Wigner functions for angle and orbital angular momentum: Operators and dynamics

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Recently a paper on the construction of consistent Wigner functions for cylindrical phase spaces $\mathbb{S}^1 \times \mathbb{R}$, i.e., for the canonical pair angle and orbital angular momentum, was published [H. A. Kastrup, Phys. Rev. A **94**, 062113 (2016)] in which the main properties of these functions are derived and discussed and their usefulness is illustrated with examples. The present paper is a continuation which compares the properties of the Wigner functions for cylindrical phase spaces with those of the well-known Wigner functions for planar phase spaces in more detail. Furthermore, the mutual (Weyl) correspondence between Hilbert space operators and their phase-space functions is discussed. The $\star$ product formalism is shown to be completely implementable. In addition, basic dynamical laws for Wigner and Moyal functions are derived as generalized Liouville and energy equations. They are very similar to those in the planar case but also show characteristic differences.

I. INTRODUCTION

In a recent paper [1] mathematically and physically consistent Wigner functions for cylindrical phase spaces $\mathbb{S}^1 \times \mathbb{R}$ were proposed, based to a large extent on group theoretical considerations: Replacing the angle θ—which characterizes the $\mathbb{S}^1$ part—with the pair $(\cos\theta, \sin\theta)$ allows for a consistent quantization in terms of unitary representations of the Euclidean group E(2) of the plane. This group comes into play because the Poisson brackets of $\cos\theta$ and $\sin\theta$ and the angular momentum p obey the Lie algebra of E(2) so that the corresponding quantum mechanical self-adjoint operators (observables) C, S, and L become the generators of unitary representations of E(2) [2].

In view of the many applications of Wigner and Moyal functions for planar phase spaces (see, e.g., Refs. [3–14]), a similarly well-founded theoretical framework for cylindrical phase spaces may open paths to new applications in physics, mathematics, informatics, and technologies. A few simple typical examples are discussed in Ref. [1]. In physics, additional applications to fractional orbital angular momenta [2] or pendulum-type systems might be of interest.

Considering the structural similarities between the well-known Wigner-Moyal functions for planar phase spaces and those for cylindrical phase spaces proposed in Ref. [1], it is a strong challenge to apply the many methods and tools developed for the former to the cylindrical case. Let me briefly recall the main result in Ref. [1]: The Wigner functions $V_\psi(\theta, p) = (\psi, V(\theta, p)\psi)$, the more general Moyal functions $V_{\psi_2 \psi_1}(\theta, p) = (\psi_2, V(\theta, p)\psi_1)$, and the associated Wigner Hermitian operator (matrix) $V(\theta, p) = (V_{mn}(\theta, p))$ are constructed by a kind of ordered group-averaging within a Hilbert space $L^2(S^1, d\varphi/2\pi)$ with the scalar product

$$(\psi_2, \psi_1)_{\mathbb{S}^1} = \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \psi_2^*(\varphi) \psi_1(\varphi) \qquad (1)$$

and the basis

$$e_n(\varphi) = e^{in\varphi}, \quad (e_m, e_n)_{\mathbb{S}^1} = \delta_{mn}, \quad m, n \in \mathbb{Z}, \qquad (2)$$

where δ_{mn} is the usual Kronecker symbol. [Generally one has a basis $e_{n,\delta}(\varphi) = \exp[i(n + \delta)]$, $\delta \in [0, 1)$, where δ characterizes a covering group of the rotation group SO(2) [1,2]. Here we consider only the case $\delta = 0$.]

We then have

$$V_{mn}(\theta, p) = (e_m, V(\theta, p)e_n)_{\mathbb{S}^1} \qquad (3)$$

and

$$\psi(\varphi) = \sum_{n \in \mathbb{Z}} c_n e_n(\varphi), \quad c_n = (e_n, \psi)_{\mathbb{S}^1}. \qquad (4)$$

The matrix elements $V_{mn}(\theta, p)$ have the explicit form [1]

$$V_{mn}(\theta, p) = \frac{1}{2\pi} e^{i(n-m)\theta} \int_{-\pi}^{\pi} \frac{d\vartheta}{2\pi} e^{i[(n+m)/2 - p/\hbar]\vartheta} \qquad (5)$$

$$= \frac{1}{2\pi} e^{i(n-m)\theta} \operatorname{sinc}\pi[p/\hbar - (m+n)/2],$$

$$\operatorname{sinc} x \equiv \frac{\sin x}{x} \qquad (6)$$

and yield the Wigner function—a bilinear form—for a wave function $\psi(\varphi)$,

$$V_\psi(\theta, p) = \sum_{m,n \in \mathbb{Z}} c_m^* V_{mn}(\theta, p) c_n$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\vartheta}{2\pi} e^{-ip\vartheta/\hbar} \psi^*(\theta - \vartheta/2)\psi(\theta + \vartheta/2). \qquad (7)$$

Many properties of $V_{mn}(\theta, p)$ are discussed in Ref. [1]. Those which are essential for the present paper are listed below.

In Ref. [1] it was pointed out—without going into detail—that many properties associated with the matrix, (5), and the Wigner function, (7), correspond to structurally equivalent ones for the usual Wigner functions on the plane $\mathbb{R}^2$:

$$W_\phi(q, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \, e^{-ip\xi/\hbar} \phi^*(q - \xi/2)\phi(q + \xi/2). \qquad (8)$$

(The letter V, introduced in Ref. [1] for the "cylindrical" Wigner function, stands for "v"ortex or "v"ariant of the "planar" W.)

Superficially expressions (7) and (8) look very similar: The integral in Eq. (7) appears as a restriction from the one over $\mathbb{R}$ in Eq. (8) to the one over the finite interval $[-\pi,\pi]$. The implications are, however, quite nontrivial: Whereas there exists a well-defined position operator Q in the quantum version of the coordinate q, no such operator exists for the angle θ [15], and the angle has to be replaced by the equivalent pair $(\cos\theta, \sin\theta)$ before quantization. Furthermore, the translation $q \to q + a$ or $\xi \to \xi + a$ form an invariance group of the integral, (8). The corresponding transformations for integral (7) are rotations from the group SO(2): $\theta \to \theta + \alpha$ or $\vartheta \to \vartheta + \alpha$. For the associated quantum mechanics this means that the angular momentum operator L, the canonically conjugate variable of the angle, has a discrete spectrum, whereas the momentum operator P and position operator Q of the phase plane generally have a continuous one. In addition, the quantum mechanics (representation theory) of the group SO(2) has subtleties the translations do not have: SO(2) is infinitely connected—you can wrap a string around a cylinder an infinite number of times—which implies an infinite number of covering groups and, therefore, allows for fractional angular momenta [2] and so for additional physical effects.

The close correspondence between quite a number of relations of the two types of Wigner functions is made more explicit in Sec. II. Furthermore, as the discussions of relations between operators in Hilbert space and their corresponding functions in phase space are still incomplete in Ref. [1], they are discussed in more detail in Sec. III. Section IV contains an explicit analysis of the dynamics—time evolution and energy equations—of the Wigner-Moyal functions in the cylindrical case.

II. COMPARISONS OF WIGNER FUNCTIONS FOR PLANAR AND CYLINDRICAL PHASE SPACES

In this section well-known results for the planar case (see, e.g., Refs. [4,6], and [16–20]) are compared to those established in Ref. [1]. Additional *new* results for the cylindrical case are discussed in Secs. III and IV and compared to the corresponding properties of the planar case.

In the cylindrical case operators A are generally and conveniently represented by infinite matrices:

$$A = (A_{mn} = (e_m, Ae_n)_{\mathbb{S}^1}), \quad m,n \in \mathbb{Z}. \qquad (9)$$

In the following, wave functions in the Hilbert space on $\mathbb{R}$ are denoted $\phi(x)$; those on $\mathbb{S}^1$, by $\psi(\varphi)$.

Another remark concerns the different incorporations of Planck's constant $\hbar$: Comparing the *integrals* (8) and (13) shows that the former is dimensionless. The prefactor $1/(2\pi\hbar)$ serves the following purpose: The integration measure in a planar phase space, $dq\,dp$, has the dimension of an action. It is usually made dimensionless by dividing it by $2\pi\hbar = h$. However, only the product $dqdp$ itself has the dimension of an action. So one cannot associate that denominator with dq or dp alone. But there are situations where one would like to integrate over q or p separately. In order to avoid the problem mentioned one inserts the factor $1/(2\pi\hbar$ into the integrand, (8). In this way we get the (dimensionless)

normalization

$$\int_{\mathbb{R}^2} dq\, dp\, W_\phi(q,p) = 1. \qquad (10)$$

On the other hand, in the case of a cylindrical phase space with measure $d\theta\, dp$ the angular momentum p alone has the dimension of an action. Thus, one makes dp dimensionless by dividing it by $\hbar$ and associates the remaining factor $1/(2\pi)$ with the dimensionless $d\theta$. For convenience we write

$$d\bar{p} \equiv d(p/\hbar) \qquad (11)$$

and absorb the factor $1/(2\pi)$ into the integrand, (7). We then have the normalization

$$\int_{-\pi}^{\pi} d\theta \int_{-\infty}^{\infty} d\bar{p}\, V_\psi(\theta,p) = 1. \qquad (12)$$

In order to keep the appearance of $\hbar$ otherwise explicit in the following, we do not in general replace $p/\hbar$ in functions or integrands with $\bar{p}$. This serves the discussion of classical limits $\hbar \to 0$.

