Universal locality of quantum thermal susceptibility

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The ultimate precision of any measurement of the temperature of a quantum system is the inverse of the local quantum thermal susceptibility [A. De Pasquale *et al.*, Nat. Commun. **7**, 12782 (2016)] of the subsystem with which the thermometer interacts. If this subsystem can be described with the canonical ensemble, such quantity reduces to the variance of the local Hamiltonian, which is proportional to the heat capacity of the subsystem. However, the canonical ensemble might not apply in the presence of interactions between the subsystem and the rest of the system. In this work, we address this problem in the framework of locally interacting quantum systems. We prove that the local quantum thermal susceptibility of any subsystem is close to the variance of its local Hamiltonian, provided the volume-to-surface ratio of the subsystem is much larger than the correlation length. This result greatly simplifies the determination of the ultimate precision of any local estimate of the temperature and rigorously determines the regime where interactions can affect this precision.

I. INTRODUCTION

Current technology permits the realization of extremely small thermometers [1–4], conceived in order to carry out the challenging task of controlling the thermodynamical behavior of physical systems at the spatial resolution of the micro- and nanometer length scales. In this regime, the quantum correlations shared among the subcomponents of the system can play a nontrivial role in the measurement of the system temperature. Determining the ultimate precision of these procedures represents a fundamental issue from many perspectives, such as the development and control of the products of the nanotechnologies [5–8] or, for instance, in the study of microorganisms [9]. From a mathematical point of view, the action of a generic thermometer aimed to determine the inverse temperature β of a global system S can be modeled by a positive operator-valued measurement (POVM) [10,11] on the subsystem A with which the thermometer directly interacts and which typically is a small fraction of S.

Let $\hat{\rho}(\beta)$ be the reduced state of the subsystem A associated to the state of the global system S at inverse temperature β. The selected POVM will act on the state $\hat{\rho}(\beta)$, producing an estimate $\tilde{\beta}$ for β which represents the value returned by our thermometer. The ultimate lower bound to the mean-square error of any such estimator is set by the quantum Cramér-Rao bound [12–14] (see also [15–17]),

$$\Delta^2 \, \tilde{\beta} \geqslant 1/F(\beta), \qquad (1)$$

with $F(\beta)$ being the quantum Fisher information of the family of states $\{\hat{\rho}(\beta)\}_\beta$. The functional $F(\beta)$ does not depend upon the specific choice of the detection process, instead it characterizes how small variations in the inverse temperature β of the global system S influence the local state of A: accordingly, in Ref. [18], it was used to gauge the local thermal susceptibility of composite systems, i.e., large values of $F(\beta)$ corresponding to models where the equilibrium temperature of S is better perceived by its components.

If the subsystem A can be described with the canonical ensemble, i.e.,

$$\hat{\rho}(\beta) = e^{-\beta \hat{H}_A} \big/ \mathrm{Tr}_A e^{-\beta \hat{H}_A}, \qquad (2)$$

where $\hat{H}_A$ is the local Hamiltonian, then the local quantum thermal susceptibility of the subsystem A coincides with the variance of its local Hamiltonian, which is proportional to its heat capacity [19–22],

$$F(\beta) = \mathrm{Tr}_A\big[\hat{H}_A^2 \, \hat{\rho}(\beta)\big] - \{\mathrm{Tr}_A[\hat{H}_A \, \hat{\rho}(\beta)]\}^2, \qquad (3)$$

and we recover the well-known result of classical thermodynamics [23–25]. However, the assumption (2) is not always justified in the presence of interactions between the subsystem A and the rest of the system [26].

In this paper, we address this problem in the framework of locally interacting quantum systems [26] and we prove that (3) still holds in the high-temperature regime for any subsystem A whose volume-to-surface ratio is much larger than the correlation length (see Corollary 1 below). Locally interacting quantum systems constitute a very general framework that encompasses all the fundamental spin models, such as the Ising model [27], the Heisenberg model [28], the Potts model [29], the n-vector model [30], the Hubbard model [31], and the Majumdar-Ghosh model [32,33]. Our finding rigorously proves that when the correlation length is larger than the volume-to-surface ratio of the subsystem, local interactions do not spoil (3) and hence they do not significantly affect the precision of local measurements of the temperature. In this case, the local quantum thermal susceptibility is determined by the expectation values of local operators (the local Hamiltonian and its square), and is therefore a local quantity itself. This locality is universal since the volume-to-surface ratio of the subsystem does not depend on the microscopic details of the subsystem or of the Hamiltonian.

