PHYSICAL REVIEW A **95**, 052107 (2017)

Solutions for bosonic and fermionic dissipative quadratic open systems

Chu Guo and Dario Poletti
Singapore University of Technology and Design, 8 Somapah Road, 487372 Singapore
(Received 7 October 2016; revised manuscript received 11 April 2017; published 10 May 2017)

We show how to solve a large class of Lindblad master equations for noninteracting particles on L sites. In the first portion we concentrate on bosonic particles, while in the second we will address fermionic particles. In both cases we show how to reduce the problem to diagonalizing an $L \times L$ non-Hermitian matrix. In particular, for boundary dissipative driving of a uniform chain, the matrix is a tridiagonal bordered Toeplitz matrix which can be solved analytically for the normal master modes and their relaxation rates (rapidities). In the regimes in which an analytical solution cannot be found, our approach can still provide a speedup in the numerical evaluation. For bosonic particles, we use this numerical method to study the relaxation gap at nonequilibrium phase transitions in a boundary driven bosonic ladder with synthetic gauge fields. We conclude by showing how to construct the nonequilibrium steady state. The analysis for fermionic particles closely follows that of bosons, but with important differences due to the different commutation rules.

DOI: 10.1103/PhysRevA.95.052107

I. INTRODUCTION

Quantum systems in contact with an environment display a very rich physics including emergence of nonequilibrium phase transitions [1–3] which can be used to engineer particularly interesting nonequilibrium steady states (NESSs). Moreover, also the relaxation towards an asymptotic or steady state can, for example, manifest a nontrivial dynamics, from power law to stretched exponentials and even aging [4–7]. A particularly important class of open quantum systems is that of boundary driven systems, in which a current may be induced by the coupling, only at the extremities, of the system to the environment. This class of systems is of particular relevance in the study of quantum transport.

The knowledge of analytical solutions for open quantum systems would allow us to build a better intuition of the physics of these systems and to test numerical methods. The NESSs of a boundary driven Heisenberg model can, in many regimes, be computed analytically using a matrix product ansatz as shown in [8–13] (for a review see [14]). In [15] an exact solution for a diffusive XX chain is presented using a cleverly designed ansatz. For a boundary driven bosonic noninteracting system, in [16] the authors showed how to analytically compute the local densities and the current. In the seminal article [17], which is particularly relevant to our work, Prosen showed that for a quadratic open fermionic model with L sites, solving for the relaxation rates of the quantum Lindblad equation can be reduced to the diagonalization of a $4L \times 4L$ antisymmetric matrix, which can be further reduced to the diagonalization of a $2L \times 2L$ general matrix [18]. A similar method was also applied to quadratic open bosonic model [19].

Here we build on this strategy while focusing on a boundary dissipative driven quadratic bosonic or fermionic system with number-conserving Hamiltonians. We are able to reduce the problem to the diagonalization of a $L \times L$ matrix which can thus be studied numerically more effectively [20]. Moreover we show that in many physically relevant cases the matrix to be diagonalized is a tridiagonal bordered Toeplitz matrix for which analytical expressions for the eigenvalues and eigenvectors are known. We also show examples for which we can explicitly write the relaxation rates (rapidities) of all the normal master modes of the Lindblad master equation (uniform noninteracting bosonic chain and XX chain). This can be used, for example, to compute the relaxation gap, that is the rapidity of the slowest decaying normal master mode. Based on the equations we found, we propose an efficient algorithm to compute the rapidities. We use this method to study the scaling of the relaxation gap in a system with two phase transitions of different nature. We show that the scaling of the relaxation gap is different in the two cases. We also show that our approach can be used to give an expression for the steady state. We should stress that imposing fermionic anticommutation relations between the normal master modes requires a few further steps compared to the bosonic case, and while the structure of the equations found may be similar, the matrices involved have important differences.

This paper is divided in two large sections: in the first we study the bosonic case and in the second the fermionic case. To help the reader understand the material without having to go back and forth between the sections, we have chosen to write the parts on the bosons and that on the fermions almost independently and with a similar structure: in Secs. II A and III A we introduce the bosonic or fermionic model we study. In Secs. II B and III B, we show how to diagonalize the Lindblad master equation and obtain the normal master modes. In Secs. II C and III C we show how to solve analytically the boundary driven bosonic or XX chain and study the scaling of the relaxation gap. In Secs. II D and III D we analytically construct similarity transformations which map the vacuum state to the steady state of the system. In Sec. II C 1, we introduce an efficient numerical algorithm to compute the quadratic observables such as currents and densities which we then use to study the relaxation gap of systems which present nonequilibrium phase transitions. This is exemplified only for bosonic particles as the algorithm is based solely on the structure of the equations. In Sec. IV, we draw our conclusions.

II. BOSONIC PARTICLES

A. Model

We consider an open quantum system of L sites with bosonic particles. Its dynamics is described by the quantum

Lindblad master equation [21,22] with Lindbladian $\mathcal{L}$,

$$\frac{d}{dt}\hat{\rho} = \mathcal{L}(\hat{\rho}) = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \mathcal{D}(\hat{\rho}). \quad (1)$$

Here $\hat{\rho}$ is the density operator of the system, $\hat{H}$ is the Hamiltonian, and $\mathcal{D}$, the dissipator, describes the dissipative part of the evolution. The Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \sum_{m,n=1}^{L} \mathbf{h}_{m,n}\hat{\alpha}_m^\dagger\hat{\alpha}_n, \quad (2)$$

where $\mathbf{h}$ is an $L \times L$ Hermitian matrix, and $\hat{\alpha}_j^\dagger$ ($\hat{\alpha}_j$) creates (annihilates) one boson on site j. The dissipative part is given by

$$\mathcal{D}(\hat{\rho}) = \sum_{i,j=1}^{L}[\mathbf{\Lambda}^+_{i,j}(\hat{\alpha}_i^\dagger\hat{\rho}\hat{\alpha}_j - \hat{\alpha}_j\hat{\alpha}_i^\dagger\hat{\rho}) + \mathbf{\Lambda}^-_{i,j}(\hat{\alpha}_i\hat{\rho}\hat{\alpha}_j^\dagger - \hat{\alpha}_j^\dagger\hat{\alpha}_i\hat{\rho}) + \text{H.c.}], \quad (3)$$

where $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$ are $L \times L$ Hermitian and non-negative matrices. In the trivial case of only one site, the dissipator has the familiar form $\mathcal{D}(\hat{\rho}) = \Lambda^+(\hat{\alpha}^\dagger\hat{\rho}\hat{\alpha} - \hat{\alpha}\hat{\alpha}^\dagger\hat{\rho}) + \Lambda^-(\hat{\alpha}\hat{\rho}\hat{\alpha}^\dagger - \hat{\alpha}^\dagger\hat{\alpha}\hat{\rho}) + \text{H.c.}$, where $\Lambda^\pm$ is the heating or cooling rate. In general, one can make a unitary transformation to the creation and annihilation operators so that $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$ become diagonal.

