ by an

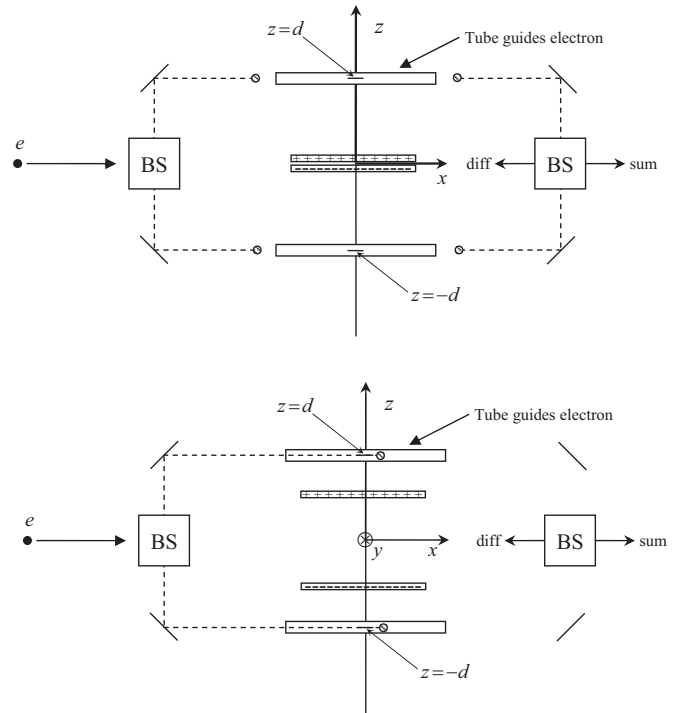


FIG. 4. Shown on top is the setup for electric AB effect, when the plates are together, before and after the electron nears the center of the plates. Here BS denotes beam splitter. (The left beam splitter’s output differs from the right beam splitter’s output, as illustrated.) On the bottom is an illustration of the plates with separation $\approx D$ while the electron moves for a time interval T . (Figures are not to scale; the plates and tubes should be greatly stretched in the $\pm x$ direction.)

external field. (While the plates acquire a phase shift during their separation and rejoining motions, we may take the time scale to be brief enough that it may be neglected compared to the phase shift acquired during the interval T .) Eventually the electron reaches the edge of the plates and, by means of two more mirrors reflecting the packets toward each other and another beam splitter, the packets emerge with their sum in one direction and their difference in another direction, each direction containing a detector.

The Hamiltonian over the time interval $(0, T)$ for electron passage above (H_+) or below (H_-) the capacitor is

$$H_{\pm} = \frac{P_e^2}{2m} + \frac{P_U^2}{2M} + \frac{P_L^2}{2M} + \frac{1}{2}[\sigma^2 A \pm e\sigma](Z_U - Z_L). \quad (75)$$

Here (X_e, P_e) , (Z_U, P_U) , and (Z_L, P_L) are the conjugate operators for the electron motion in the x direction and upper plate and lower plate motion in the z direction, respectively, and A is the plate area. This gives the force on each capacitor plate as due to the other plate and the electron and of course no force on the electron.

B. Calculations

One can take the point of view that the capacitor is a classical object and only the electron should be quantized. Then one may truncate this Hamiltonian by throwing away the plate potential energy and kinetic energy operator terms and setting $Z_U = D/2$ and $Z_L = -D/2$, their classical positions

to excellent accuracy. The solution of Schrödinger's equation with the Hamiltonian

$$H_{\pm\text{electron}} = \frac{P_e^2}{2m} + \frac{1}{2}[\pm e\sigma]D \quad (76)$$

is easily seen to have the form

$$\psi_{\pm}(x, T) = \psi'(x, T)e^{-i(1/2)[\pm e\sigma]DT}, \quad (77)$$

where $\psi'(x, T)$ is a free-particle wave function [this is of course what is obtained from Eq. (8) or Eqs. (13) and (30) applied to this problem]. Here $\psi_+(x, T)$ [$\psi_-(x, T)$] is the wave function for the electron packet that passes above (below) the capacitor plates. By inspection, the electric AB phase shift, the difference of phases for the two traverses, is seen to be $-e\sigma DT$.