A. Wigner functions proper

1. Expectation values of operators and operator-related phase-space functions

The basic postulate for the concept of Wigner functions— here, in the planar case—is to express the expectation value $\langle A\rangle_\phi$ of an operator A with respect to a state ϕ in a Hilbert space with scalar product

$$(\phi_2,\phi_1)_{\mathbb{R}} = \int_{-\infty}^{\infty} dx\, \phi_2^*(x)\phi_1(x) \qquad (13)$$

as an integral over phase space with density $W_\phi(q,p)$,

$$\langle A\rangle_\phi = (\phi, A\phi)_{\mathbb{R}} = \int_{\mathbb{R}^2} dq\, dp\, \tilde{A}(q,p)\, W_\phi(q,p), \qquad (14)$$

where, in Dirac's notation,

$$\tilde{A}(q,p) = \int_{-\infty}^{\infty} d\xi\, e^{-ip\xi/\hbar}\langle q + \xi/2|A|q - \xi/2\rangle \qquad (15)$$

is the phase-space function (Weyl symbol) associated with the Hilbert-space operator A.

Expression (15) for $\tilde{A}(q,p)$ may be rewritten in a way which resembles the corresponding cylindrical one very closely: Introducing the operator [4,21,22]

$$\Delta(q,p) = \int_{-\infty}^{\infty} d\xi\, e^{-ip\xi/\hbar}|q - \xi/2\rangle\langle q + \xi/2|, \qquad (16)$$

the function $\tilde{A}(q,p)$ can be written as

$$\tilde{A}(q,p) = \text{tr}[A \cdot \Delta(q,p)], \qquad (17)$$

where

$$\text{tr}(B) = \int_{-\infty}^{\infty} dx\, \langle x|B|x\rangle. \qquad (18)$$

Expectation values of operators with respect to a density operator ρ are treated below where traces of operators and their products are discussed.

If A is an operator in a Hilbert space with scalar product (1), then—according to Ref. [1]—its expectation value with

respect to a wave function $\psi(\varphi)$ is given by an associated integral over the cylindrical phase space,

$$\langle A \rangle_\psi = (\psi, A\psi)_{\mathbb{S}^1} = \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta \, \tilde{A}(\theta, p) \, V_\psi(\theta, p),$$
(19)

where now

$$\tilde{A}(\theta, p) = 2\pi \, \mathrm{tr}[A \cdot V(\theta, p)].$$
(20)

As $V(\theta + 2\pi, p) = V(\theta, p)$ [see Eq. (5)], the function $\tilde{A}(\theta, p)$ has the same periodicity. Furthermore, if A is self-adjoint, i.e., $A_{mn}^* = A_{nm}$, like $V(\theta, p)$, then $\tilde{A}(\theta, p)$ is real.

For the elementary operators

$$C = \cos\varphi, \quad S = \sin\varphi, \quad L = \frac{\hbar}{i}\partial_\varphi,$$
(21)

e.g., one has—according to Eq. (9)—the matrix elements

$$C_{mn} = \frac{1}{2}(\delta_{m(n+1)} + \delta_{m(n-1)}),$$

$$S_{mn} = \frac{1}{2i}(\delta_{m(n+1)} - \delta_{m(n-1)}),$$

$$L_{mn} = \hbar \, m \, \delta_{mn}.$$
(22)

Using the relation

$$\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\beta} = \delta(\beta) \quad \text{for} \quad \beta \in [-\pi, +\pi],$$
(23)

one obtains, according to Eq. (20),

$$\tilde{C}(\theta, p) = 2\pi \, \mathrm{tr}[C \cdot V(\theta, p)] = \cos\theta,$$
(24)

$$\tilde{S}(\theta, p) = 2\pi \, \mathrm{tr}[S \cdot V(\theta, p)] = \sin\theta,$$
(25)

$$\tilde{L}(\theta, p) = 2\pi \, \mathrm{tr}[L \cdot V(\theta, p)] = p.$$
(26)

As a real 2π-periodic potential $U(\varphi)$ of a Hamilton operator

$$H(\varphi, L) = \gamma \, L^2 + U(\varphi)$$
(27)

can be expanded in terms of the multiplication operators

$$C^{(k)} = \cos k\varphi, \quad S^{(k)} = \sin k\varphi, \, k \in \mathbb{Z},$$
(28)

the following relations are useful too:

$$\tilde{C^{(k)}}(\theta, p) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot C^{(k)}] = \cos k\theta,$$
(29)

$$\tilde{S^{(k)}}(\theta, p) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot S^{(k)}] = \sin k\theta,$$
(30)

$$\tilde{U}(\theta, p) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot U] = U(\theta),$$
(31)

$$\tilde{L^2}(\theta, p) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot L^2] = p^2,$$
(32)

$$\tilde{H}(\theta, p) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot H] = H(\theta, p).$$
(33)

In the same manner one gets for the "mixed" products, $L \cdot C$ and $C \cdot L$,

$$\widetilde{LC}(\theta, p) = 2\pi \, \mathrm{tr}[L \cdot C \cdot V(\theta, p)] = p\cos\theta + i\frac{\hbar}{2}\sin\theta,$$
(34)

$$\widetilde{CL}(\theta, p) = 2\pi \, \mathrm{tr}[C \cdot L \cdot V(\theta, p)] = p\cos\theta - i\frac{\hbar}{2}\sin\theta,$$
(35)

so that

$$\widetilde{[L,C]}(\theta, p) = 2\pi \, \mathrm{tr}\{[L,C] \cdot V(\theta, p)\} = i\hbar\sin\theta,$$
(36)

$$\widetilde{\{L,C\}}(\theta, p) = 2\pi \, \mathrm{tr}[\{L,C\} \cdot V(\theta, p)] = 2p\cos\theta.$$
(37)

Equation (36) corresponds to the Lie algebra relation $[L,C] = \hbar i S$ of the operators, (21), or the Poisson bracket $\{p, \cos\theta\} = \sin\theta$ [1,2]. The occurrence of $\hbar$ in Eq. (36) shows its quantum mechanical character. In the limit $\hbar \to 0$ the operators L and C commute, i.e., they become "classical."

The anticommutator function, (37), does not contain any $\hbar$ and is purely classical.

The general case of phase-space functions for products of operators is discussed in Sec. III below. The phase-space function $\tilde{V}(\theta, p)$ of the operator (matrix) $V(\theta_1, p_1)$ itself is

$$\tilde{V}(\theta, p) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot V(\theta_1, p_1)]$$

$$= \delta(\theta - \theta_1) \, \mathrm{sinc}[\pi(p - p_1)/\hbar].$$
(38)

The corresponding relation for the planar space operator, (16), is, using Eq. (17),

$$\tilde{\Delta}(q, p) = \mathrm{tr}[\Delta(q, p) \cdot \Delta(q_1, p_1)]$$

$$= 2\pi\hbar \, \delta(q - q_1) \, \delta(p - p_1).$$
(39)

The inverse problem, namely, constructing the Hilbert-space operator A from a given phase-space function $\tilde{A}(\theta, p)$, is addressed in Sec. III.

2. Marginal distributions and transition probabilities

As the Wigner functions, (8) and (7), can be negative for certain subsets of their respective phase spaces, they are not proper probability distributions. That $W_\phi(q, p)$ may be negative can be seen from the relation

$$W_\phi(q = 0, p = 0) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \, \phi^*(-\xi/2)\phi(\xi/2).$$
(40)

So, if $\phi(x)$ is an odd function, the right-hand side of Eq. (40) is negative. This is so, e.g., for odd n eigenfunctions of the harmonic oscillator [10,13]. Similar arguments apply to the Wigner function, (7). In addition, it follows from Eq. (6) that the Wigner function $V_{mm}(\theta, p)$ of the basic function $e_m(\varphi)$ is the sinc function, which has negative regions [1].

Contrary to the Wigner functions themselves the marginal distributions,

$$\int_{-\infty}^{\infty} dp \, W_\phi(q, p) = |\phi(q)|^2, \quad \int_{-\infty}^{\infty} dq \, W_\phi(q, p) = |\hat{\phi}(q)|^2,$$
(41)

where $\hat{\phi}(p)$ is the Fourier transform of $\phi(q)$,

$$\hat{\phi}(p) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} dq\, \phi(q)\, e^{-iqp/\hbar}, \qquad (42)$$

are the usual nonnegative quantum mechanical ones. (Definition (42) follows the conventions of Ref. [6].) This is considered to be an essential requirement for the properties of Wigner functions [6].