B. Solving the master equation

1. Reshaping the density operator in a different representation

We perform a one-to-one mapping from the density operator basis elements $|n_1,n_2,\dots n_L\rangle\langle n_1',n_2',\dots n_L'|$ to a state vector basis (with $2L$ sites) $|n_1,\dots n_L,n_1',\dots n_L'\rangle$ (see for example [18,23–25]). From this, the operator $\hat{\alpha}_i$ acting on site i on the left of the density matrix is mapped to $\hat{a}_i$ acting on the state vector on the ith site, while the operator $\hat{\alpha}_i$ acting on the right of the density matrix is mapped to $\hat{a}_{L+i}^\dagger$ acting on the state vector. We denote the density operator to be $|\rho\rangle$ in our representation.

2. Master equation in our representation

$\mathcal{L}$ in Eq. (1) can thus be written as

$$\mathcal{L} = \begin{pmatrix}\mathbf{a}^\dagger_{1\to L}\\ \mathbf{a}_{L+1\to 2L}\end{pmatrix}^t \mathbf{M}\begin{pmatrix}\mathbf{a}_{1\to L}\\ \mathbf{a}^\dagger_{L+1\to 2L}\end{pmatrix} + \begin{pmatrix}\mathbf{a}_{1\to L}\\ \mathbf{a}^\dagger_{L+1\to 2L}\end{pmatrix}^t \mathbf{M}^t\begin{pmatrix}\mathbf{a}^\dagger_{1\to L}\\ \mathbf{a}_{L+1\to 2L}\end{pmatrix} + \text{tr}(\mathbf{\Lambda}^{-t} - \mathbf{\Lambda}^+), \quad (4)$$

where $\mathbf{M}$ is a $2L \times 2L$ matrix,

$$\mathbf{M} = \begin{pmatrix}\mathbf{K} & \mathbf{\Lambda}^+\\ \mathbf{\Lambda}^{-t} & \mathbf{K}^\dagger\end{pmatrix}. \quad (5)$$

Here $\mathbf{K} = (-i\mathbf{h}/\hbar - \mathbf{\Lambda}^+ - \mathbf{\Lambda}^{-t})/2$, where with $\mathbf{A}^t$ we indicate the transpose of the matrix $\mathbf{A}$. We have also used the notation $\mathbf{a}_{1\to L}$ to denote the column vector made of operators $\hat{a}_1,\hat{a}_2,\dots,\hat{a}_L$ and $\mathbf{a}^\dagger_{1\to L}$ for the column vector made of $\hat{a}_1^\dagger,\hat{a}_2^\dagger,\dots,\hat{a}_L^\dagger$; the same applies for $\mathbf{a}_{L+1\to 2L}$ and $\mathbf{a}^\dagger_{L+1\to 2L}$. Indeed, we note that the Liouvillian $\mathcal{L}$ can be written in the simple form of Eq. (4) because we study a number conserving Hamiltonian. If the Hamiltonian is not number conserving, the coefficient matrix will be a $4L \times 4L$ matrix.

3. Normal master modes of the master equation

In general $\mathbf{M}$ is not Hermitian and it cannot always be diagonalized, however in the following we start from the assumption that we know a transformation which can diagonalize $\mathbf{M}$ and preserves bosonic commutation relations. This assumption is *a posteriori* verified in all the cases we considered. This transformation is given by the matrices $\mathbf{W}_1$ and $\mathbf{W}_2$ as follows:

$$\begin{pmatrix}\mathbf{a}_{1\to L}\\ \mathbf{a}^\dagger_{L+1\to 2L}\end{pmatrix} = \mathbf{W}_1\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}'_{L+1\to 2L}\end{pmatrix}, \quad (6)$$

$$\begin{pmatrix}\mathbf{a}^\dagger_{1\to L}\\ \mathbf{a}_{L+1\to 2L}\end{pmatrix} = \mathbf{W}_2\begin{pmatrix}\mathbf{b}'_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}, \quad (7)$$

whereas for $\mathbf{a}_{1\to L}$ and $\mathbf{a}^\dagger_{1\to L}$, $\mathbf{b}_{1\to L}$ means the column vector made of operators $\hat{b}_1,\hat{b}_2,\dots,\hat{b}_L$ and $\mathbf{b}'_{1\to L}$ means the column vector made of $\hat{b}_1',\hat{b}_2',\dots,\hat{b}_L'$. Again the same notation applies for $\mathbf{b}_{L+1\to 2L}$ and $\mathbf{b}'_{L+1\to 2L}$.

Using this transformation we get

$$\mathcal{L} = \begin{pmatrix}\mathbf{b}'_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^t \mathbf{W}_2^t\mathbf{M}\mathbf{W}_1\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}'_{L+1\to 2L}\end{pmatrix} + \begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}'_{L+1\to 2L}\end{pmatrix}^t \mathbf{W}_1^t\mathbf{M}^t\mathbf{W}_2\begin{pmatrix}\mathbf{b}'_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix} + \text{tr}(\mathbf{\Lambda}^{-t} - \mathbf{\Lambda}^+). \quad (8)$$

The bosonic commutation relation can be written as

$$\left[\begin{pmatrix}\mathbf{a}_{1\to L}\\ \mathbf{a}^\dagger_{L+1\to 2L}\end{pmatrix},\begin{pmatrix}\mathbf{a}^\dagger_{1\to L}\\ \mathbf{a}_{L+1\to 2L}\end{pmatrix}^t\right] = \mathbf{Z}_L,$$

and requiring for the bosonic commutation relation to apply also to the $\hat{b}$ we get

$$\left[\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}'_{L+1\to 2L}\end{pmatrix},\begin{pmatrix}\mathbf{b}'_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^t\right] = \mathbf{Z}_L$$

and hence

$$\mathbf{Z}_L = \mathbf{W}_1\mathbf{Z}_L\mathbf{W}_2^t \iff \mathbf{W}_2 = \mathbf{Z}_L\mathbf{W}_1^{t\,-1}\mathbf{Z}_L. \quad (9)$$