However, one may rewrite the Hamiltonian (75) as (omitting the self-energy of the capacitor since it makes the same contribution to the phase for the upper and lower electron traverses)

$$H_{\pm} = \frac{P_e^2}{2m} + \left[\frac{P_U^2}{2M} \pm \frac{1}{2}e\sigma Z_U \right] + \left[\frac{P_L^2}{2M} \mp \frac{1}{2}e\sigma Z_L \right]. \quad (78)$$

In this form, the wave function is the product of the free-electron wave function, associated with no phase shift, and the upper and lower plate wave functions, which do have associated phase shifts. We use our solution (8) in Sec. II [or Eqs. (13) and (30) in Sec. III] of the problem of finding the wave function for one-dimensional motion in an electric potential to obtain the wave function in terms of the classical variables:

$$\begin{aligned} \psi_{\pm}(x, z_U, z_L, T) \\ = \psi_{el}(x) N e^{-[(z_U - D/2)^2/4\sigma^2]} e^{\mp i(1/2)e\sigma T[z_U - D/2]} e^{\mp i(1/4)eD\sigma T} \\ \times e^{-[(z_L + D/2)^2/4\sigma^2]} e^{\pm i(1/2)e\sigma T[z_L + D/2]} e^{\mp i(1/4)eD\sigma T}. \end{aligned} \quad (79)$$

In obtaining (79) we have used $|p_{U,L,cl}(T)| = \frac{1}{2}e\sigma T$, $z_{U,cl}(T) \approx D/2$ and $z_{L,cl}(T) \approx -D/2$, and e.g., for the upper traverse situation, $-\int_0^T dt' V(z_U(t')) = -\int_0^T dt' \frac{e\sigma}{2} \frac{D}{2} = -\frac{1}{4}eD\sigma T$.

Because of the narrowness of the Gaussian wave packets in Eq. (79), $z_U \approx D/2$ and $z_L \approx -D/2$, so it is only the last factor in each wave function expression that contributes to the phase shift. For the upper traverse, from ψ_+ , the contribution of both plates is then $2(-\frac{1}{4}eD\sigma T) = -\frac{1}{2}eD\sigma T$. Adding the contribution of the lower traverse gives $4(-\frac{1}{4}eD\sigma T) = -eD\sigma T$.

It might seem from this solution that we have settled the question as to what causes the phase shift: Here the phase shift is completely due to the capacitor. However, one can also write this Hamiltonian as

$$\begin{aligned} H_{\pm} = \left[\frac{P_e^2}{2m} \pm \frac{1}{2}e\sigma D \right] + \left[\frac{P_U^2}{2M} \pm \frac{1}{2}e\sigma \left(Z_U - \frac{D}{2} \right) \right] \\ + \left[\frac{P_L^2}{2M} \mp \frac{1}{2}e\sigma \left(Z_L + \frac{D}{2} \right) \right]. \end{aligned} \quad (80)$$

In this case, the phase shift belongs completely to the electron part of the wave function and the capacitor plate part has no phase shift associated with it [the wave function for each plate is the wave function in Eq. (79) with additional phase factors

that exactly cancel the phase that arose from the dynamics]. Alternatively, one could add and subtract a constant term in such a way that any fraction of the phase shift belongs to the electron wave function and the rest belongs to the capacitor. It is clear that these are alternative ways of calculating the same phase shift, with no reason apparent here to prefer one over the other.

C. Electric force in the electric AB effect

In the example just discussed, the phase shift in both cases was calculated using potentials. In one case there is the electron in the non-field-producing potential of the capacitor. In the other case there are the capacitor plates in the electric field of the electron.

The conceptual contrast, no force vs force, would be even greater if the phase-shift calculation for the capacitor plates was expressed not in terms of the potential, but in terms of the electron's electric force on the capacitor plates. This could not be done in the above example because the external force, required to separate the plates and reunite them, intrudes.