The situation is slightly more complicated for the Wigner function $V_\psi(\theta, p)$. Here we have

$$\int_{-\infty}^{\infty} d\bar{p}\, V_\psi(\theta, p) = \frac{1}{2\pi} |\psi(\theta)|^2, \qquad (43)$$

$$\int_{-\pi}^{\pi} d\theta\, V_\psi(\theta, p) = \sum_{n\in\mathbb{Z}} |c_n|^2 \operatorname{sinc} \pi(p/\hbar - n) \equiv \omega_\psi(p), \qquad (44)$$

$$\int_{-\infty}^{\infty} d\bar{p}\, \omega_\psi(p)\, \operatorname{sinc} \pi(p/\hbar - m) = |c_m|^2, \quad m \in \mathbb{Z}. \quad (45)$$

Equation (45), which yields the quantum mechanical marginal probabilities $|c_m|^2$ for the quantized angular momentum, follows from Eq. (44) as a consequence of the orthonormality relations

$$\int_{-\infty}^{\infty} d\bar{p}\, \operatorname{sinc} \pi(m - \bar{p})\, \operatorname{sinc} \pi(n - \bar{p}) = \delta_{mn}. \qquad (46)$$

More details, especially about the interpolating role of the sinc functions, can be found in Ref. [1].

If the Wigner functions $W_{\phi_1}(q, p)$ and $W_{\phi_2}(q, p)$ for two states ϕ_1 and ϕ_2 are known, then one can calculate the probability for the transitions $\phi_1 \leftrightarrow \phi_2$,

$$|(\phi_2, \phi_1)_{\mathbb{R}}|^2 = 2\pi\hbar \int_{\mathbb{R}^2} dq\, dp\, W_{\phi_2}(q, p)\, W_{\phi_1}(q, p), \quad (47)$$

which implies

$$\int_{\mathbb{R}^2} dq\, dp\, W_\phi^2(q, p) = \frac{1}{2\pi\hbar}. \qquad (48)$$

An analog relation holds for the cylindrical case [1],

$$|(\psi_2, \psi_1)_{\mathbb{S}^1}|^2 = 2\pi \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta\, V_{\psi_2}(\theta, p)\, V_{\psi_1}(\theta, p), \quad (49)$$

and therefore

$$\int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta\, V_\psi^2(\theta, p) = \frac{1}{2\pi}. \qquad (50)$$

3. Traces of operators

It is possible to discuss the trace of the product of two Hilbert-space operators in terms of function (15) or (20) without having treated the operators themselves: If A and B are two operators in a Hilbert space with the scalar product, (13), then

$$\operatorname{tr}(A \cdot B) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} dq\, dp\, \tilde{A}(q, p)\, \tilde{B}(q, p). \qquad (51)$$

This relation can be obtained by using expression (15) or (17) under the last integral (see, e.g., Ref. [4]).

If A is a density operator ρ and B a self-adjoint observable O, we get from Eq. (51) the expectation value of O with respect to ρ,

$$\langle O \rangle_\rho = \operatorname{tr}(\rho \cdot O) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} dq\, dp\, \tilde{\rho}(q, p)\, \tilde{O}(q, p), \quad (52)$$

with, according to Eq. (15),

$$\tilde{\rho}(q, p) = \int_{-\infty}^{\infty} d\xi\, e^{-ip\xi/\hbar} \langle q + \xi/2 | \rho | q - \xi/2 \rangle. \qquad (53)$$

Inserting for ρ the projection operator $P_\phi = |\phi\rangle\langle\phi|$ the right-hand side becomes $2\pi\hbar$ times the Wigner function $W_\phi(q, p)$ from Eq. (8). Thus,

$$W_\rho(q, p) = \frac{1}{2\pi\hbar} \tilde{\rho}(q, p) = \frac{1}{2\pi\hbar} \operatorname{tr}[\rho\, \Delta(q, p)] \qquad (54)$$

is the generalization of the Wigner function from that for a pure state to that of a mixed state.

Another useful relation is obtained for $A = \rho_1$ and $B = \rho_2$:

$$\operatorname{tr}(\rho_1 \cdot \rho_2) = 2\pi\hbar \int_{\mathbb{R}^2} dq\, dp\, W_{\rho_1}(q, p)\, W_{\rho_2}(q, p). \qquad (55)$$

If $\rho_2 = \rho_1 = \rho$, then

$$2\pi\hbar \int_{\mathbb{R}^2} dq\, dp\, W_\rho^2(q, p) = \operatorname{tr}(\rho^2) \leqslant 1. \qquad (56)$$

Relations completely analog to those in Eqs. (51)–(56) are derived in Ref. [1] for the cylindrical case [note the difference in normalization: $\operatorname{tr}[\Delta(q, p)] = 1$, $\operatorname{tr}[V(\theta, p)] = 1/(2\pi)$]:

$$\operatorname{tr}(A \cdot B) = 2\pi \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta$$
$$\times\, \operatorname{tr}[A \cdot V(\theta, p)] \operatorname{tr}[B \cdot V(\theta, p)]. \qquad (57)$$

Application to the expectation value of an operator O for a given density operator ρ gives

$$\langle O \rangle_\rho = \operatorname{tr}(\rho \cdot O)$$
$$= 2\pi \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta\, \operatorname{tr}[\rho \cdot V(\theta, p)] \operatorname{tr}[O \cdot V(\theta, p)]. \qquad (58)$$

Here $V_\rho(\theta, p) = \operatorname{tr}[\rho \cdot V(\theta, p)]$ is the Wigner function for a given ρ.

In addition, we have

$$\int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta\, V_{\rho_2}(\theta, p)\, V_{\rho_1}(\theta, p) = \frac{1}{2\pi} \operatorname{tr}(\rho_2 \cdot \rho_1). \quad (59)$$

Other properties of $V_\rho(\theta, p)$ are discussed in Ref. [1].

4. Recovering the wave function

If the Wigner function $W_\phi(q, p)$ or $V_\psi(\theta, p)$ of the wave function $\phi(x)$ or $\psi(\varphi)$ is given, then the wave functions may be retrieved up to an overall constant phase [14]: Multiplying the Wigner function, (8), by $\exp(ip\xi_1/\hbar)$ and integrating over $p/\hbar$ yields a delta function $\delta(\xi - \xi_1)$, which leads to

$$\phi^*(q - \xi_1/2)\phi(q + \xi_1/2) = \int_{-\infty}^{\infty} d(p/\hbar)\, e^{ip\xi_1/\hbar} W_\phi(q, p). \qquad (60)$$

Finally, first, putting $q = \xi_1/2$ and then renaming ξ_1 as q gives

$$\phi^*(0)\,\phi(q) = \int_{-\infty}^{\infty} d(p/\hbar)\, e^{ipq/\hbar}\, W_\phi(q/2,p). \quad (61)$$

Here $\phi^*(0)$ is a fixed complex number the modulus of which can be absorbed into the normalization of $\phi(q)$ and the constant phase of which has no physical significance.

Exactly the same reasoning leads, in the cylindrical case, to the result

$$\psi^*(0)\,\psi(\theta) = 2\pi \int_{-\infty}^{\infty} d\bar{p}\, e^{ip\theta/\hbar}\, V_\psi(\theta/2,p). \quad (62)$$

B. Moyal functions

Wigner functions (8) and (7) are special cases of bilinear forms for wave functions in which both wave functions $\psi_1(\varphi)$ and $\psi_2(\varphi)$ or $\phi_1(x)$ and $\phi_2(x)$, respectively, are identified. The general forms are

$$W_{\phi_2\phi_1}(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi\, e^{-ip\xi/\hbar} \phi_2^*(q-\xi/2)\phi_1(q+\xi/2) \quad (63)$$

and

$$V_{\psi_2\psi_1}(\theta,p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\vartheta}{2\pi}\, e^{-ip\vartheta/\hbar} \psi_2^*(\theta-\vartheta/2)\psi_1(\theta+\vartheta/2). \quad (64)$$

These so-called Moyal functions [23] have a number of interesting properties of which a few are listed here. We start with $W_{\phi_2\phi_1}(q,p)$, relying mainly on Refs. [16] and [17]:

$$\int_{-\infty}^{\infty} dp\, W_{\phi_2\phi_1}(q,p) = \phi_2^*(q)\,\phi_1(q), \quad (65)$$

$$\int_{-\infty}^{\infty} dq\, W_{\phi_2\phi_1}(q,p) = \hat{\phi}_2^*(p)\,\hat{\phi}_1(p), \quad (66)$$

$$\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, W_{\phi_2\phi_1}(q,p) = (\phi_2,\phi_1), \quad (67)$$

$$\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, W_{\phi_2\phi_1}^*(q,p)\, W_{\phi_4\phi_3}(q,p)$$
$$= \frac{1}{2\pi\hbar}(\phi_1,\phi_3)\,(\phi_2,\phi_4)^*. \quad (68)$$