Here we have used

$$\mathbf{Z}_L = \begin{pmatrix}\mathbf{1}_L & 0\\ 0 & -\mathbf{1}_L\end{pmatrix}, \quad (10)$$

where we have denoted $\mathbf{1}_l$ for an identity matrix of size l. In the following we will also use the matrices

$$\mathbf{X}_L = \begin{pmatrix}0 & \mathbf{1}_L\\ \mathbf{1}_L & 0\end{pmatrix} \quad (11)$$

and

$$\mathbf{Y}_L = -i\begin{pmatrix}0 & \mathbf{1}_L\\ -\mathbf{1}_L & 0\end{pmatrix}. \quad (12)$$

Matrices in Eqs. (10)–(12), being given by a tensor product between Pauli matrices and identity, satisfy the relations

$$\mathbf{Z}_L^2 = \mathbf{1}_{2L}, \quad \mathbf{X}_L^2 = \mathbf{1}_{2L}, \tag{13}$$

$$\mathbf{Y}_L^2 = \mathbf{1}_{2L}, \quad \mathbf{Z}_L\mathbf{X}_L = -\mathbf{X}_L\mathbf{Z}_L = i\mathbf{Y}_L. \tag{14}$$

It follows that

$$\begin{aligned}\mathcal{L} =& \begin{pmatrix}\mathbf{b}'_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^t \mathbf{Z}_L\mathbf{W}_1^{-1}\mathbf{Z}_L\mathbf{M}\mathbf{W}_1\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}'_{L+1\to 2L}\end{pmatrix}\\ &+\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}'_{L+1\to 2L}\end{pmatrix}^t \mathbf{W}_1^t\mathbf{M}^t\mathbf{Z}_L\mathbf{W}_1^{t\,-1}\mathbf{Z}_L\begin{pmatrix}\mathbf{b}'_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}\\ &+\mathrm{tr}(\mathbf{\Lambda}^{-t}-\mathbf{\Lambda}^+)\\ =& \begin{pmatrix}\mathbf{b}'_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^t \mathbf{Z}_L(\mathbf{W}_1^{-1}\mathbf{Z}_L\mathbf{M}\mathbf{W}_1)\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}'_{L+1\to 2L}\end{pmatrix}\\ &+\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}'_{L+1\to 2L}\end{pmatrix}^t (\mathbf{W}_1^{-1}\mathbf{Z}_L\mathbf{M}\mathbf{W}_1)^t\mathbf{Z}_L\begin{pmatrix}\mathbf{b}'_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}\\ &+\mathrm{tr}(\mathbf{\Lambda}^{-t}-\mathbf{\Lambda}^+).\end{aligned} \tag{15}$$

This implies that the problem of finding the normal modes of the system reduces to finding a $\mathbf{W}_1$ such that $\mathbf{Z}_L\mathbf{M}$ can be diagonalized, that is,

$$\mathbf{W}_1^{-1}(\mathbf{Z}_L\mathbf{M})\mathbf{W}_1 = \mathrm{diag}(\beta_1,\beta_2,\ldots,\beta_{2L}), \tag{16}$$

where $\mathrm{diag}(\vec{v})$ is a diagonal matrix with the elements of the vector $\vec{v}$ on its diagonal. It would then be possible to write the following compact form for $\mathcal{L}$:

$$\begin{aligned}\mathcal{L} =& 2\sum_{i=1}^{L}(\beta_i\hat{b}'_i\hat{b}_i - \beta_{L+i}\hat{b}'_{L+i}\hat{b}_{L+i})\\ &+\sum_{i=1}^{L}(\beta_i-\beta_{L+i}) + \mathrm{tr}(\mathbf{\Lambda}^{-t}-\mathbf{\Lambda}^+).\end{aligned} \tag{17}$$

4. Diagonalizing $\mathbf{Z}_L\mathbf{M}$

Here we will explicitly construct the eigenvalues and eigenvectors of the matrix $\mathbf{Z}_L\mathbf{M}$. From Eq. (5) we note the relation

$$\mathbf{X}_L\mathbf{M}\mathbf{X}_L = \mathbf{M}^\dagger; \tag{18}$$

we find that if

$$x = \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}$$

is a right eigenvector of $\mathbf{Z}_L\mathbf{M}$ with eigenvalue ω, then $x^\dagger\mathbf{Y}_L$ is a left eigenvector of $\mathbf{Z}_L\mathbf{M}$ with eigenvalue $-\omega^*$. In fact, using Eq. (13),

$$\begin{aligned}\mathbf{Z}_L\mathbf{M}x = \omega x &\to x^\dagger\mathbf{M}^\dagger\mathbf{Z}_L = \omega^* x^\dagger\\ &\to x^\dagger\mathbf{X}_L\mathbf{X}_L\mathbf{M}^\dagger\mathbf{X}_L = \omega^* x^\dagger\mathbf{Z}_L\mathbf{X}_L\\ &\to x^\dagger\mathbf{X}_L\mathbf{M} = \omega^* x^\dagger\mathbf{Z}_L\mathbf{X}_L.\end{aligned}$$

This implies, using Eq. (14), that $x^\dagger\mathbf{X}_L\mathbf{Z}_L\mathbf{Z}_L\mathbf{M} = x^\dagger\mathbf{X}_L\mathbf{M} = \omega^* x^\dagger\mathbf{Z}_L\mathbf{X}_L$, which means $x^\dagger\mathbf{Y}_L\mathbf{Z}_L\mathbf{M} = -\omega^* x^\dagger\mathbf{Y}_L$. Thus it follows that

$$x^\dagger\mathbf{Y}_L(\mathbf{Z}_L\mathbf{M}) = -\omega^* x^\dagger\mathbf{Y}_L, \tag{19}$$

i.e., $x^\dagger\mathbf{Y}_L$ is a left eigenvector of $\mathbf{Z}_L\mathbf{M}$.