Accordingly, we consider a modified problem such that an external force does not act. Instead, we suppose that, at $t = 0$, the plates that coincide at $z = 0$ are suddenly given velocities $\pm v_0$ in the z direction. They then move, under their mutual forces and the force of the electron, to a maximum separation, which we will call D_{\pm} (D_+ for the upper traverse and D_- for the lower traverse), and fall back to $z = 0$. Taking e to be positive for definiteness, then $D_+ < D_-$. Therefore, for the upper traverse, the plates return to $z = 0$ at time T_+ , which is slightly earlier than the time of return T_- for the lower electron traverse.

The plates encounter a decelerating potential at $z \approx 0$ designed to slow them down to a near stop and keep them close to each other (an idea introduced by Vaidman [5] in his example of the electric AB effect) until after the electron is finally detected. Since the potentials are $\sim z$ and $z \approx 0$, there is negligible phase-shift contribution thereafter. The width of the wave packets is chosen to be large enough that the positive-plate wave packets from the two traverses finally overlap almost completely (similarly of course for the negative plate wave packets), which is necessary if there is to be maximal interference.

The calculation is governed by the same Hamiltonian we have already discussed and the details of that calculation are in Appendix D. However, they are not essential to the point being made here. We illustrate it by considering the phase contributed by the positively charged plate's motion in the electric field of the electron undergoing the upper traverse using Eq. (39), namely,

$$\begin{aligned} \Phi &= -\int_0^T dt V(z_U(t)) = -\int_0^T dt \int_0^t dt' \frac{d}{dt'} V(z_U(t')) \\ &= -\int_0^T dt \int_0^t dt' \frac{dz_U(t')}{dt'} \frac{d}{dz_U(t')} V(z_U(t')) \\ &= \int_0^T dt \int_0^t dz_U(t') F(z_U(t')) \end{aligned} \quad (81)$$

[we have utilized $V(0) = 0$], where $F(z_U(t))$ is the force on the capacitor plate exerted by the electron. Thus, this phase

contribution has been expressed in terms of the work done by the electron on the upper plate.

VII. CONCLUDING REMARKS

Here is a summary of the results in this paper:

(i) We obtained an exact solution for the magnitude and phase of a localized wave packet describing a particle moving in a vector and scalar potential corresponding to non-spatially-varying forces.

(ii) We calculated an approximate solution for the magnitude and phase of a particle moving in an *arbitrary* vector and scalar potential.

(iii) We attained an approximate solution of the problem where both the electron and solenoid are quantized, employing a variational technique, obtaining Schrödinger equations for the electron and solenoid, showing how an *extra phase* arises.

(iv) We applied (ii) and (iii) to an interference experiment, where one localized particle interacts with a second, or interacts with $N > 1$ other localized particles, obtaining the magnitude and phase shift of the interference term.

(v) We obtained a fully quantum mechanical verification of Vaidman's semiclassical argument that the electron acting on the quantized solenoid particles gives the magnetic AB phase shift.

(vi) We answered the vexing question of why this does not result in twice the usual magnetic AB phase shift, the phase shift from the solenoid acting on the electron added to the equal phase shift from the electron acting on the solenoid. The

answer lies in (ii)–(iv): For the problem of the jointly quantized electron and solenoid, the extra phase provides the negative of these phase shifts.

(vii) We developed a time-average approach to understanding the contribution of a solenoid piece to the phase shift that gives simple intuition into the nature of the buildup of the phase in the magnetic AB effect.

(viii) We devised a treatment of the electric AB effect, the exactly soluble problem of interference when a quantized electron interacts with quantized capacitor plates. We verified Vaidman's semiclassical argument that the electric AB phase shift arises from the electron exerting forces on quantized capacitor plates. We showed how an extra phase is responsible for providing the electric AB phase shift and not twice that value.

We have seen, in both effects, two conceptual pictures, that the AB phase shift arises from the electron moving in non-field-producing potentials and that the AB phase shift arises from the electric force exerted by the electron (on the solenoid particles or on the charged capacitor plates) as was argued by Vaidman. As far as our examples are concerned, there is no reason to prefer one of these conceptual pictures over the other.