Correspondingly, we have for the cylindrical case [1]

$$\int_{-\infty}^{\infty} d\bar{p}\, V_{\psi_2\psi_1}(\theta,p) = \frac{1}{2\pi}\psi_2^*(\theta)\,\psi_1(\theta), \quad (69)$$

$$\int_{-\pi}^{\pi} d\theta\, V_{\psi_2\psi_1}(\theta,p) = \sum_{m\in\mathbb{Z}} c_m^{(2)*}\, \mathrm{sinc}\,\pi(p/\hbar - m)c_m^{(1)}, \quad (70)$$

$$\int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta\, V_{\psi_2\psi_1}(\theta,p) = (\psi_2,\psi_1), \quad (71)$$

$$\int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta\, V_{\psi_2\psi_1}^*(\theta,p)\, V_{\psi_4\psi_3}(\theta,p)$$
$$= \frac{1}{2\pi}(\psi_1,\psi_3)\,(\psi_2,\psi_4)^*. \quad (72)$$

As for Eqs. (41) and (44), the main difference between the two cases can be seen from relations (66) and (70): the classical angular momentum p is continuous, whereas the quantum mechanical one is discrete and the sinc function provides the interpolation.

Again using relation (46) yields the analog to Eq. (66) (n corresponds to p):

$$\int_{-\infty}^{\infty} d\bar{p}\, \mathrm{sinc}\,\pi(n-\bar{p}) \int_{-\pi}^{\pi} d\theta\, V_{\psi_2\psi_1}(\theta,p) = c_n^{(2)*}\, c_n^{(1)}. \quad (73)$$

III. OPERATORS FROM THEIR PHASE-SPACE FUNCTION

A. The cylindrical phase space

In Eqs. (17) and (20) we have seen how a phase-space function can be associated with a given Hilbert-space operator, for the planar case and for the cylindrical one as well. There is, of course, the question: Given the phase-space functions $\tilde{A}(q,p)$ and $\tilde{A}(\theta,p)$, how can one construct the corresponding Hilbert-space operators? The answer is well known for a planar phase space (see below) and is now first discussed for the cylindrical phase space. Here it is sufficient to determine all matrix elements $A_{mn} = (e_m, A e_n)$ of an operator A [we drop the index "$\mathbb{S}^1$" of the scalar product (1) in this subsection]. Essential use is made of the important relation [1]

$$2\pi \int_{-\pi}^{\pi} d\theta \int_{-\infty}^{\infty} d\bar{p}\, V_{kl}(\theta,p)\, V_{mn}(\theta,p) = \delta_{kn}\delta_{lm}. \quad (74)$$

In the present subsection this relation is used more consequently than in Sec. IV B of Ref. [1].

1. Single operators

Using expansion (4) and Eqs. (9) and (74) we get

$$(\psi_2, A\psi_1) = \sum_{kl} c_k^{(2)*} A_{kl} c_l^{(1)}$$
$$= 2\pi \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta \sum_{klmn} c_m^{(2)*}$$
$$\times V_{mn}(\theta,p) c_n^{(1)} A_{kl} V_{lk}(\theta,p)$$
$$= \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta\, V_{\psi_2\psi_1}(\theta,p)\, \tilde{A}(\theta,p),$$
$$\tilde{A}(\theta,p) = 2\pi\, \mathrm{tr}[A \cdot V(\theta,p)], V(\theta,p) = (V_{mn}(\theta,p)), \quad (75)$$

implying

$$A = (A_{mn}) = \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta\, V(\theta,p)\, \tilde{A}(\theta,p). \quad (76)$$

This formula allows us to calculate the matrix elements A_{mn} once the function $\tilde{A}(\theta,p)$ is given. If $\tilde{A}(\theta,p)$ is real, then A is self-adjoint.

A simple example is $\tilde{A}(\theta,p) = p^2$ [see Eq. (32)]. Integrating $V_{mn}(\theta,p)$ in Eq. (5) over θ gives a factor $2\pi\delta_{mn}$. It remains that

$$A_{mn} = \delta_{mn} \int_{-\infty}^{\infty} d\bar{p}\, p^2 \int_{-\pi}^{\pi} \frac{d\vartheta}{2\pi} e^{-ip\vartheta/\hbar} e^{im\vartheta} \quad (77)$$
$$= \delta_{mn} \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} \frac{d\vartheta}{2\pi} \left[\left(-\hbar^2 \partial_\vartheta^2\right) e^{-ip\vartheta/\hbar}\right] e^{im\vartheta}$$
$$= \delta_{mn} \int_{-\pi}^{\pi} d\vartheta \left[-\hbar^2 \partial_\vartheta^2 \delta(\vartheta)\right] e^{im\vartheta}$$
$$= \hbar^2 m^2 \delta_{mn} = (e_m, L^2 e_n), \quad (78)$$

with L from Eq. (21).

2. Product of operators I: Convolution

Of considerable interest is the correspondence between the product $A \cdot B$ of two Hilbert-space operators and their phase-space function $\widetilde{AB}(\theta)$ (the usual "dot" between the operators is omitted for their phase-space functions): According to the general equation, (76), we have

$$A \cdot B = \int_{-\infty}^{\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta \, V(\theta, p) \, \widetilde{AB}(\theta, p). \qquad (79)$$

Again using relation (74) one can verify that

$$\widetilde{AB}(\theta, p) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot A \cdot B]$$
$$= \int_{-\infty}^{\infty} d\bar{p}_1 \int_{-\pi}^{\pi} d\theta_1 \, G_A(\theta, p; \theta_1, p_1) \tilde{B}(\theta_1, p_1),$$
$$G_A(\theta, p; \theta_1, p_1) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot A \cdot V(\theta_1, p_1)]. \qquad (80)$$

Here the phase-space function of the product $A \cdot B$ is a kind of convolution of those for the single operators. Compare Ref. [24] for a similar relation in the planar case.

3. Product of operators II: ⋆ product

It turns out that the procedure which is employed in the planar case [4,6,16–20] can be used in the cylindrical one too. Starting again with the relation

$$\widetilde{AB}(\theta, p) = 2\pi \, \mathrm{tr}[V(\theta, p) \cdot A \cdot B], \qquad (81)$$

and using representation (76) for A and B inside the trace yields

$$\widetilde{AB}(\theta, p) = 2\pi \int_{-\infty}^{\infty} d\bar{p}_1 \, d\bar{p}_2 \int_{-\pi}^{\pi} d\theta_1 \, d\theta_2$$
$$\times \mathrm{tr}[V(\theta, p) \cdot V(\theta_1, p_1) \cdot V(\theta_2, p_2)]$$
$$\times \tilde{A}(\theta_1, p_1) \, \tilde{B}(\theta_2, p_2). \qquad (82)$$

Inserting for V the integral representation, (5), and using relation (23), one obtains for the trace

$$\mathrm{tr}[V(\theta, p) \cdot V(\theta_1, p_1) \cdot V(\theta_2, p_2)] \qquad (83)$$
$$= \frac{4}{(2\pi)^3} e^{-2i[p(\theta_1 - \theta_2) + p_1(\theta_2 - \theta) + p_2(\theta - \theta_1)]/\hbar}. \qquad (84)$$

Observing that

$$[p(\theta_1 - \theta_2) + p_1(\theta_2 - \theta) + p_2(\theta - \theta_1)]$$
$$= -[(\theta_1 - \theta)b - \alpha(p_1 - p)],$$
$$\alpha = \theta_2 - \theta, \; b = p_2 - p, \; d\theta_2 \, dp_2 = d\alpha \, db,$$
$$\tilde{B}(\theta + \alpha, p + b) = e^{(\alpha \partial_\theta + b \partial_p)} \tilde{B}(\theta, p) \quad \text{(Taylor series)},$$
$$\alpha \, e^{-2i\alpha(p_1 - p)/\hbar} = -\frac{i\hbar}{2} \partial_p e^{-2i\alpha(p_1 - p)/\hbar},$$
$$b \, e^{2ib(\theta_1 - \theta)/\hbar} = \frac{i\hbar}{2} \partial_\theta e^{2ib(\theta_1 - \theta)/\hbar} \qquad (85)$$

yields

$$\widetilde{AB}(\theta, p) = \frac{4}{(2\pi)^2} \int_{-\infty}^{\infty} d\bar{p}_1 \, d\bar{b} \int_{-\pi}^{\pi} d\theta_1 \, d\alpha \qquad (86)$$
$$\times \tilde{A}(\theta_1, p_1)\{e^{2i[(\theta_1 - \theta)b - (p_1 - p)\alpha]/\hbar} e^{\frac{\hbar}{2i}\Lambda(\theta, p)} \tilde{B}(\theta, p)\}, \qquad (87)$$

$$\Lambda(\theta, p) = \overleftarrow{\partial}_p \overrightarrow{\partial}_\theta - \overleftarrow{\partial}_\theta \overrightarrow{\partial}_p, \qquad (88)$$

where $\overleftarrow{\partial}_p$ and $\overleftarrow{\partial}_\theta$ act to the left, here on exponentials.