II. THE SETUP

We work in the framework of locally interacting many-body quantum systems, extensively discussed in the review given in Ref. [26]. We hence consider a lattice characterized by a finite set $\mathcal{V}$ of sites (the "quantum particles" of the model). The global Hilbert space $\mathcal{H}$ is the tensor product of the Hilbert spaces $\mathcal{H}_{\{x\}}$ associated to each site: $\mathcal{H} = \bigotimes_{x \in \mathcal{V}} \mathcal{H}_{\{x\}}$. The edges of the lattice, which represent the interactions among the particles, are identified with the subsystems $X \subset \mathcal{V}$ that are the support of one term $\hat{H}_X$ of the system Hamiltonian,

$$\hat{H} = \sum_{X \in \mathcal{E}} \hat{H}_X \qquad (4)$$

(the support of an operator $\hat{O}$ is defined as the smallest subset $Y \subset \mathcal{V}$ such that $\hat{O}$ acts like the identity outside Y, i.e., $\hat{O} = \hat{O}_Y \otimes \hat{\mathbb{1}}_{\mathcal{V} \setminus Y}$). In Eq. (4), the symbol $\mathcal{E}$ indicates the set formed by all of the edges that compose the lattice, which fully determines the geometry of the model. In particular, the distance $d(x, y)$ between two sites $x, y \in \mathcal{V}$ is the length d of the shortest sequence of elements $X_1, \ldots, X_d \in \mathcal{E}$ such that $x \in X_1$, $y \in X_d$, and $X_i \cap X_{i+1} \neq \emptyset$ for $i = 1, \ldots, d - 1$. Consequently, the distance between a site $x \in \mathcal{V}$ and a subsystem $Y \subset \mathcal{V}$ and the distance between two subsystems $X, Y \subset \mathcal{V}$ are given by

$$d(x, Y) := \min_{y \in Y} d(x, y), \quad d(X, Y) := \min_{x \in X} d(x, Y), \quad (5)$$

respectively. Another important notion is the boundary edge set X_∂ of a subsystem $X \subset \mathcal{V}$: this is the subset of $\mathcal{E}$ formed by the (boundary) edges which overlap with both X and its complement, i.e., $X_\partial := \{Y \in \mathcal{E} : Y \cap X \neq \emptyset, \, Y \cap (\mathcal{V} \setminus X) \neq \emptyset\}$.

The key property of locally interacting many-body quantum systems described by a Hamiltonian of the form (4) is that they admit a critical temperature T^* above which the correlation between any two observables decays exponentially with the distance between their supports [26,34]. Namely, for any $|\beta| < \beta^* = 1/(K_b T^*)$, with K_b being the Boltzmann constant, there exists a correlation length $\xi(\beta) > 0$ such that for any two subsystems $X, Y \subset \mathcal{V}$ and any two observables $\hat{O}_X, \hat{O}_Y$ with support in X and Y, respectively, we have

$$|\langle \hat{O}_X \, \hat{O}_Y \rangle_\beta - \langle \hat{O}_X \rangle_\beta \langle \hat{O}_Y \rangle_\beta|$$
$$\leqslant 4[\xi(\beta) + 1] \min(|X_\partial|, |Y_\partial|) \|\hat{O}_X\|_\infty \|\hat{O}_Y\|_\infty e^{-\frac{d(X,Y)}{\xi(\beta)}}, \quad (6)$$

with $|Z|$ denoting the number of elements of the set Z.