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APPENDIX A: WAVE FUNCTION IN SEC. II

In this Appendix we do the algebra involved in obtaining the wave function for a charged object, moving on a one-dimensional path, subject to a time-dependent electric field.

1. Wave function in the momentum representation

Putting the momentum space wave function (6),

$$\psi(p, t) \sim e^{-(\sigma^2 + it/2m)p^2 + \beta(t)p + i\gamma(t) - \beta_R^2(t)/4\sigma^2}, \quad (\text{A1})$$

with initial conditions $\gamma(0) = 0$ and $\beta(0) = 2p_0\sigma^2 - ix_0$ into the momentum space Schrödinger equation (4) and equating coefficients of p and 1 (the coefficient of p^2 agrees on both sides of the equation) yields

$$\begin{aligned} i \frac{d}{dt} \beta &= -\frac{q}{mc} A - 2iqV'\sigma^2 + \frac{qt}{m} V' \\ -\frac{d}{dt} \gamma &= \frac{i}{2\sigma^2} \beta_R \frac{d}{dt} \beta_R + \frac{q^2}{2mc^2} A^2 + iqV'\beta + qg(t) \\ &= \frac{i\beta_R}{2\sigma^2} \left[\frac{d}{dt} \beta_R + qV'2\sigma^2 \right] + \frac{q^2}{2mc^2} A^2 - qV'\beta_I + qg(t) = \frac{q^2}{2mc^2} A^2 - qV'\beta_I + qg(t), \end{aligned} \quad (\text{A2})$$

where the bracketed term in the penultimate equation vanishes because of the imaginary part of the first equation. Solving for β and putting in the initial conditions, results in

$$\beta_R = 2\sigma^2[p_0 - qU] = 2\sigma^2 p_{cl}(t), \quad \beta_I = -x_0 + \frac{q}{m}[A/c - W]. \quad (\text{A3})$$

Now note that

$$x_{cl}(t) = x_0 + \frac{p_0}{m}t - \frac{q}{mc}A - \frac{q}{m}[tU(t) - W(t)] = \frac{t}{m}(p_0 - qU) - \left[-x_0 + \frac{q}{m}(A/c - W) \right] = \frac{t}{2m\sigma^2}\beta_R - \beta_I, \quad (\text{A4})$$

from which follows

$$\beta_I = -x_{cl}(t) + \frac{p_{cl}(t)t}{m}. \quad (\text{A5})$$

In Eq. (A2) for γ , dropping terms quadratic in the potentials leaves only

$$-\frac{d}{dt}\gamma = qx_0V'(t) + qg(t) = qV(t) - v_0tV'(t) \quad \text{or} \quad \gamma = -q \int_0^t dt' V(t') + v_0W(t), \quad (\text{A6})$$

where we have used $g(t) \equiv V(t) - (x_0 + v_0t)V'(t)$.

2. Wave function in the position representation

The Fourier transform of (6) is

$$\begin{aligned} \psi(x,t) &\sim \int dp e^{ipx} e^{-(\sigma^2+it/2m)p^2+\beta(t)p+i\gamma(t)-\beta_R^2(t)/4\sigma^2} \\ &= e^{(ix+\beta)^2/4[\sigma^2+it/2m]} e^{i\gamma} e^{-\beta_R^2(t)/4\sigma^2} \\ &= \exp\left(\frac{[-(x+\beta_I)^2+\beta_R^2+2i\beta_R(x+\beta_I)][\sigma^2-it/2m] - (\beta_R^2/4\sigma^2)4[\sigma^4+(t/2m)^2]}{4[\sigma^4+(t/2m)^2]}\right) e^{i\gamma}. \end{aligned} \quad (\text{A7})$$

We will assume that the mass of the object is large enough so that there is negligible spreading of the wave packet over time T , so $\sigma^4 + (t/2m)^2 \approx \sigma^4$.