Integration over $\bar{b}$ gives a delta function $\delta(\theta_1 - \theta)$ which allows us to carry out the θ_1 integration. There remains the integral

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\bar{p}_1 \int_{-\pi}^{\pi} d\alpha \, \tilde{A}(\theta, p_1) \, e^{-2i(p_1 - p)\alpha/\hbar}. \qquad (89)$$

Inserting into

$$\tilde{A}(\theta, p_1) = 2\pi \, \mathrm{tr}[A \cdot V(\theta, p_1)] \qquad (90)$$

representation (5) for $V(\theta, p_1)$ and carrying out the integrations gives, for integral (89),

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\bar{p}_1 \int_{-\pi}^{\pi} d\alpha \, \tilde{A}(\theta, p_1) \, e^{-2i(p_1 - p)\alpha/\hbar} = \tilde{A}(\theta, p), \qquad (91)$$

so that, finally,

$$\widetilde{AB}(\theta, p) = \tilde{A}(\theta, p) \, e^{\frac{\hbar}{2i}\Lambda(\theta, p)} \tilde{B}(\theta, p) \qquad (92)$$

$$\equiv \tilde{A}(\theta, p) \star \tilde{B}(\theta, p) \qquad (93)$$
$$= \tilde{B}(\theta, p) \, e^{-\frac{\hbar}{2i}\Lambda(\theta, p)} \tilde{A}(\theta, p),$$
$$\widetilde{A^2}(\theta, p) = \tilde{A}^2(\theta, p).$$

This is the complete analog to the corresponding formula in the planar case (see the next subsection). Thus, the $\star$ product formalism of the planar phase space [25–36] can be carried over to the cylindrical one. Only a few important examples are mentioned here:

From Eq. (92) one obtains, for the commutator and anticommutator,

$$\widetilde{[A, B]}(\theta, p) = 2i \, \tilde{A}(\theta, p) \sin\left[\frac{\hbar}{2}\Lambda(\theta, p)\right] \tilde{B}(\theta, p), \qquad (94)$$

$$\widetilde{\{A, B\}}(\theta, p) = 2 \, \tilde{A}(\theta, p) \cos\left[\frac{\hbar}{2}\Lambda(\theta, p)\right] \tilde{B}(\theta, p). \qquad (95)$$

At first order of $\hbar$ the right-hand side of Eq. (94) becomes—as usual—$i\hbar$ times the Poisson bracket of functions $\tilde{A}$ and $\tilde{B}$, and the right-hand side of (95) at zeroth order becomes twice their product. The expression for the commutator is of special importance for the time evolution of an operator A or that of a Wigner function which is determined by the commutator $[H, A]$, etc. (see Sec. IV, below).

Equation (92) can also be written in terms of the Bopp operators (shifts) [37,38], which are especially helpful for explicit calculations,

$$\widetilde{A \cdot B}(\theta, p) = \tilde{A}(\theta, p) \star \tilde{B}(\theta, p) = \tilde{A}(\tilde{\theta}, \tilde{p}) \cdot \tilde{B}(\theta, p), \qquad (96)$$

where

$$\tilde{\theta} = \theta - \frac{\hbar}{2i}\partial_p, \quad \tilde{p} = p + \frac{\hbar}{2i}\partial_\theta. \tag{97}$$

As a simple application of Eq. (96) consider the example discussed in Sec. II A above, $\tilde{L}(\theta,p) = p$ and $\tilde{C}(\theta,p) = \cos\theta$: Using Eq. (96) we get

$$\widetilde{LC}(\theta,p) = \left(p + \frac{\hbar}{2i}\partial_\theta\right)\cos\theta = p\cos\theta - \frac{\hbar}{2i}\sin\theta \tag{98}$$

and

$$\begin{aligned}\widetilde{CL}(\theta,p) &= \cos\left(\theta - \frac{\hbar}{2i}\partial_p\right)p \\ &= \left[\cos\theta\,\cos\left(\frac{\hbar}{2i}\partial_p\right) + \sin\theta\,\sin\left(\frac{\hbar}{2i}\partial_p\right)\right]p \\ &= p\cos\theta + \frac{\hbar}{2i}\sin\theta,\end{aligned} \tag{99}$$

which coincide with Eqs. (34) and (35). In both cases, (98) and (99), the relation

$$\cos(x-y) = \cos x\,\cos y + \sin x\,\sin y \tag{100}$$

has been used.

B. The planar phase space

We recall only very briefly a few essential elements of constructing Hilbert-space operators from functions in the planar phase space, namely, mainly those which correspond closely to relations in the cylindrical case discussed in the preceding subsection. Note the many references relating to the planar case given there.

Like the operator $V(\theta,p)$ in the cylindrical case the operator $\Delta(q,p)$ from Eq. (16) can play a crucial role for the planar one: Its importance for the Wigner function in the planar phase space was emphasized by Leaf [21,22]. de Groot and Suttorp give a very recommendable exposition of that approach in their textbook [4].

The inversion of relation (17) is

$$A = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\,dp\,\Delta(q,p)\,\tilde{A}(q,p), \tag{101}$$

which is the analog to Eq. (76). As $\Delta(q,p)$ can also be written as

$$\Delta(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du\,dv\,e^{-i[(Q-q)u+(P-p)v]/\hbar}, \tag{102}$$

where Q and P are the usual position and momentum operators, the operator A may also be (Weyl) represented as

$$A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du\,dv\,e^{-i(Qu+Pv)/\hbar}\,\hat{A}(u,v), \tag{103}$$

where $\hat{A}(u,v)$ is the Fourier transform of $\tilde{A}(q,p)$:

$$\tilde{A}(q,p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du\,dv\,\hat{A}(u,v)e^{-i(qu+pv)/\hbar}. \tag{104}$$

The important operator $\Delta(q,p)$ is hardly mentioned or used in most of the literature on the planar Wigner function. Probably the main reason is that the above formulas can be rewritten by

using the operator relations

$$[P,Q] = \frac{\hbar}{i},\; e^{A+B} = e^A \cdot e^B \cdot e^{-[A,B]/2}. \tag{105}$$

They lead to

$$\begin{aligned}A = \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{\infty} &dq\,dp\,du\,dv\,\tilde{A}(q,p)\\ &\times e^{iuv/(2\hbar)}e^{-i(Q-q)u/\hbar}\cdot e^{-i(P-p)v/\hbar}.\end{aligned} \tag{106}$$

Replacing the product uv with the differential operators $-\hbar^2\partial_q\,\partial_p$ and integrating partially one obtains

$$\begin{aligned}A = \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{\infty} &dq\,dp\,du\,dv\\ &\times e^{-i(Q-q)u/\hbar}\cdot e^{-i(P-p)v/\hbar}\exp\left[\frac{\hbar}{2i}\partial_q\partial_p\right]\tilde{A}(q,p).\end{aligned} \tag{107}$$

Here the operator ordering is such that the position operators are to the left of the momentum ones.

The phase-space function $\widetilde{AB}(q,p)$ of the product $A \cdot B$ of two operators is given by

$$\begin{aligned}\widetilde{AB}(q,p) &= \tilde{A}(q,p)\,\exp[\hbar\Lambda(q,p)/(2i)]\tilde{B}(q,p),\\ \Lambda(q,p) &= \overleftarrow{\partial}_p\,\overrightarrow{\partial}_q - \overleftarrow{\partial}_q\,\overrightarrow{\partial}_p\,.\end{aligned} \tag{108}$$

The expression, (92), for the cylindrical case is the analog of relation (108), which was first derived by Groenewold [25].

There are, of course, many more important properties of Wigner functions for the planar phase space, which have been worked out over decades (see the numerous references in Sec. III A). For most of them their analogues in the cylindrical case await new applications.

IV. DYNAMICAL AND ENERGY EQUATIONS

Quantum mechanics provides—among others—two basic elements for the description of atomic systems. First, it gives a description of their time evolution, i.e., their dynamics, either—in the Schrödinger picture—in terms of a Schrödinger equation for the wave function of the system or—in the Heisenberg picture—in terms of equations of motion for the operators ("observables") which characterize the system.

Second, it allows, in principle, for calculation of the eigenvalues and eigenstates of observables, especially the energy, which are important for a quantitative description of the system. In addition, in the realm of continuous (higher) energies it provides the basis for the description of, e.g., stationary scattering processes for a given energy. In this second case, which concerns the stationary structure of physical systems, the time-independent Schrödinger equation is an important tool.