In what follows, we shall exploit the above inequality by focusing on models where the number of edges at distance $n \in \mathbb{N}$ from a given site $x \in \mathcal{V}$ increases, at most, quadratically with n, i.e., there exists a constant $M > 0$ such that

$$|\{Y \in \mathcal{E} : d(x, Y) = n\}| \leqslant M(n + 1)^2, \qquad (7)$$

for any $x \in \mathcal{V}$ and any $n \in \mathbb{N}$. For a D-dimensional lattice, the left-hand side of (7) scales as the surface of the sphere of radius n, i.e., as n^{D-1}. Equation (7) reproduces the scaling for $D = 3$. For $D < 3$, (7) still holds since the actual scaling is tighter. The case $D > 3$ is unphysical. However, replacing $(n + 1)^2$ with $(n + 1)^{D-1}$ in (7) would only add an awkward D-dependent prefactor to our results.

III. UNIVERSAL LOCALITY

In the presence of interactions, the reduced density matrix of the subsystem A might not be the Gibbs state (2). Quantum statistical mechanics [26,35] assumes that this reduced density matrix is the partial trace of the Gibbs state of the global system,

$$\hat{\rho}(\beta) = \text{Tr}_B \hat{\omega}(\beta), \qquad (8)$$

where B is the remaining part of the system,

$$\hat{\omega}(\beta) = e^{-\beta \hat{H}} / \text{Tr} \, e^{-\beta \hat{H}}, \qquad (9)$$

and $\hat{H}$ is the global Hamiltonian. We stress that the assumption (8) does not require the global system to be in a Gibbs state. It holds for almost all global pure states chosen from a given energy shell [36,37] and it is always satisfied for almost any time if the global Hamiltonian satisfies the eigenstate thermalization hypothesis [26,38–67] (see also [68]).

We then suppose that the locally interacting many-body quantum system characterized by the Hamiltonian (4) is initialized in the Gibbs state (9), whose inverse temperature β we aim to recover via local measurements performed on a subset $A \subset \mathcal{V}$ of the sites. Let then $B = \mathcal{V} \setminus A$ be the complement of A to $\mathcal{V}$, $\mathcal{A} := \{X \in \mathcal{E} : X \subset A\}$ the set of the edges contained into A, and $\hat{H}_A := \sum_{X \in \mathcal{A}} \hat{H}_X$, the local Hamiltonian of A, i.e., the contribution to H which act locally on A. For any $R \in \mathbb{N}$, we define

$$\mathcal{C}_R := \{X \in \mathcal{E} \setminus \mathcal{A} : d(X, A) < R\} \qquad (10)$$

the set of the edges not contained into A and with distance from A less than R, and $\hat{H}_{\mathcal{C}_R} := \sum_{X \in \mathcal{C}_R} \hat{H}_X$ the sum of the Hamiltonian terms in $\mathcal{C}_R$ (see Fig. 1). For R much smaller than the dimension of A, the edges of $\mathcal{C}_R$ are a thin layer of width R that lies on the boundary of A. Hence, we expect $|\mathcal{C}_R|$ to be proportional both to R and to the surface of A, i.e.,

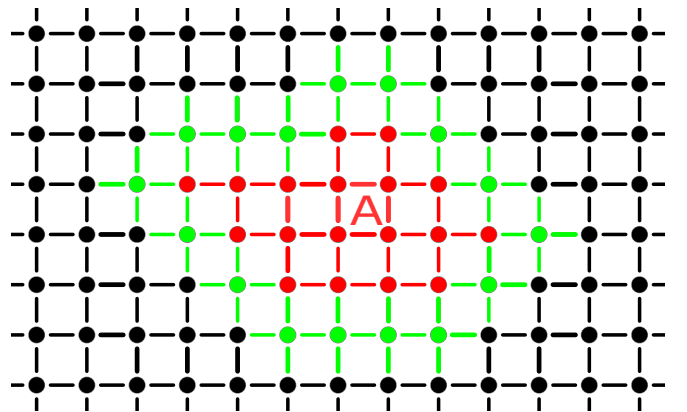$$|\mathcal{C}_R| \propto R |A_\partial|. \qquad (11)$$

FIG. 1. A 2D square lattice with two-site first-neighbor interactions. In red are the sites of A and the edges of $\mathcal{A}$. In green are the sites x with $d(x, A) = 1$ and the edges of $\mathcal{C}_1$. The boundary edges of A are the green edges connecting a red with a green site. In black are all the other sites and edges. Here, $N_\partial = 6$.