a. Real part of the exponent

We have for the real part of the exponent

$$\begin{aligned} &4\sigma^4 \text{Re}\{[-(x+\beta_I)^2+\beta_R^2+2i\beta_R(x+\beta_I)][\sigma^2-it/2m] - (\beta_R^2/4\sigma^2)4[\sigma^4+(t/2m)^2]\} \\ &= -(x+\beta_I)^2\sigma^2 + 2\beta_R(x+\beta_I)t/2m - (\beta_R t/2m\sigma)^2 \\ &= \sigma^2\{-x^2 + 2x[\beta_R(t/2m\sigma^2) - \beta_I] - [\beta_R(t/2m\sigma^2) - \beta_I]^2\} \\ &= -\sigma^2[x - x_{cl}(t)]^2. \end{aligned} \quad (\text{A8})$$

Thus, the mean position follows the classical trajectory.

b. Imaginary part of the exponent

The imaginary part of the exponent, using expressions (A3) and (A1) for β_R and β_I (and omitting γ for the moment), is

$$\begin{aligned} \text{Im} &= \{[-(x+\beta_I)^2+\beta_R^2](-t/2m\sigma^2) + 2\beta_R(x+\beta_I)\}/4\sigma^2 \\ &= \frac{t}{8m\sigma^2} \frac{[x - x_{cl}(t) + p_{cl}(t)t/m]^2}{\sigma^2} - \frac{p_{cl}^2(t)t}{2m} + p_{cl}(t)[x - x_{cl}(t) + p_{cl}(t)t/m] \\ &= \frac{t}{8m\sigma^2} \frac{[x - x_{cl}(t) + p_{cl}(t)t/m]^2}{\sigma^2} + \frac{p_{cl}^2(t)t}{2m} + p_{cl}(t)[x - x_{cl}(t)]. \end{aligned} \quad (\text{A9})$$

That the first term in Eq. (A9) is negligibly smaller than the following two terms follows below, from the negligible spread condition $t/m\sigma^2 \ll 1$ and that the real part of the exponent makes $[x - x_{cl}(t)]^2/\sigma^2$ be of order 1:

$$\begin{aligned} \text{Im} &= \frac{t}{8m\sigma^2} \frac{[x - x_{cl}(t)]^2}{\sigma^2} + \frac{p_{cl}^2(t)t}{2m} \left[1 + \left(\frac{t}{2m\sigma^2}\right)^2\right] + p_{cl}(t)[x - x_{cl}(t)] \left[1 + \left(\frac{t}{2m\sigma^2}\right)^2\right] \\ &\approx \frac{t}{8m\sigma^2} + \frac{p_{cl}^2(t)t}{2m} + p_{cl}(t)[x - x_{cl}(t)] \approx \frac{p_{cl}^2(t)t}{2m} + p_{cl}(t)[x - x_{cl}(t)]. \end{aligned} \quad (\text{A10})$$

The last approximation arises since, in order that the momentum of the wave packet be well defined, many oscillations of the wave function should lie within the packet width:

$$\frac{1}{8m\sigma^2} \ll \frac{p_{cl}^2(t)}{2m} = \frac{k_{cl}^2}{2m}. \quad (\text{A11})$$

Combining the exponents in (A6), (A8), and (A10), we have finally arrived at the wave packet expression

$$\psi(x,t) = N e^{-[[x-x_{cl}(t)]^2/4\sigma^2]} e^{ip_{cl}(t)[x-x_{cl}(t)]} e^{ip_{cl}^2(t)t/2m} \exp\left(-iq \int_0^t dt' V(t') + iv_0W(t)\right). \quad (\text{A12})$$

**APPENDIX B: GENERAL CASE PHASE OF SEC. III REDUCES TO SPECIAL CASE PHASE OF SEC. II
FOR A SOLELY TIME-DEPENDENT ELECTRIC FIELD**

If the force depends only upon time, i.e., if $V'(x,t) = V'(t)$ and $A(x,t) = A(t)$, the phase angle is given by Eq. (8) in Sec. II:

$$\theta(x,t) = p_{cl}(t)[x - x_{cl}(t)] + \frac{t}{2m} p_{cl}^2(t) - q \int_0^t dt' V(t') + v_0 q W(t) \quad (\text{B1})$$