Similarly, one can ask, How is the time evolution of a system, from a given initial state, described by its associated Wigner function? and Are there appropriate structure equations corresponding to the eigenvalue, etc., equations of the conventional time-independent quantum mechanics? The question of time evolution was answered by Wigner, in his very

first paper on the subject, for planar phase spaces [3] in terms of a generalized Liouville equation. The problem was later discussed extensively by Moyal [23]. For further development of this subject see the reviews [6,8,10,39,40].

Discussion of the time-independent structure or energy "eigenvalue" equations for Wigner functions has its origin in applications to statistical mechanics systems, the topic of Wigner's seminal paper [3]: For a quantum canonical ensemble in equilibrium the quantum mechanical time evolution operator

$$U(t) = e^{-itH/\hbar} \tag{109}$$

is replaced with

$$\Omega(\beta) = e^{-\beta H}, \quad \beta = \frac{1}{k_B T}, \tag{110}$$

where H is the associated Hamilton operator. The operator Ω provides the density operator

$$\rho(\beta) = \Omega(\beta)/Z(\beta), \quad Z(\beta) = \text{tr}(\Omega) \tag{111}$$

of the system and obeys the Bloch equation [41]

$$\partial_\beta \Omega = -H \cdot \Omega = -\Omega \cdot H. \tag{112}$$

Wigner functions related to Ω have been discussed in Refs. [26] and [42]. A brief summary is contained in Ref. [6].

Comparing Eqs. (109) and (110) we note the correspondence

$$-t/\hbar \longleftrightarrow i\beta. \tag{113}$$

In modern approaches [43–45] this relation is treated under the topic "KMS condition," which deals with the analytic continuation between the two regions in the complex t plane.

A. Time evolution of the Wigner function for a density matrix

For the time dependence of a density operator $\rho(t)$ von Neumann's equation holds:

$$i\hbar \partial_t \rho(t) = [H, \rho(t)]. \tag{114}$$

In order to translate this into the corresponding equation for the phase-space function,

$$\tilde{\rho}(\theta, p; t) = 2\pi V_\rho(\theta, p; t) = 2\pi \, \text{tr}[\rho(t) \cdot V(\theta, p)], \tag{115}$$

we need the phase-space function $\tilde{H}(\theta, p)$ of the Hamilton operator, (27). According to Eq. (33) this is given by

$$\tilde{H}(\theta, p) = \gamma \, p^2 + U(\theta). \tag{116}$$

In the following we use for $U(\theta)$ the pendulum potential

$$U(\theta) = -A \, \cos \theta, \tag{117}$$

and for $\rho(t)$ the diagonal form [1],

$$\rho(t) = (\rho_{mn}(t)), \, \rho_{mn}(t) = \lambda_m(t) \, \delta_{mn},$$
$$\lambda \geqslant 0, \quad \sum_{m \in \mathbb{Z}} \lambda_m(t) = 1. \tag{118}$$

This gives

$$V_\rho(\theta, p; t) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \lambda_m(t) \, \text{sinc} \, \pi(p/\hbar - m), \tag{119}$$

$$\text{sinc} \, \pi(p/\hbar - m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\vartheta \, e^{-i(p/\hbar - m)\vartheta}. \tag{120}$$

Thus, $V_\rho(\theta, p; t)$ is independent of θ. The more general θ-dependent case is treated in the next subsection, where additional remarks on the pendulum can be found, too.

Using relations (96), (97), and (100) we have

$$\left(p + \frac{\hbar}{2i} \partial_\theta \right)^2 \cdot V_\rho(p; t) = p^2 \, V_\rho(p; t),$$

$$\cos \left(\theta - \frac{\hbar}{2i} \partial_p \right) V_\rho(p; t) = \left[\cos \theta \, \cos \left(\frac{\hbar}{2i} \partial_p \right) \right.$$
$$\left. + \sin \theta \sin \left(\frac{\hbar}{2i} \partial_p \right) \right] V_\rho(p; t),$$

$$\widetilde{HV}_\rho(\theta, p) = \left\{ \gamma \, p^2 - A \left[\cos \theta \, \cos \left(\frac{\hbar}{2i} \partial_p \right) \right. \right.$$
$$\left. \left. + \sin \theta \sin \left(\frac{\hbar}{2i} \partial_p \right) \right] \right\} V_\rho(p; t), \tag{121}$$

and

$$\text{sinc} \, \pi [p/\hbar + \frac{1}{2i} \partial_\theta - m] \cos \theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\vartheta \, e^{-i[p/\hbar - m + \frac{1}{2i} \partial_\theta]\vartheta} \cos \theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\vartheta \, \cos(\theta - \vartheta/2) e^{-i(p/\hbar - m)\vartheta}$$

$$= \left[\cos \theta \, \cos \left(\frac{\hbar}{2i} \partial_p \right) \right.$$
$$\left. - \sin \theta \sin \left(\frac{\hbar}{2i} \partial_p \right) \right] \text{sinc} \, \pi(p/\hbar - m),$$

$$\widetilde{V_\rho H}(\theta, p) = \left\{ \gamma p^2 - A \left[\cos \theta \, \cos \left(\frac{\hbar}{2i} \partial_p \right) \right. \right.$$
$$\left. \left. - \sin \theta \sin \left(\frac{\hbar}{2i} \partial_p \right) \right] \right\} V_\rho(p; t), \tag{122}$$

so that

$$\widetilde{[H, V_\rho]} = -2A \sin \theta \left[\sin \left(\frac{\hbar}{2i} \partial_p \right) \right] V_\rho(p; t). \tag{123}$$

Combined with Eq. (114) this yields

$$\partial_t V_\rho(p; t) = -\frac{2A}{i\hbar} \sin \theta \, \sin \left(\frac{\hbar}{2i} \partial_p \right) V_\rho(p; t). \tag{124}$$

With

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{125}$$

and

$$A \sin \theta = \partial_\theta U(\theta), \tag{126}$$

Eq. (124) can be written as

$$[\partial_t - \partial_\theta U(\theta)\,\partial_p]V_\rho(p;t)$$

$$= \partial_\theta U(\theta)\left[\sum_{n=1}^\infty \left(\frac{\hbar}{2}\right)^{2n}\frac{\partial_p^{2n+1}}{(2n+1)!}\right]V_\rho(p;t). \quad (127)$$

Note that the left-hand side of Eq. (127) is independent of $\hbar$, whereas the right-hand side represents quantum effects.

B. Generalized Liouville equation

The time evolution of the model discussed in the preceding subsection is an example [see the left-hand side of Eq. (123)] of the general time evolution equation, derived in Ref. [1], for a Moyal function $V_{\psi_2\psi_1}(\theta,p;t)$:

$$\partial_t V_{\psi_2\psi_1}(\theta,p;t) = \frac{1}{\hbar}(\psi_2(t),K(\theta,p)\psi_1(t))_{\mathbb{S}^1}, \quad (128)$$

$$K(\theta,p) = i[H,V(\theta,p)]. \quad (129)$$

In the following the righ-hand side of this equation is worked out in detail directly, partially following Ref. [10].

We start from

$$i\hbar\partial_t\psi(\theta;t) = H\psi(\theta;t), \quad H = H_0 + U(\theta), \quad (130)$$

$$H_0 = \frac{1}{2mr_0^2}L^2 = -\varepsilon\frac{d^2}{d\theta^2}, \quad \varepsilon = \frac{\hbar^2}{2mr_0^2}, \quad (131)$$

where mr_0^2 is the moment of inertia of a point mass rotating at a distance r_0 around an axis.

In the following we abbreviate:

$$\psi_2(\theta-\vartheta/2)\equiv\psi_2, \quad \psi_1(\theta+\vartheta/2)\equiv\psi_1. \quad (132)$$

It follows from Eqs. (64) and (130) that

$$i\hbar\partial_t V_{\psi_2\psi_1} = \frac{1}{2\pi}\int_{-\pi}^\pi\frac{d\vartheta}{2\pi}e^{-ip\vartheta/\hbar}[-(H\psi_2^*)\psi_1 + \psi_2^*(H\psi_1)]. \quad (133)$$

Starting with the H_0 part, observing that

$$\frac{d\psi_2}{d\theta} = -2\frac{d\psi_2}{d\vartheta}, \quad \frac{d\psi_1}{d\theta} = 2\frac{d\psi_1}{d\vartheta} \quad (134)$$

and then integrating partially with respect to ϑ under the integral, (133), yields

$$i\hbar\partial_t V_{\psi_2\psi_1}(\theta,p;t) = -\frac{2i\varepsilon p}{\hbar}\partial_\theta V_{\psi_2\psi_1}(\theta,p;t)$$

$$-\frac{2\varepsilon}{(2\pi)^2}\partial_\theta b_{\psi_2\psi_1}(\theta,p,\vartheta=\pm\pi;t), \quad (135)$$

where the boundary "potential"

$$b_{\psi_2\psi_1}(\theta,p,\vartheta=\pm\pi;t)$$

$$= \frac{1}{(2\pi)^2}[e^{-ip\vartheta/\hbar}\psi_2^*(\theta-\vartheta/2;t)\,\psi_1(\theta+\vartheta/2;t)]_{\vartheta=-\pi}^{\vartheta=\pi}. \quad (136)$$

is the difference in the values of the integrand of $V_{\psi_2\psi_1}(\theta,p)$ at the boundaries $\vartheta=\pm\pi$. This term vanishes in the planar case because the integrand vanishes at ∞. It does not in general vanish here.