We can now state our main result. For $|\beta| < \beta^*$, the local quantum thermal susceptibility of the system $F(\beta)$ is close to the variance of the local Hamiltonian, as follows:

Theorem 1. For any $R \geqslant 2\xi(\beta) + 1$,

$$|\sqrt{F(\beta)} - \sqrt{\text{Var}_\beta \hat{H}_A}|$$
$$\leqslant \sqrt{\text{Var}_\beta \hat{H}_{C_R}} + 40JM|A|\sqrt{N_\partial}[\xi(\beta)+1]^{\frac{3}{2}} R^2 e^{-R/2\xi(\beta)},$$
$$\text{(12)}$$

with $J := \max_{X \in \mathcal{E}} \|\hat{H}_X\|_\infty$ being the interaction strength of the system Hamiltonian $\hat{H}$ and $N_\partial := \max_{X \in \mathcal{E}} |X_\partial|$ being the maximum number of edges sharing sites with a given one.

Proof. For any operator $\hat{O}$, we denote with $\langle \hat{O} \rangle_\beta := \text{Tr}[\hat{O} \hat{\omega}(\beta)]$ its expectation value with respect to the global Gibbs state. We remind the reader that the quantum Fisher information [13] of a family of states $\{\hat{\rho}(\beta)\}_\beta$ can be expressed as $F(\beta) := \|\frac{d}{d\beta}\hat{\rho}(\beta)\|^2_{\hat{\rho}(\beta)}$, where given $\hat{\rho}$ a generic density matrix and $\hat{O}$ a generic operator, $\|\hat{O}\|_{\hat{\rho}}$ represents the Bures norm [69],

$$\|\hat{O}\|_{\hat{\rho}} = \sqrt{\sum_{i,j} \frac{2|\langle i|\hat{O}|j\rangle|^2}{p_i + p_j}}, \quad \text{(13)}$$

with $\{|i\rangle\}$ being the eigenvectors of $\hat{\rho}$ and p_i, their corresponding eigenvalues. Let then

$$\mathcal{F}_R := \{X \in \mathcal{E} : d(X,A) \geqslant R\} \quad \text{(14)}$$

be the set of the edges with distance at least R from A, such that $\mathcal{E} = \mathcal{A} \cup \mathcal{C}_R \cup \mathcal{F}_R$, and $\hat{H} = \hat{H}_A + \hat{H}_{C_R} + \sum_{X \in \mathcal{F}_R} \hat{H}_X$. Given $\hat{\rho}(\beta)$ the density matrix of A defined as in (8), we can then write

$$\frac{d}{d\beta}\hat{\rho}(\beta) = \hat{A}(\beta) + \hat{C}_R(\beta) + \sum_{X \in \mathcal{F}_R} \hat{B}_X(\beta), \quad \text{(15)}$$

where indicating with $\{\cdot,\cdot\}$ the anticommutator, we have

$$\hat{A}(\beta) = \{\langle \hat{H}_A \rangle_\beta - \hat{H}_A, \hat{\rho}(\beta)\}/2, \quad \text{(16)}$$

$$\hat{C}_R(\beta) = \text{Tr}_B[\{\langle \hat{H}_{C_R} \rangle_\beta - \hat{H}_{C_R}, \hat{\omega}(\beta)\}]/2, \quad \text{(17)}$$

and for any $X \in \mathcal{F}_R$,

$$\hat{B}_X(\beta) = \text{Tr}_B[\{\langle \hat{H}_X \rangle_\beta - \hat{H}_X, \omega(\hat{\beta})\}]/2. \quad \text{(18)}$$

From the triangular inequality for the Bures norm, it then follows that

$$\left| \left\| \frac{d}{d\beta}\hat{\rho}(\beta) \right\|_{\hat{\rho}(\beta)} - \sqrt{\text{Var}_\beta \hat{H}_A} \right|$$
$$\leqslant \|\hat{C}_R(\beta)\|_{\hat{\rho}(\beta)} + \sum_{X \in \mathcal{F}_R} \|\hat{B}_X(\beta)\|_{\hat{\rho}(\beta)}, \quad \text{(19)}$$