Moreover if x_1 is a right eigenvector of $\mathbf{Z}_L\mathbf{M}$ with eigenvalue ω_1, and x_2 is a right eigenvector of $\mathbf{Z}_L\mathbf{M}$ with eigenvalue ω_2, then if $\omega_1+\omega_2^*\neq 0$ it follows that $x_1^\dagger\mathbf{Y}_L x_2 = 0$. In fact

$$\begin{aligned}\mathbf{Z}_L\mathbf{M}x_1 &= \omega_1 x_1;\\ \mathbf{Z}_L\mathbf{M}x_2 &= \omega_2 x_2,\end{aligned}$$

then

$$\begin{aligned}x_1^\dagger\mathbf{Y}_L\mathbf{Z}_L\mathbf{M} &= -\omega_1^* x_1^\dagger\mathbf{Y}_L;\\ \mathbf{Z}_L\mathbf{M}x_2 &= \omega_2 x_2;\\ &\to (\omega_1^*+\omega_2)x_1^\dagger\mathbf{Y}_L x_2 = 0.\end{aligned}$$

Since the eigenvalues of $\mathbf{Z}_L\mathbf{M}$ always appear in pairs, we could list the eigenvalues and the corresponding eigenvectors of $\mathbf{Z}_L\mathbf{M}$ as $\omega_1,\omega_2,\ldots,\omega_L,-\omega_1^*,\ldots,\omega_L^*$, with the matrix $\mathbf{W}_1$ composed in each column by the right eigenvectors $\mathbf{W}_1 = (\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_{2L})$. Then following Eq. (19) we know that $\vec{x}_{L+j}^\dagger\mathbf{Y}_L$ is the left eigenvector of $\mathbf{Z}_L\mathbf{M}$ correponding to ω_j, and $\vec{x}_j^\dagger\mathbf{Y}_L$ is the left eigenvector corresponding to $-\omega_j^*$, for $1\leqslant j\leqslant L$. Therefore the left eigenvectors of $\mathbf{Z}_L\mathbf{M}$ constitute the matrix $\mathbf{X}_L\mathbf{W}_1^\dagger\mathbf{Y}_L$. We can now choose to renormalize the right eigenvectors as

$$i\mathbf{X}_L\mathbf{W}_1^\dagger\mathbf{Y}_L\mathbf{W}_1 = \mathbf{Z}_L \Leftrightarrow \mathbf{Y}_L\mathbf{W}_1^\dagger\mathbf{Y}_L\mathbf{W}_1 = -\mathbf{1}_{2L} \tag{20}$$

so that we have

$$\mathbf{W}_1^{-1} = -\mathbf{Y}_L\mathbf{W}_1^\dagger\mathbf{Y}_L, \tag{21}$$

$$\mathbf{W}_2 = -\mathbf{X}_L\mathbf{W}_1^*\mathbf{X}_L. \tag{22}$$

At this point we define a new $L\times L$ matrix $\mathbf{P}$, which satisfies

$$\mathbf{P} = \mathbf{K}+\mathbf{\Lambda}^+ = (-i\mathbf{h}/\hbar + \mathbf{\Lambda}^+ - \mathbf{\Lambda}^{-t})/2, \tag{23}$$

for which we assume to have the eigendecomposition

$$\mathbf{P}\mathbf{W}_P = \mathbf{W}_P\boldsymbol{\lambda}_P, \tag{24}$$

where $\mathbf{W}_P$ and $\boldsymbol{\lambda}_P$ are eigenvectors and eigenvalues. Then we find that the $2L\times L$ matrix formed by $(\frac{\mathbf{W}_P}{\mathbf{W}_P})$ constitutes L right eigenvectors of $\mathbf{Z}_L\mathbf{M}$, corresponding to $\boldsymbol{\lambda}_P$, and the $L\times 2L$ matrix $(\mathbf{W}_P^\dagger\ ,\ -\mathbf{W}_P^\dagger)$ constitutes L left eigenvectors of $\mathbf{Z}_L\mathbf{M}$, corresponding to $-\boldsymbol{\lambda}_P^*$. This can be shown from

$$\mathbf{Z}_L\mathbf{M}\begin{pmatrix}\mathbf{W}_P\\ \mathbf{W}_P\end{pmatrix} = \begin{pmatrix}\mathbf{P}\mathbf{W}_P\\ \mathbf{P}\mathbf{W}_P\end{pmatrix} = \begin{pmatrix}\mathbf{W}_P\\ \mathbf{W}_P\end{pmatrix}\boldsymbol{\lambda}_P$$

and

$$\begin{aligned}(\mathbf{W}_P^\dagger, -\mathbf{W}_P^\dagger)\mathbf{Z}_L\mathbf{M} &= (-\mathbf{W}_P^\dagger\mathbf{P}^\dagger, \mathbf{W}_P^\dagger\mathbf{P}^\dagger)\\ &= -\boldsymbol{\lambda}_P^*(\mathbf{W}_P^\dagger, -\mathbf{W}_P^\dagger).\end{aligned}$$

Hence by denoting the remaining L right eigenvectors of $\mathbf{Z}_L\mathbf{M}$ as $(\frac{\mathbf{C}}{\mathbf{D}})$, where $\mathbf{C}$, $\mathbf{D}$ are $L\times L$ matrices, we know that they form the right eigenvectors with eigenvalues $-\boldsymbol{\lambda}_P^*$, which are paired with the left eigenvectors ($\mathbf{W}_P^\dagger \;\; -\mathbf{W}_P^\dagger$). Also ($-\mathbf{D}^\dagger \;\; \mathbf{C}^\dagger$) will be the left eigenvectors corresponding to the eigenvalues $\boldsymbol{\lambda}_P$, which are paired with the right eigenvectors ($\frac{\mathbf{W}_P}{\mathbf{W}_P}$).

Therefore $\mathbf{W}_1$ and $\mathbf{W}_2$ can be written more explicitly as

$$\mathbf{W}_1 = \begin{pmatrix} \mathbf{W}_P & \mathbf{C} \\ \mathbf{W}_P & \mathbf{D} \end{pmatrix}, \quad \mathbf{W}_1^{-1} = \begin{pmatrix} -\mathbf{D}^\dagger & \mathbf{C}^\dagger \\ \mathbf{W}_P^\dagger & -\mathbf{W}_P^\dagger \end{pmatrix}, \tag{25}$$

$$\mathbf{W}_2 = -\begin{pmatrix} \mathbf{D}^* & \mathbf{W}_P^* \\ \mathbf{C}^* & \mathbf{W}_P^* \end{pmatrix}, \quad \mathbf{W}_2^{-1} = \begin{pmatrix} \mathbf{W}_P^t & -\mathbf{W}_P^t \\ -\mathbf{C}^t & \mathbf{D}^t \end{pmatrix}, \tag{26}$$

which means

$$\mathbf{a}_{1\to L} = \mathbf{W}_P \mathbf{b}_{1\to L} + \mathbf{C}\mathbf{b}'_{L+1\to 2L}; \tag{27a}$$

$$\mathbf{a}^\dagger_{L+1\to 2L} = \mathbf{W}_P \mathbf{b}_{1\to L} + \mathbf{D}\mathbf{b}'_{L+1\to 2L}; \tag{27b}$$

$$\mathbf{a}^\dagger_{1\to L} = -\mathbf{D}^*\mathbf{b}'_{1\to L} - \mathbf{W}^*_P \mathbf{b}_{L+1\to 2L}; \tag{27c}$$