For $\psi_2(\varphi) = \psi_1(\varphi) = \psi(\varphi)$ one can write

$$b_\psi(\theta,p,\vartheta=\pm\pi;t)$$

$$= \frac{2i}{(2\pi)^2}\text{Im}[e^{-ip\pi/\hbar}\psi^*(\theta-\pi/2)\psi(\theta+\pi/2)]. \quad (137)$$

A simple example is

$$\psi(\varphi) = \frac{1}{\sqrt{2}}(e^{im\varphi} + e^{in\varphi}), \quad (138)$$

for which the boundary term turns out to be

$$b_\psi(\theta,p,\vartheta=\pm\pi;t) = \frac{2i}{(2\pi)^2}\sin\pi[(m+n)/2 - p]$$

$$\times\{\cos(m-n)\theta + \cos\pi[(m-n)/2]\}. \quad (139)$$

Differentiating with respect to θ [see Eq. (135)] leaves a term proportional to $\sin(m-n)\theta$, which vanishes for $m=n$ but not in general. It vanishes also if $p-(m+n)/2 = k\pi, k\in\mathbb{Z}$.

We next turn to the contribution of the potential $U(\theta)$ on the right-hand side of Eq. (133) [recall Eq. (132)]:

$$\frac{1}{2\pi}\int_{-\pi}^\pi\frac{d\vartheta}{2\pi}e^{-ip\vartheta/\hbar}[-U(\theta-\vartheta/2)\psi_2^*]\psi_1$$

$$+\psi_2^*[U(\theta+\vartheta/2)\psi_1]$$

$$= \frac{1}{2\pi}\int_{-\pi}^\pi\frac{d\vartheta}{2\pi}e^{-ip\vartheta/\hbar}[U(\theta+\vartheta/2) - U(\theta-\vartheta/2)]\psi_2^*\psi_1. \quad (140)$$

A real periodic potential, $U(\theta+2\pi) = U(\theta)$, can be expanded in a Fourier series:

$$U(\theta) = \sum_{k=0}^\infty U_k(\theta), \quad U_k(\theta) = A_k\cos k\theta + B_k\sin k\theta. \quad (141)$$

It follows that

$$U_k(\theta+\vartheta/2) - U_k(\theta-\vartheta/2)$$

$$= 2\sin(k\vartheta/2)(-A_k\sin k\theta + B_k\cos k\theta)$$

$$= 2\sin(k\vartheta/2)\,\partial_\theta U_k(\theta)/k. \quad (142)$$

Inserting this into Eq. (140) gives

$$2[\partial_k U(\theta)/k]\frac{1}{2\pi}\int_{-\pi}^\pi\frac{d\vartheta}{2\pi}e^{-ip\vartheta/\hbar}\sin(k\vartheta/2)\psi_2^*\psi_1. \quad (143)$$

As

$$(k\vartheta/2)\,e^{-ip\vartheta/\hbar} = (k/2)i\hbar\partial_p e^{-ip\vartheta/\hbar}, \quad (144)$$

the contribution, (143), can be written as

$$2[\partial_k U(\theta)/k]\sin[(k/2)i\hbar\partial_p]V_{\psi_2\psi_1}(\theta,p), \quad (145)$$

with $\sin z$ as in Eq. (125).

Adding the term, (145), to Eq. ((135)) and dividing by $i\hbar$ yields

$$\partial_t V_{\psi_2\psi_1}(\theta,p;t) = -\frac{2\varepsilon\,p}{\hbar^2}\partial_\theta V_{\psi_2\psi_1}(\theta,p;t)$$
$$+\frac{2}{ik\hbar}[\partial_k U(\theta)]\sin[(k/2)i\hbar\partial_p]V_{\psi_2\psi_1}(\theta,p)$$
$$-\frac{2\varepsilon}{\hbar i}\partial_\theta b_{\psi_2\psi_1}(\theta,p,\vartheta=\pm\pi;t). \qquad (146)$$

Inserting ε from Eq. (131) and separating all the terms which are independent of $\hbar$ finally gives

$$\left\{\partial_t + \frac{p}{mr_0^2}\partial_\theta - \partial_\theta U_k(\theta)\,\partial_p\right\}V_{\psi_2\psi_1}(\theta,p;t)$$
$$= \left\{\partial_\theta U_k(\theta)\sum_{n=1}^{\infty}\frac{(k\hbar/2)^{2n}}{k(2n+1)!}\partial_p^{2n+1}\right\}V_{\psi_2\psi_1}(\theta,p;t)$$
$$+\frac{\hbar}{mr_0^2 i}\partial_\theta b_{\psi_2\psi_1}(\theta,p,\vartheta=\pm\pi;t). \qquad (147)$$

Some remarks on the result follow: The generalized Liouville equation, (147), holds, of course, also, if $\psi_2(\varphi)=\psi_1(\varphi)$; i.e., for the Wigner function proper $V_\psi(\theta,p;t)$,

$$\{\partial_t + \frac{p}{mr_0^2}\partial_\theta - \partial_\theta U_k(\theta)\,\partial_p\}V_\psi(\theta,p;t)$$
$$= \{\partial_\theta U_k(\theta)\sum_{n=1}^{\infty}\frac{(k\hbar/2)^{2n}}{k(2n+1)!}\partial_p^{2n+1}\}V_\psi(\theta,p;t)$$
$$+\frac{\hbar}{mr_0^2 i}\partial_\theta b_\psi(\theta,p,\vartheta=\pm\pi;t), \qquad (148)$$

where b_ψ is given by Eq. (137). If $V_\psi(\theta,p;t)$ is independent of θ, Eq. (148) reduces to Eq. (127).

It is remarkable that no higher derivatives of the potential $U_k(\theta)$ enter the right-hand side of Eq. (147), contrary to the planar case, where the corresponding potential term has the form [46]

$$\sum_{n=1}^{\infty}\frac{(-1)^n(\hbar/2)^{2n}}{(2l+1)!}\big[\partial_q^{2n+1}U(q)\big]\partial_p^{2n+1}W_\phi(q,p;t). \qquad (149)$$

The boundary "potential" b_ψ on the right-hand side of Eq. (147) is proportional to $\hbar$, i.e., it is a quantum effect. In addition, it makes the otherwise homogeneous PDE an inhomogeneous one.

If the time dependence of the wave functions $\psi(\theta;t)$ in Eq. (130) can be separated as

$$\psi(\theta;t) = e^{-iEt/\hbar}u(\theta), \qquad (150)$$

then the time derivative in Eq. (147) takes the form

$$\partial_t V_{\psi_2\psi_1}(\theta,p;t) = \frac{E_1-E_2}{i\hbar}V_{\psi_2\psi_1}(\theta,p;t). \qquad (151)$$

Thus, for $\psi_2(\varphi)=\psi_1(\varphi)$ and a stationary Wigner function $V_\psi(\theta,p)$, the generalized Liouville equation, (148),

reduces to

$$\left\{\frac{p}{mr_0^2}\partial_\theta - \partial_\theta U_k(\theta)\,\partial_p\right\}V_\psi(\theta,p)$$
$$= \left\{\partial_\theta U_k(\theta)\sum_{n=1}^{\infty}\frac{(k\hbar/2)^{2n}}{k(2n+1)!}\partial_p^{2n+1}\right\}V_\psi(\theta,p)$$
$$+\frac{\hbar}{mr_0^2 i}\partial_\theta b_\psi(\theta,p,\vartheta=\pm\pi). \qquad (152)$$

As discussed in Sec. IV A, an important example of a potential $U(\theta)$ of Eq. (141) is that of a pendulum [47,48],

$$U(\theta) = U_1(\theta) = -A\cos\theta, \qquad (153)$$

where $A = mg\,r_0$ for a rotating or oscillating point mass in a constant gravitational field $\vec{g}$, $A = q\,Er_0$ for such a point mass with charge q in a homogeneous electric field $\vec{E}$, and $A = p\,E$ for a (rotating) dipole $\vec{p}$ in an external electrical field $\vec{E}$.

C. An energy equation for the Wigner-Moyal function

We have seen in the preceding subsection that a time dependence, (150), for the wave functions $\psi_{1,2}(\theta;t)$ in the principal dynamical equation, (147), leads to a stationary equation, (152), in the limit $\psi_2 \to \psi_1 = \psi$ which no longer contains an energy parameter E. One can change this with two related arguments.