[neglecting some terms squared in the potentials (i.e., $\sim q^2$), which is the order we employ]. We wish to show here that the phase expression (30) in Sec. III,

$$\theta(x,t) = p_{cl}(t)[x - x_{cl}(t)] + \int_0^t dt' \frac{1}{2m} p_{cl}^2(t') - q \int_0^t dt' V(x_{cl}(t'), t'), \quad (\text{B2})$$

for the general case of the electric field depending on position and time reduces to (B1) when there is only time dependence. In that case,

$$p_{cl}(t) = m v_{cl}(t) + \frac{q}{c} A(x_{cl}(t), t) \rightarrow m v_{cl}(t) + \frac{q}{c} A(t) = m v_0 - U(t),$$

$$V(x_{cl}(t'), t') \rightarrow V(t'). \quad (\text{B3})$$

[The latter is the zeroth-order term in the expansion of the scalar potential $V(x,t)$ in $x - x_{cl}(t)$.] Then we have

$$\begin{aligned} \int_0^t dt' \frac{1}{2m} p_{cl}^2(t') &= \int_0^t dt' \frac{1}{2m} [m v_0 - q U(t')]^2 = \frac{m}{2} v_0^2 t - v_0 q \int_0^t dt' U(t') \\ &= \frac{m}{2} v_0^2 t - v_0 q [t U(t) - W(t)] = \frac{1}{2m} p_{cl}^2(t) t + v_0 q W(t). \end{aligned} \quad (\text{B4})$$

When (B4) is inserted into (B2), we obtain (B1).

APPENDIX C: CALCULATIONS WITH EXPLICATION INVOLVED IN THE TIME-AVERAGE APPROACH

In Sec. VD we argued that one gets the correct contribution to the phase associated with the motion of a piece, subjected to the vector potential of the electron, by recognizing that, effectively, there is always a piece at angle ϕ that experiences the time-averaged vector potential:

$$\begin{aligned} \langle A \rangle_T &\equiv \frac{1}{T} \int_0^T dt A(t) = \frac{1}{T} \int_0^T dt \frac{e}{cS} \hat{\phi} \cdot \mathbf{u} \\ &= \frac{u}{\pi R} \int_0^T dt (\pm u) e \left[\frac{\cos[\phi - \phi'(t)]}{cD} + \frac{aR \cos^2[\phi - \phi'(t)]}{cD^3} \right] = \pm \frac{eu}{cD} \left[\frac{2}{\pi} \cos \phi + \frac{aR}{2D^2} \right], \end{aligned} \quad (\text{C1})$$

where $S \equiv \sqrt{a^2 + R^2 + z^2 - 2aR \cos(\phi - \phi')}$ has been expanded to first order in a , $D \equiv \sqrt{R^2 + z^2}$, and $uT = \pi R$ has been employed, as has $d\phi'(t) = \pm \frac{u}{R} dt$ [+ for the right traverse where $\phi'(t) = \frac{u}{R}t - \frac{\pi}{2}$ and - for the left traverse where $\phi'(t) = -\frac{u}{R}t - \frac{\pi}{2}$].

Now, since the vector potential is constant along the path of the piece and noting that the distance traveled by each piece during T is $v_0 T = v_0 \frac{\pi R}{u}$, the phase contributed by a piece at angle ϕ is given in Eq. (73),

$$\Lambda_{\pm} = \int_{\text{path}} ds \frac{q}{c} \langle A \rangle_T = \pm v_0 \frac{eq\pi R}{c^2 D} \left[\frac{2}{\pi} \cos \phi + \frac{aR}{2D^2} \right]. \quad (\text{C2})$$

Integrating over all the pieces using q in Eq. (64) gives for the total phase for a path and a cylinder:

$$P_{\pm} = Q \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-L/2}^{L/2} \frac{dz}{L} \Lambda_{\pm} = Q \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-L/2}^{L/2} \frac{dz}{L} \frac{1}{D^3} (\pm) v_0 \frac{eq\pi a R^2}{2c^2} = \pm \frac{Qev_0 a \pi}{Lc^2}. \quad (\text{C3})$$

The phase difference between the two paths gives twice the magnitude of this. Accounting for both cylinders requires another factor of 2, resulting in the standard AB phase shift. This result is the same as that obtained in Sec. VC, where the integral over angle was considered first, followed by the integrals over time: Here the integrals are taken in reverse order.