$$\mathbf{a}_{L+1\to 2L} = -\mathbf{C}^*\mathbf{b}'_{1\to L} - \mathbf{W}^*_P \mathbf{b}_{L+1\to 2L} \tag{27d}$$

and the inverse equation

$$\mathbf{b}_{1\to L} = -\mathbf{D}^\dagger \mathbf{a}_{1\to L} + \mathbf{C}^\dagger \mathbf{a}^\dagger_{L+1\to 2L}; \tag{28a}$$

$$\mathbf{b}'_{L+1\to 2L} = \mathbf{W}^\dagger_P \mathbf{a}_{1\to L} - \mathbf{W}^\dagger_P \mathbf{a}^\dagger_{L+1\to 2L}; \tag{28b}$$

$$\mathbf{b}'_{1\to L} = \mathbf{W}^t_P \mathbf{a}^\dagger_{1\to L} - \mathbf{W}^t_P \mathbf{a}_{L+1\to 2L}; \tag{28c}$$

$$\mathbf{b}_{L+1\to 2L} = -\mathbf{C}^t \mathbf{a}^\dagger_{1\to L} + \mathbf{D}^t \mathbf{a}_{L+1\to 2L}. \tag{28d}$$

Noticing that $\sum \lambda_{P,i} = \text{tr}(\mathbf{P}) = [-i\ \text{tr}(\mathbf{h}/\hbar) - \text{tr}(\boldsymbol{\Lambda}^{-t} - \boldsymbol{\Lambda}^{+})]/2$ and since the $(\lambda_{P,1}, \ldots \lambda_{P,L}, -\lambda^*_{P,1}, \cdots - \lambda^*_{P,L})$ correspond to $(\beta_1, \ldots \beta_{2L})$, the eigenvalues of $\mathbf{Z}_L\mathbf{M}$, we get the following identity:

$$\sum_{i=1}^{L}(\beta_i - \beta_{L+i}) = \sum_{i=1}^{L}(\lambda_{P,i} + \lambda^*_{P,i}) = \text{tr}(\boldsymbol{\Lambda}^+ - \boldsymbol{\Lambda}^{-t}), \tag{29}$$

which exactly cancels the last term in the expression of $\mathcal{L}$ in Eq. (17).

We can then write $\mathcal{L}$ as

$$\mathcal{L} = 2\sum_{i=1}^{L}\lambda_{P,i}\hat{b}'_i\hat{b}_i + 2\sum_{i=1}^{L}\lambda^*_{P,i}\hat{b}'_{L+i}\hat{b}_{L+i}. \tag{30}$$

The state $|\rho_{ss}\rangle$ which annihilates all the operator $\mathbf{b}_{1\to 2L}$ is the steady state because $\mathcal{L}|\rho_{ss}\rangle = 0$. The $\hat{b}_i$ are the normal master modes of the Lindblad master equation and the $\lambda_{P,i}$ the rapidities.

5. Nonpositivity of the eigenvalues of the Lindbladian

To prove the eigenvalues of $\mathcal{L}$ are nonpositive, it is sufficient to prove that all the eigenvalues of the matrix $\mathbf{P}$ are nonpositive. The proof is similar as in [18]. Assuming $\mathbf{P}x = \omega x$, therefore $x^\dagger\mathbf{P}^\dagger = \omega^* x^\dagger$, then we have

$$x^\dagger(\mathbf{P}^\dagger + \mathbf{P})x = x^\dagger\mathbf{P}^\dagger x + x^\dagger\mathbf{P}x = 2\mathcal{R}(\omega)x^\dagger x, \tag{31}$$

where $\mathcal{R}(\omega)$ means the real part of ω. Moreover $\mathbf{P}^\dagger + \mathbf{P} = \boldsymbol{\Lambda}^+ - \boldsymbol{\Lambda}^{-t}$ hence all the eigenvalues of the matrix on the right-hand side have to be nonpositive for the master equation to have a steady state. Hence we can conclude that

$$\mathcal{R}(\omega) \leqslant 0, \tag{32}$$

i.e., the real part of eigenvalues of the Lindbladian is nonpositive.

6. Computing the expectation value $\langle\hat{\alpha}_i^\dagger\hat{\alpha}_j\rangle$

We denote $|\mathbf{1}\rangle = \sum_{i_1,i_2,\ldots,i_L}|i_1,i_2,\ldots,i_L,i_1,i_2,\ldots,i_L\rangle$ to be the state vector resulting from the mapping of an identity operator, and $\langle\mathbf{1}|$ to be its transpose. Computing the expectation value of observable $\hat{O}$ on the steady state $\hat{\rho}_{ss}$, which is $\text{tr}(\hat{O}\hat{\rho}_{ss})$, is equivalent to the expression $\langle\mathbf{1}|\hat{O}|\rho_{ss}\rangle$ where we have simply rewritten the trace of an operator times the density operator in the enlarged space. Of course the operator $\hat{O}$ has also been mapped to the new space. In order to compute quadratic expectation values such as $\text{tr}(\hat{\alpha}_i^\dagger\hat{\alpha}_j\hat{\rho}_{ss}) = \langle\mathbf{1}|\hat{a}_i^\dagger\hat{a}_j|\rho_{ss}\rangle$ it is convenient to rewrite the eigenequation Eq. (16) in a different form.

To do so we start from $\mathbf{W}_1^{-1}\ \mathbf{W}_1 = \mathbf{1}_{2L}$ and using Eq. (25), we have

$$\mathbf{D} = \mathbf{C} - \mathbf{W}_P^{\dagger\ -1}; \tag{33}$$

$$\mathbf{C} = \mathbf{W}_P\mathbf{Q}, \tag{34}$$

where $\mathbf{Q}$ is a $L \times L$ Hermitian matrix. Therefore $\mathbf{W}_1$ can also be written as

$$\mathbf{W}_1 = \begin{pmatrix} \mathbf{W}_P & \mathbf{W}_P\mathbf{Q} \\ \mathbf{W}_P & \mathbf{W}_P\mathbf{Q} - \mathbf{W}_P^{\dagger\ -1} \end{pmatrix}. \tag{35}$$

From Eq. (16), which can be written now as

$$\mathbf{Z}_L\mathbf{M}\mathbf{W}_1 = \mathbf{W}_1\begin{pmatrix} \boldsymbol{\lambda}_P & 0 \\ 0 & -\boldsymbol{\lambda}_P^* \end{pmatrix}, \tag{36}$$

and using Eq. (35), together with $\Omega = \mathbf{W}_P\mathbf{Q}\mathbf{W}_P^\dagger$ we have

$$\mathbf{P}\Omega + \Omega\mathbf{P}^\dagger = \boldsymbol{\Lambda}^+. \tag{37}$$

Solving this equation for Ω will prove very useful in the following.