where we used the fact that $\|\hat{A}(\beta)\|_{\hat{\rho}(\beta)} = \sqrt{\text{Var}_\beta \hat{H}_A}$. Next we invoke the fact that the Bures metric is contractive with respect to completely positive trace-preserving (CPTP) maps [11]. Accordingly, we can bound the first term on the right-hand side of (19) as follows:

$$\|\hat{C}_R(\beta)\|^2_{\hat{\rho}(\beta)} \leqslant \left\| \tfrac{1}{2}\{\langle \hat{H}_{C_R} \rangle_\beta - \hat{H}_{C_R}, \hat{\omega}\} \right\|^2_{\hat{\omega}(\beta)} = \text{Var}_\beta \hat{H}_{C_R(\beta)}.$$
$$\text{(20)}$$

For the second term, instead we use the fact that for all operators $\hat{O}$ and states $\hat{\rho}$, one has $\|\hat{O}\|^2_{\hat{\rho}} \leqslant \text{Tr}[\hat{\rho}^{-\frac{1}{2}} \hat{O} \hat{\rho}^{-\frac{1}{2}} \hat{O}]$, which follows from (13) since $p_i + p_j \geqslant 2\sqrt{p_i p_j}$ for any $p_i, p_j \geqslant 0$. Accordingly, for any $X \in \mathcal{F}_R$, we can write

$$\|\hat{B}_X(\beta)\|^2_{\hat{\rho}(\beta)}$$
$$\leqslant \text{Tr}[\hat{\rho}(\beta)^{-\frac{1}{2}} \hat{B}_X(\beta) \hat{\rho}(\beta)^{-\frac{1}{2}} \hat{B}_X(\beta)]$$
$$= \langle \hat{\rho}(\beta)^{-\frac{1}{2}} \hat{B}_X(\beta) \hat{\rho}(\beta)^{-\frac{1}{2}}(\langle \hat{H}_X \rangle_\beta - \hat{H}_X)\rangle_\beta$$
$$\leqslant 8[\xi(\beta)+1]N_\partial J \|\hat{\rho}(\beta)^{-\frac{1}{2}} \hat{B}_X \hat{\rho}(\beta)^{-\frac{1}{2}}\|_\infty e^{-d(X,A)/\xi(\beta)},$$
$$\text{(21)}$$

where in the second step we have used (18) and in the last step we have applied (6) to the observables $\hat{O}_A = \hat{\rho}(\beta)^{-\frac{1}{2}}\hat{B}_X(\beta)\hat{\rho}(\beta)^{-\frac{1}{2}}$ and $\hat{O}_X = \langle \hat{H}_X \rangle_\beta - \hat{H}_X$ supported into A and X, respectively, bounding $|X_\partial|$ with N_∂ and using the inequality

$$\|\langle \hat{H}_X \rangle_\beta - \hat{H}_X\|_\infty \leqslant 2J. \quad \text{(22)}$$

Now we notice that for any vector $|\psi\rangle$ in the Hilbert space of A, we have

$$|\langle \psi|\hat{\rho}(\beta)^{-\frac{1}{2}} \hat{B}_X(\beta) \hat{\rho}(\beta)^{-\frac{1}{2}}|\psi\rangle|$$
$$= |\text{Tr}_B[(\langle \hat{H}_X \rangle_\beta - \hat{H}_X)\langle \psi|\hat{\rho}(\beta)^{-\frac{1}{2}} \hat{\omega}(\beta) \hat{\rho}(\beta)^{-\frac{1}{2}}|\psi\rangle]|$$
$$\leqslant 2J\|\langle \psi|\hat{\rho}(\beta)^{-\frac{1}{2}} \hat{\omega}(\beta) \hat{\rho}(\beta)^{-\frac{1}{2}}|\psi\rangle\|_1 = 2J\langle \psi|\psi\rangle, \quad \text{(23)}$$

where $\hat{H}_X$ is meant as an operator on the Hilbert space of B, and where we have used (18) and (22). Then, $\|\hat{\rho}(\beta)^{-\frac{1}{2}} \hat{B}_X(\beta)\hat{\rho}(\beta)^{-\frac{1}{2}}\|_\infty \leqslant 2J$, and (21) implies

$$\|\hat{B}_X(\beta)\|_{\hat{\rho}} \leqslant 4J\sqrt{N_\partial[\xi(\beta)+1]}e^{-d(X,A)/2\xi(\beta)}. \quad \text{(24)}$$