**APPENDIX D: ANALYSIS OF THE PHASE SHIFT IN THE ELECTRIC AB EFFECT
FOR THE SETUP DISCUSSED IN SEC. VIC**

To calculate the phase shift with the Hamiltonian (78), we must find the classical motion of the electron and the capacitor plates. The electron experiences no force and the plates experience a constant force with the potential energy $\frac{\sigma}{2} [\sigma A \pm e][z_U - z_L]$,

where z_U and z_L are the upper and lower capacitor plate positions. The upper sign in \pm refers to the electron's traverse above the capacitor (trajectory A) and the lower sign refers to the electron's traverse below the capacitor (trajectory B).

The solutions of the equations of motion are, for the electron and plates,

$$v_e = u, \quad x_e = ut, \quad v_U = -v_L = v_0 - \frac{\sigma}{2M}[\sigma A \pm e]t, \quad z_U = -z_L = v_0t - \frac{\sigma}{4M}[\sigma A \pm e]t^2. \quad (D1)$$

The largest separation of the plates for the two possible traverses and the time it takes the plates to return to $z = 0$ are

$$D_{\pm} = \frac{1}{4}v_0T_{\pm}, \quad T_{\pm} = \frac{4Mv_0}{\sigma[\sigma A \pm e]}. \quad (D2)$$

Taking e to be positive for simplicity, then $T_+ < T_-$. We take $e \ll \sigma A$, so the time difference is small, but we will see that nonetheless it has an important effect. The phase shift is then

$$\begin{aligned} \Phi^A - \Phi^B &\equiv - \int_0^T dt [V^A(t) - V^B(t)] = - \int_0^{T_+} dt \frac{\sigma}{2}[\sigma A + e][z_U(t) - z_L(t)]_+ + \int_0^{T_-} dt \frac{\sigma}{2}[\sigma A + e][z_U(t) - z_L(t)]_- \\ &= - \frac{\sigma e v_0}{6}[T_-^2 + T_+^2] + \frac{\sigma^2 A v_0}{6}[T_-^2 - T_+^2] \approx - \frac{1}{3}\sigma e v_0 \bar{T}^2 + \frac{2}{3}\sigma e v_0 \bar{T}^2 = \frac{1}{3}\sigma e v_0 \bar{T}^2, \end{aligned} \quad (D3)$$

where we have written $\bar{T} \equiv \frac{4Mv_0}{\sigma^2 A}$ and the approximation is to drop terms $\sim e/\sigma A$ compared to 1. The first term in the third line of (D3) is the negative phase shift due to the plates moving in the potential of the electron. The second term in that line is a positive phase shift due to one plate moving in the potential of the other. That phase shift would vanish if the plates spent the same amount of time on their trajectories when the electron was above them as below them. However, that is not the case, due to the force of the electron on the plates.

Perhaps surprisingly, the net phase shift is positive. This is because, although the time difference in travel is relatively small, the self-potential energy of the plates is relatively large compared to their potential energy in the field of the electron and the phase shift due to the larger potential energy wins out.

Again, as in Sec. [VIB](#), although it appears here that the phase shift is totally due to the motion of the plates under the force of the electron, one can add and subtract a term from the Hamiltonian (78), rewriting it as

$$H_{\pm} = \left[\frac{P_e^2}{2m} + \frac{\sigma}{2}[\sigma A \pm e][z_U(t) - z_L(t)]_{\pm} \right] + \left[\frac{P_U^2}{2M} + \frac{\sigma}{2}[\sigma A \pm e][Z_U - z_U(t)]_{\pm} \right] + \left[\frac{P_L^2}{2M} - \frac{\sigma}{2}[\sigma A \pm e][Z_L - z_L(t)]_{\pm} \right]. \quad (D4)$$

Now the electron part of the Hamiltonian has in it the time-varying spatially constant potential it experiences and this is completely responsible for the phase shift, while the capacitor plate part of the Hamiltonian makes no contribution to the phase shift.

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