Using Eqs. (27) we get

$$\hat{a}'_i = -\sum_{k=1}^{L}\mathbf{D}^*_{i,k}\hat{b}'_k - \sum_{k=1}^{L}\mathbf{W}_P{}^*_{i,k}\hat{b}_{L+k}, \tag{38}$$

$$\hat{a}_j = \sum_{k=1}^{L}\mathbf{W}_P{}_{j,k}\hat{b}_k + \sum_{k=1}^{L}\mathbf{C}_{j,k}\hat{b}'_{L+k} \tag{39}$$

for $1 \leqslant i,j \leqslant L$. Using this we can write

$$\begin{aligned}\hat{a}'_i\hat{a}_j = &-\sum_{k,m=1}^{L}\mathbf{D}^*_{i,k}\mathbf{W}_P{}_{j,m}\hat{b}'_k\hat{b}_m - \sum_{k,m=1}^{L}\mathbf{D}^*_{i,k}\mathbf{C}_{j,m}\hat{b}'_k\hat{b}'_{L+m} \\ &- \sum_{k,m=1}^{L}\mathbf{W}_P{}^*_{i,k}\mathbf{W}_P{}_{j,m}\hat{b}_{L+k}\hat{b}_m \\ &- \sum_{k,m=1}^{L}\mathbf{W}_P{}^*_{i,k}\mathbf{C}_{j,m}\hat{b}_{L+k}\hat{b}'_{L+m}.\end{aligned} \tag{40}$$

We then show that $\langle\mathbf{1}|$ is annihilated by all the operators $\mathbf{b}'_{1\to 2L}$. It is actually sufficient to prove it for all the $\mathbf{b}'_{1\to L}$ because the

$\mathbf{b}'_{L+1\rightarrow 2L}$ have the same structure. Taking $1 \leqslant i \leqslant L$, and using Eq. (28), we have

$$\begin{aligned}\langle \mathbf{1}|b'_i &= \sum_{i_1,i_2,\ldots,i_L} \langle \mathbf{1}| \left(\sum_{k=1}^{L} \mathbf{W}^t_{P\,i,k} \hat{a}^\dagger_k - \sum_{k=1}^{L} \mathbf{W}^t_{P\,i,k} \hat{a}_{L+k} \right) \\ &= \sum_{k=1}^{L} \mathbf{W}^t_{P\,i,k} \sum_{i_1,i_2,\ldots,i_L} \langle \mathbf{1}|(\hat{a}^\dagger_k - \hat{a}_{L+k}) \\ &= \sum_{k=1}^{L} \mathbf{W}^t_{P\,i,k} \sqrt{i_k} \sum_{i_1,i_2,\ldots,i_L} \langle \ldots, i_k - 1, \ldots, i_k, \ldots | \\ &\quad - \sum_{k=1}^{L} \mathbf{W}^t_{P\,i,k} \sqrt{i_k+1} \sum_{i_1,i_2,\ldots,i_L} \langle \ldots, i_k, \ldots, i_k + 1, \ldots | \\ &= \sum_{k=1}^{L} \mathbf{W}^t_{P\,i,k} \sqrt{i_k+1} \sum_{i_1,i_2,\ldots,i_L} \langle \ldots, i_k, \ldots, i_k + 1, \ldots | \\ &\quad - \sum_{k=1}^{L} \mathbf{W}^t_{P\,i,k} \sqrt{i_k+1} \sum_{i_1,i_2,\ldots,i_L} \langle \ldots, i_k, \ldots, i_k + 1, \ldots | \\ &= 0. \end{aligned} \tag{41}$$

Hence, computing the trace of Eq. (40) we find that only the last term does not vanish and gives

$$\begin{aligned}\langle \mathbf{1}|\hat{a}'_i \hat{a}_j|\rho_{ss}\rangle &= -\langle \mathbf{1}| \sum_{k,m=1}^{L} \mathbf{W}_{P\,i,k}^{*} \mathbf{C}_{j,m} \hat{b}_{L+k} \hat{b}'_{L+m} |\rho_{ss}\rangle \\ &= -\sum_{k,m=1}^{L} \mathbf{W}_{P\,i,k}^{*} \mathbf{C}_{j,m} \delta_{k,l} = -(\mathbf{C}\mathbf{W}_P{}')_{j,i} \\ &= -(\mathbf{W}_P \mathbf{Q} \mathbf{W}'_P)_{j,i} = -\Omega_{j,i}. \end{aligned} \tag{42}$$

The observable matrix $\mathbf{O}_{i,j} = \mathrm{tr}(\hat{\rho}\hat{\alpha}^\dagger_i \hat{\alpha}_j) = \langle \mathbf{1}|\hat{a}^\dagger_i \hat{a}_j|\rho_{ss}\rangle$ is then given by

$$\mathbf{O} = -\Omega^t. \tag{43}$$

C. Exact solution of a boundary driven bosonic chain

Here we apply our method to directly obtain the spectrum of Eq. (4) for a class of linear chains (LCs) which can then be solved analytically in the limit of a long chain. We consider a linear lattice of L sites in which each site can have identical bosons and which is driven at boundaries (similar, for example, to [26]). The Lindbladian $\mathcal{L}_{\rm LC}$ then becomes

$$\mathcal{L}_{\rm LC}(\hat{\rho}) = -\frac{i}{\hbar}[\hat{H}_{\rm LC}, \hat{\rho}] + \mathcal{D}_{\rm LC}(\hat{\rho}) \tag{44}$$

with

$$\hat{H}_{\rm LC} = -J \sum_{l=1}^{L-1} (\hat{\alpha}^\dagger_l \hat{\alpha}_{l+1} + \hat{\alpha}^\dagger_{l+1} \hat{\alpha}_l) \tag{45}$$

and

$$\begin{aligned}\mathcal{D}_{\rm LC}(\hat{\rho}) = \sum_{l=1,L} [&\Lambda^+_l (\hat{\alpha}^\dagger_l \hat{\rho} \hat{\alpha}_l - \hat{\alpha}_l \hat{\alpha}^\dagger_l \hat{\rho}) \\ &+ \Lambda^-_l (\hat{\alpha}_l \hat{\rho} \hat{\alpha}^\dagger_l - \hat{\alpha}^\dagger_l \hat{\alpha}_l \hat{\rho}) + \text{H.c.}],\end{aligned} \tag{46}$$

where Λ^+_l and Λ^-_l are respectively the raising and lowering rates at site l, while J is the tunneling amplitude.