Recalling (14), we can hence write

$$\sum_{X \in \mathcal{F}_R} \|\hat{B}_X(\beta)\|_{\hat{\rho}(\beta)}$$
$$\leqslant 4J\sqrt{N_\partial[\xi(\beta)+1]}$$
$$\times \sum_{n=R}^\infty |\{X \in \mathcal{E} : d(X,A) = n\}|e^{-n/2\xi(\beta)}$$
$$\leqslant 40JM|A|\sqrt{N_\partial}[\xi(\beta)+1]^{\frac{3}{2}} R^2 e^{-R/2\xi(\beta)}, \quad \text{(25)}$$

where in the last step we used the hypothesis (7), stating that the number of edges at distance $n \in \mathbb{N}$ from A increases at most quadratically in n, i.e.,

$$|\{X \in \mathcal{E} : d(X,A) = n\}| \leqslant M|A|(n+1)^2 \quad \text{(26)}$$

(indeed, from (5), if $d(X,A) = n$, there exists at least one site $x \in A$ with $d(x,A) = n$), and the inequality $\sum_{n=R}^\infty (n+1)^2 e^{-n/2\xi} \leqslant 10R^2(\xi+1)e^{-R/2\xi}$, which holds for any $\xi > 0$ and $R \geqslant 2\xi + 1$. The inequality (12) finally follows by replacing (20) and (25) in (19). ∎

The result of Theorem 1 can be made more explicit by bounding the term $\text{Var}_\beta \hat{H}_{C_R}$, for example, by showing that the variance of the local Hamiltonian of a subsystem scales linearly with the size of the subsystem.

Theorem 2. Variance of local Hamiltonian. Let $\mathcal{G} \subset \mathcal{E}$ be a set of edges and let $\hat{H}_\mathcal{G} := \sum_{X \in \mathcal{G}} \hat{H}_X$ be the sum of the

Hamiltonian terms in $\mathcal{G}$. Then, for any $|\beta| < \beta^*$, the variance of $\hat{H}_\mathcal{G}$ increases at most linearly with the size of $\mathcal{G}$, i.e.,

$$\text{Var}_\beta \, \hat{H}_\mathcal{G} \leqslant 8N_\partial \, J^2 |\mathcal{G}| N \, M(\xi(\beta) + 1)^4, \qquad (27)$$

where $N := \max_{X \in \mathcal{E}} |X|$.

Proof. Reminding the reader of the definition of J and N_∂ and using Eq. (6), we can write

$$\text{Var}_\beta \, \hat{H}_\mathcal{G} = \sum_{X,Y \in \mathcal{G}} (\langle \hat{H}_X \, \hat{H}_Y \rangle_\beta - \langle \hat{H}_X \rangle_\beta \langle \hat{H}_Y \rangle_\beta)$$

$$\leqslant 4[\xi(\beta) + 1]N_\partial \, J^2 \sum_{X,Y \in \mathcal{G}} e^{-d(X,Y)/\xi(\beta)}$$

$$= 4[\xi(\beta) + 1]N_\partial \, J^2$$

$$\times \sum_{X \in \mathcal{G}} \sum_{n=0}^{\infty} |\{Y \in \mathcal{G} : d(X,Y) = n\}| e^{-n/\xi(\beta)}, \qquad (28)$$

which yields (27) by invoking the assumption (7) as in (26) and the inequality $\sum_{n=0}^{\infty} (n+1)^2 e^{-n/\xi} \leqslant 2(\xi + 1)^3$. ∎

Corollary 1. Universal locality of quantum thermal susceptibility. For any $R \geqslant 2\xi(\beta) + 1$,

$$|\sqrt{F(\beta)} - \sqrt{\text{Var}_\beta \hat{H}_A}|$$

$$\leqslant 2J\sqrt{2MNN_\partial}[\xi(\beta) + 1]^2 \times \sqrt{|\mathcal{C}_R|}$$

$$+ 40JM|A|\sqrt{N_\partial}[\xi(\beta) + 1]^{\frac{3}{2}} \times R^2 e^{-R/2\xi(\beta)}. \qquad (29)$$