If one replaces $\psi_2^*(\theta;t)$ with $\psi_2^*(\theta;-t)$, then E_2 in Eq. (151) changes sign. However, this operation is not the established Wigner time inversion, which involves a simultaneous complex conjugation of the wave function. Therefore, $\psi_2^*(\theta;-t)$ is equivalent to $\psi_2(\theta;t)$. Thus, if one replaces $\psi_2^*(\theta;t)$ merely with its complex conjugate, then E_2 in Eq. (151) changes sign too. So, either of these procedures can make the difference $E_1 - E_2$ in Eq. (151) into the sum $E_1 + E_2$. Another related argument makes use of the exchange, (113): if one replaces $it/\hbar$ in the Schrödinger equation with β, then one achieves the same goal, namely, an equation for $E_1 + E_2$.

The discussion to obtain such an energy-type equation for Wigner functions was—implicitly—started by Baker [49], explicitly discussed by Fairlie [50], and then pursued by others [10,31,51,52]. We now derive the corresponding equation for the cylindrical phase space:

If we have the stationary Schrödinger equations

$$H\psi_j = E_j\psi_j, \quad j = 1,2, \qquad (154)$$

with H as in Eqs. (130) and (43), then

$$(E_1 + E_2)V_{\psi_2\psi_1}(\theta,p)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\vartheta}{2\pi}e^{-ip\vartheta/\hbar}[(H\psi_2^*)\psi_1 + \psi_2^*(H\psi_1)]. \qquad (155)$$

As in the case of Eq. (133) we first use relations (134) in order to transform the H_0 part under the last integral by partial integration and obtain

$$\left\{\frac{2\varepsilon}{\hbar^2}p^2 - \frac{\varepsilon}{2}\partial_\theta^2\right\}V_{\psi_2\psi_1}(\theta,p)$$
$$-2i\frac{\varepsilon\,mr_0^2}{\hbar}\left[e^{-ip\vartheta/\hbar}\left\{\frac{p}{mr_0^2}\rho_{21}(\theta;\vartheta) + j_{21}(\theta;\vartheta)\right\}\right]_{\vartheta=-\pi}^{\vartheta=\pi},$$
$$(156)$$

where

$$\rho_{21}(\theta;\vartheta) = \psi_2^*(\theta - \vartheta/2)\psi_1(\theta + \vartheta/2), \qquad (157)$$

$$j_{21}(\theta;\vartheta) = \frac{\hbar}{2imr_0^2}\{\psi_2^*(\theta - \vartheta/2)\,\partial_\theta\psi_1(\theta + \vartheta/2) \\ - (\partial_\theta\psi_2^*(\theta - \vartheta/2))\,\psi_1(\theta + \vartheta/2)\}. \quad (158)$$

If the solutions $\psi_{1,2}(\theta)$ of Eqs. (154) are a "stationary" consequence of the time-dependent equations,

$$i\hbar\partial_t\psi_1(\theta;t) = H\psi_1(\theta;t), \quad -i\hbar\partial_t\psi_2^*(\theta;t) = H\psi_2(\theta;t), \tag{159}$$

then the continuity equation

$$\partial_t\rho_{21}(\theta;\vartheta;t) + \partial_\theta j_{21}(\theta;\vartheta;t) = 0 \qquad (160)$$

holds. As $p/(mr_0^2)$ is the angular velocity $\dot{\theta}$, the term

$$\frac{p}{mr_0^2}\rho_{21}(\theta;\vartheta;t) \qquad (161)$$

in the boundary part of Eq. (156) represents a kind of current density too. Thus the boundary term in Eq. (156) represents a circular flow depending on the boundaries $\vartheta = \pm\pi$.

For the potential $U(\theta)$ part of H under the integral of Eq. (155) we again use the mode $U_k(\theta)$ of Eq. (141). Instead of relation (142) we now have

$$U_k(\theta + \vartheta/2) + U_k(\theta - \vartheta/2) = 2\cos(k\vartheta/2)\,U_k(\theta). \quad (162)$$

Inserting this into the potential part on the right-hand side of Eq. (155) gives the contribution

$$2U_k(\theta)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\vartheta}{2\pi}e^{-ip\vartheta/\hbar}\cos(k\vartheta/2)\,\psi_2^*\psi_1. \qquad (163)$$

Using again relation (144), Eq. (163) can be written as

$$2U_k(\theta)\cos[(k/2)i\hbar\partial_p]\,V_{\psi_2\psi_1}(\theta,p), \qquad (164)$$

with

$$\cos z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}. \qquad (165)$$

Adding the contribution, (164), to the term, (156), inserting ε from Eq. (131), and dividing the result by 2, finally, gives

$$\left\{\frac{p^2}{2mr_0^2} + U_k(\theta) - \frac{\hbar^2}{8mr_0^2}\partial_\theta^2\right\}V_{\psi_2\psi_1}(\theta,p) \\ + \left\{U_k(\theta)\sum_{n=1}^{\infty}\frac{(k\hbar/2)^{2n}}{(2n)!}\partial_p^{2n}\right\}V_{\psi_2\psi_1}(\theta,p) \\ + \frac{\hbar}{2i}\left[e^{-ip\vartheta/\hbar}\left\{\frac{p}{mr_0^2}\rho_{21}(\theta;\vartheta) + j_{21}(\theta;\vartheta)\right\}\right]_{\vartheta=-\pi}^{\vartheta=\pi} \\ = \frac{E_1 + E_2}{2}V_{\psi_2\psi_1}(\theta,p). \qquad (166)$$

For the Wigner function itself, for which $\psi_2(\varphi) = \psi_1(\varphi) = \psi(\varphi)$ and $E_2 = E_1$, the last equation takes the

form

$$\left\{\frac{p^2}{2mr_0^2} + U_k(\theta) - \frac{\hbar^2}{8mr_0^2}\partial_\theta^2\right\}V_\psi(\theta,p) \\ + \left\{U_k(\theta)\sum_{n=1}^{\infty}\frac{(k\hbar/2)^{2n}}{(2n)!}\partial_p^{2n}\right\}V_\psi(\theta,p) \\ + \frac{\hbar}{2i}\left[e^{-ip\vartheta/\hbar}\left\{\frac{p}{mr_0^2}\rho(\theta;\vartheta) + j(\theta;\vartheta)\right\}\right]_{\vartheta=-\pi}^{\vartheta=\pi} \\ = E\,V_\psi(\theta,p), \qquad (167)$$

where now

$$\rho(\theta;\vartheta) = \psi^*(\theta - \vartheta/2)\psi(\theta + \vartheta/2), \qquad (168)$$

$$j(\theta;\vartheta) = \frac{\hbar}{2imr_0^2}\{\psi^*(\theta - \vartheta/2)\,\partial_\theta\psi(\theta + \vartheta/2) \\ - (\partial_\theta\psi^*(\theta - \vartheta/2))\,\psi(\theta + \vartheta/2)\}. \qquad (169)$$

D. PDEs for Wigner functions

In the preceding two subsections we have derived PDEs for the Wigner function $V_\psi(\theta,p;t)$ or $V_\rho(\theta,p;t)$: Eq. (148) for the time evolution and Eqs. (152) and (167) for stationary systems. The question comes up immediately whether these equations can replace—at least in principle—the Schrödinger wave or Heisenberg operator equations, like path integrals in quantum mechanics or the Hamilton-Jacobi equation in mechanics.

The time evolution of the Wigner function in the planar case is considered to be equivalent to that of the Schrödinger equation [19], mainly because both are of first order in the time derivative and, therefore, have equivalent initial value problems. As the corresponding equation, (148), has additional boundary terms, (137), this problem must be analyzed anew for the cylindrical case.

As for the stationary Eqs. (152) and (167) the question arises whether they can replace the stationary Schrödinger equation, especially an eigenvalue equation. Such an eigenfunction has to obey the normalization conditions, (12) and (50), and should have properties (69)–(72) for $\psi_2 = \psi_1 = \psi$. All this has yet to be analyzed. In the planar case the corresponding eigenvalue problem has been positively solved for the harmonic oscillator [10,50]. But, of course, one has to get far beyond the harmonic oscillator in order to establish a generally attractive and convincing framework. There is much work ahead.

Equations (148), (152), and (167) have a structure which invites us to attempt semiclassical approximations and classical limits: The left-hand sides are "classsical," i.e., they do not contain any $\hbar$, whereas the right-hand sides contain a (formal) power series in $\hbar$, which appears to be useful for approximations. This was one of the main motives of Wigner already in 1932 [3]. Many investigations followed. See, e.g., Refs. [6,10], and [53–57]. However, one has to beware of essential singularities in $\hbar$ [58–61], which can spoil naive polynomial approximations.

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