In this case, the only nonzero elements of the matrix $\mathbf{h}$ are

$$\mathbf{h}_{j,j+1} = \mathbf{h}_{j+1,j} = -J. \tag{47}$$

For the dissipation we rewrite the four coefficients Λ^a_l with four new parameters:

$$\Gamma_1 = \Lambda^-_1 - \Lambda^+_1, \quad \bar{n}_1 = \frac{\Lambda^+_1}{\Gamma_1}, \tag{48}$$

$$\Gamma_L = \Lambda^-_L - \Lambda^+_L, \quad \bar{n}_L = \frac{\Lambda^+_L}{\Gamma_L}. \tag{49}$$

Therefore, all the nonzero elements of matrix $\mathbf{P}$ are

$$\mathbf{P}_{1,1} = -\frac{\Gamma_1}{2}, \quad \mathbf{P}_{L,L} = -\frac{\Gamma_L}{2}, \tag{50}$$

$$\mathbf{P}_{m,m+1} = \mathbf{P}_{m+1,m} = \frac{iJ}{2\hbar} \tag{51}$$

for $1 \leqslant m < L$. Note that $\bar{n}_1$ and $\bar{n}_L$ do not appear in the matrix $\mathbf{P}$, and hence will not affect the rapidities.

This results in the important fact that $\mathbf{P}$ is a tridiagonal matrix whose elements are constant along the diagonals (i.e., Toeplitz) except for top-left and bottom-right corners (i.e., bordered). The eigenvalues and eigenvectors of this matrix can be analytically computed [27] (for the eigendecomposition of more general tridiagonal matrices see for example [28,29]). Assuming that λ is an eigenvalue of $\mathbf{P}$, and u is the corresponding right eigenvector so that $\mathbf{P}u = \lambda u$, then we find that λ and u are given by

$$\lambda = i\frac{J}{\hbar}\cos(\theta) \tag{52}$$

and the L elements of u are

$$u_j = \frac{u_1}{\sin(\theta)} \left\{ \sin(j\theta) - i\frac{\hbar\Gamma_1}{J} \sin[(j-1)\theta] \right\}. \tag{53}$$

In Eqs. (52) and (53) θ is a complex number which satisfies the equality

$$\begin{aligned}&-\frac{J^2}{\hbar^2}\sin[(L+1)\theta] + i\frac{J}{\hbar}(\Gamma_1 + \Gamma_L)\sin(L\theta) \\ &+ \Gamma_1\Gamma_L \sin[(L-1)\theta] = 0\end{aligned} \tag{54}$$

except the trivial solutions $\theta \neq m\pi$ with $m \in Z$.

Denoting $\theta = \alpha + i\beta$, we transform the above equation into two equations of real numbers,

$$\begin{aligned}&\frac{J^2}{\hbar^2}\sin[(L+1)\alpha]\cosh[(L+1)\beta] \\ &+ \frac{J}{\hbar}(\Gamma_1 + \Gamma_L)\cos(L\alpha)\sinh(L\beta) \\ &- \Gamma_1\Gamma_L \sin[(L-1)\alpha]\cosh[(L-1)\beta] = 0,\end{aligned} \tag{55}$$

$$\begin{aligned}&\frac{J^2}{\hbar^2}\cos[(L+1)\alpha]\sinh[(L+1)\beta] \\ &- \frac{J}{\hbar}(\Gamma_1 + \Gamma_L)\sin(L\alpha)\cosh(L\beta) \\ &- \Gamma_1\Gamma_L \cos[(L-1)\alpha]\sinh[(L-1)\beta] = 0.\end{aligned} \tag{56}$$

In the following we solve the above equation approximately, in the limit $L \to \infty$. All the solutions are found for $\alpha \in [0,\pi]$ [29]. We then make the approximation that

$$\sinh[(L+1)\beta] \simeq \sinh[(L-1)\beta] \simeq \sinh[L\beta], \tag{57}$$

$$\cosh[(L+1)\beta] \simeq \cosh[(L-1)\beta] \simeq \cosh[L\beta] \tag{58}$$

with which Eqs. (55) and (56) become

$$\frac{J^2}{\hbar^2}\sin[(L+1)\alpha] + \frac{J}{\hbar}(\Gamma_1+\Gamma_L)\cos(L\alpha)\tanh(L\beta) - \Gamma_1\Gamma_L\sin[(L-1)\alpha] = 0, \tag{59}$$

$$\frac{J^2}{\hbar^2}\cos[(L+1)\alpha]\tanh(L\beta) - \frac{J}{\hbar}(\Gamma_1+\Gamma_L)\sin(L\alpha) - \Gamma_1\Gamma_L\cos[(L-1)\alpha]\tanh(L\beta) = 0. \tag{60}$$

Combining Eqs. (59) and (60) we get

$$\frac{J^2\sin[(L+1)\alpha] - \hbar^2\Gamma_1\Gamma_L\sin[(L-1)\alpha]}{\hbar J(\Gamma_1+\Gamma_L)\cos(L\alpha)} = \frac{\hbar J(\Gamma_1+\Gamma_L)\sin(L\alpha)}{\hbar^2\Gamma_1\Gamma_L\cos[(L-1)\alpha] - J^2\cos[(L+1)\alpha]}, \tag{61}$$

which can be rewritten as

$$(\kappa_1+\kappa_L)\sin(2L\alpha) + \sin[2(L-1)\alpha] + \kappa_1\kappa_L\sin[2(L+1)\alpha] = 0, \tag{62}$$

where $\kappa_1 = J^2/(\hbar^2\Gamma_1^2)$ and $\kappa_L = J^2/(\hbar^2\Gamma_L^2)$. Equation (62) can be solved analytically when $J^2 = \hbar^2\Gamma_1\Gamma_L$. In fact $\kappa_1 = \frac{1}{\kappa_L} = \kappa$ and Eq. (62) reduces to

$$\left(\kappa + \frac{1}{\kappa}\right)\sin(2L\alpha) + \sin[2(L-1)\alpha] + \sin[2(L+1)\alpha] = 0 \tag{63}$$

or equivalently $[\kappa + \frac{1}{\kappa} + 2\cos(\alpha)]\sin(2L\alpha) = 0$. Since $\kappa + 1/\kappa \geqslant 2$, the real solutions are

$$\alpha = \frac{k\pi}{L} \tag{64}$$

with $1 \leqslant k < L$. Note that the solutions of Eq. (63) $\alpha = k\pi/2L$ with k odd have been discarded because they are inconsistent with Eqs. (59) and (60).