Proof. Follows from Theorems 1 and 2. ∎

Theorem 1 and Corollary 1 imply that the local quantum thermal susceptibility of the subsystem A is close to the variance of its local Hamiltonian $\hat{H}_A$ if we can find R such that the right-hand side of (12) or (29) is much smaller than $\sqrt{\text{Var}_\beta \hat{H}_A}$. This is possible whenever

$$|\mathcal{C}_{\xi(\beta)}| \ll |\mathcal{A}|, \qquad (30)$$

i.e., recalling (11), whenever the volume-to-surface ratio of the subsystem A is much larger than the correlation length,

$$\xi(\beta)|A_\partial| \ll |\mathcal{A}|. \qquad (31)$$

Indeed, the second term in the right-hand side of (12) or (29) decreases exponentially as R increases, and it is small whenever $R \gg \xi(\beta)$. From Theorem 2, we expect $\text{Var}_\beta \hat{H}_A$ to scale linearly with $|\mathcal{A}|$. The first term in the right-hand side of (29) scales with R as $\sqrt{|\mathcal{C}_R|}$. Then, we expect it to be much smaller than $\sqrt{\text{Var}_\beta \hat{H}_A}$ whenever $|\mathcal{C}_R| \ll |\mathcal{A}|$. These conditions can both be satisfied iff (30) holds. On the contrary, if the subsystem A is smaller than the correlation length, conditions (30) and (31) do not hold. In this case, we expect that the correlations between A and the rest of the system will spoil (3), and the local quantum thermal susceptibility will no longer be close to the variance of the local Hamiltonian. This always happens when the temperature approaches a critical

point and the correlation length diverges, or when A is made only by a few sites.

Example: 1D spin chain

Let us conclude our analysis with an example. We consider a 1D spin chain with nearest-neighbor interactions. In this case, $M = 2$, $N_\partial = 2$, and $N = 2$. To this class belongs, for example, the well-known 1D Ising model [27] with Hamiltonian $\hat{H} = \sum_i (\hat{\sigma}_i^x \, \hat{\sigma}_{i+1}^x + h \, \hat{\sigma}_i^z)$, where h is the external magnetic field and $J = \sqrt{1 + h^2}$.

Our choice for the subsystem A is a set of consecutive sites. We have with this choice $|\mathcal{C}_R| = 2R$ for any $R \in \mathbb{N}$. The bound of Corollary 1 is then

$$\left| \sqrt{\frac{F(\beta)}{\text{Var}_\beta \hat{H}_A}} - 1 \right| \leqslant 8J\sqrt{2}(\xi + 1)^2$$

$$\times \left[\sqrt{\frac{R}{\text{Var}_\beta \hat{H}_A}} + \frac{10|A|R^2 \, e^{-\frac{R}{2\xi}}}{\sqrt{(\xi + 1)\text{Var}_\beta \hat{H}_A}} \right], \qquad (32)$$

where the correlation length ξ reads ([26], Eq. (163))

$$\xi(\beta) = -\frac{1}{\ln[\sqrt[3]{3}e^{2\beta J}(e^{2\beta J} - 1)]}. \qquad (33)$$

The optimal bound is the minimum of the right-hand side of (32) over $R \geqslant 2\xi + 1$. From the discussion in the previous section, this minimum is small whenever $\text{Var}_\beta \hat{H}_A \gg \xi$, i.e., whenever $|A| \gg \xi$ since $\text{Var}_\beta \hat{H}_A$ is proportional to $|A|$.

IV. CONCLUSIONS

We have proved that the local quantum thermal susceptibility of any subsystem of a locally interacting quantum system is close to the variance of its local Hamiltonian, provided the volume-to-surface ratio of the subsystem is much larger than the correlation length. This result determines the ultimate precision of any local measurement of the temperature. Moreover, it rigorously proves that the interactions between the subsystem and the remaining part of the system can affect this precision only when the correlation length becomes larger than the volume-to-surface ratio of the subsystem. This always happens when the correlation length becomes larger than the subsystem, e.g., in the proximity of a critical point when the correlation length diverges or for a subsystem made by only a few sites.

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