We then get for β

$$\tanh(L\beta) = -\frac{2\sqrt{\kappa}}{\kappa+1}\sin\frac{k\pi}{L}, \tag{65}$$

which results in

$$\beta = \frac{1}{2L}\ln\left(\frac{1-\frac{2\sqrt{\kappa}}{\kappa+1}\sin\frac{k\pi}{L}}{1+\frac{2\sqrt{\kappa}}{\kappa+1}\sin\frac{k\pi}{L}}\right). \tag{66}$$

Using Eqs. (52) and that $\theta = \alpha + i\beta$ we can write

$$\lambda = \frac{J}{\hbar}\sin(\alpha)\sinh(\beta) + i\frac{J}{\hbar}\cos(\alpha)\cosh(\beta) \tag{67}$$

and inserting the various α and β from Eqs. (64) and (66) we have an analytical solution for the eigenvalues. In Fig. 1 we compare the analytical solution Eq. (67), blue diamonds, with numerical evaluation of the spectrum for Eq. (44), red circles, for a system of length $L = 100$. The figure shows a remarkable match of the analytical and numerical spectra.

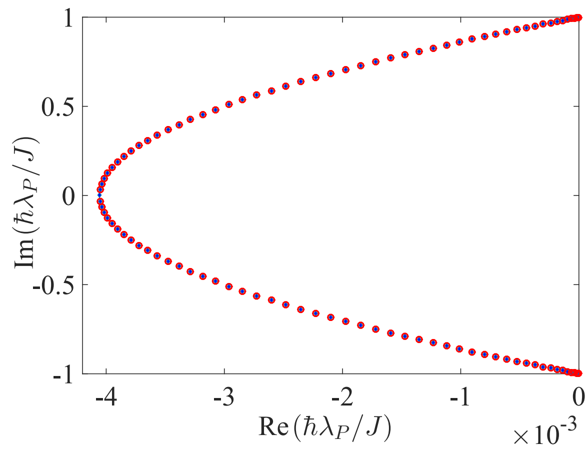

FIG. 1. Eigenvalues $\boldsymbol{\lambda}_P$ of $\mathbf{P}$ which correspond to the spectrum of the Lindbladian $\mathcal{L}_{\rm LC}$ of Eqs. (44)–(46), for a boundary driven bosonic chain of length $L = 100$, $\hbar\Gamma_1/J = 5$, $\hbar\Gamma_L/J = 1/5$. The blue diamonds are given by the numerical solution of Eq. (24), while the red circles are given by the analytical solution Eq. (67) using Eqs. (64) and (66). Re(ω) and Im(ω) mean respectively the real and the imaginary part of the complex number ω.

A natural consequence of knowing the spectrum is that it is possible to compute the relaxation gap Δ, which is given by the eigenvalues of the Lindbladian with the real part closest to zero. We thus get

$$\Delta = \frac{J}{\hbar}\sin\left(\frac{\pi}{L}\right)\sinh\left[\frac{1}{2L}\ln\left(\frac{1+\frac{2\sqrt{\kappa}}{\kappa+1}\sin\left(\frac{\pi}{L}\right)}{1-\frac{2\sqrt{\kappa}}{\kappa+1}\sin\left(\frac{\pi}{L}\right)}\right)\right] \simeq \frac{2\sqrt{\kappa}J}{\hbar(\kappa+1)L}\sin^2\left(\frac{\pi}{L}\right) \simeq \frac{2\pi^2\sqrt{\kappa}}{\hbar(\kappa+1)}\frac{J}{L^3}, \tag{68}$$

which thus scales as $1/L^3$, in agreement with the predictions in [17,30].

1. *Efficient algorithm to compute quadratic observables*

Equation (37) is a Lyapunov equation, which is a special case of Sylvester equations [31,32]. For this type of equations there exist numerical methods to efficiently solve them of order $O(L^3)$. In this section we also propose an $O(L^3)$ algorithm to solve Eq. (37) based on findings in the previous sections. We start by solving the eigenvalue decomposition problem of matrix $\mathbf{P}$, to get the left eigenvector space $\mathbf{W}^l_P$, the right eigenvector space $\mathbf{W}_P$, and the diagonal matrix of eigenvalues $\boldsymbol{\lambda}_P = \mathrm{diag}(\lambda_{P,1},\ldots\lambda_{P,L})$. At this point it is already possible to figure whether the system has any dark mode; In fact this would be manifested by the existence of eigenvalues with zero real part. If the real part of all eigenvalues of $\mathbf{P}$ is strictly smaller than 0, then the system has no dark modes, the steady state is unique, and the following algorithm can be used. We can thus write $\Omega = \mathbf{W}_P \mathbf{Q} \mathbf{W}_P^\dagger$ which gives

$$\begin{aligned}\mathbf{P}\Omega + \Omega\mathbf{P}^\dagger \\ &= \mathbf{P}\mathbf{W}_P\mathbf{Q}\mathbf{W}_P^\dagger + \mathbf{W}_P\mathbf{Q}\mathbf{W}_P^\dagger\mathbf{P}^\dagger \\ &= \mathbf{W}_P\boldsymbol{\lambda}_P\mathbf{Q}\mathbf{W}_P^\dagger + \mathbf{W}_P\mathbf{Q}\boldsymbol{\lambda}_P{}^*\mathbf{W}_P^\dagger \\ &= \boldsymbol{\Lambda}^+.\end{aligned} \tag{69}$$

We can renormalize $\mathbf{W}_P$ and $\mathbf{W}_P^l$ so that $\mathbf{W}_P^l = \mathbf{W}_P^{-1}$. From Eq. (69) we get the elements of the Hermitian matrix $\mathbf{Q}$ by

$$\begin{aligned}\boldsymbol{\lambda}_P\mathbf{Q} + \mathbf{Q}\boldsymbol{\lambda}_P^* &= \mathbf{W}_P^{-1}\boldsymbol{\Lambda}^+\left(\mathbf{W}_P^{-1}\right)^\dagger = \mathbf{W}_P^l\boldsymbol{\Lambda}^+\mathbf{W}_P^{l\,\dagger} \\ \Leftrightarrow \mathbf{Q}_{m,n} &= \frac{\left(\mathbf{W}_P^l\boldsymbol{\Lambda}^+\mathbf{W}_P^{l\,\dagger}\right)_{m,n}}{\lambda_{P,m} + \lambda_{P,n}^*}.\end{aligned} \tag{70}$$

The elements of the matrix corresponding to $\langle\hat{\alpha}_i^\dagger\hat{\alpha}_j\rangle$, i.e., $\mathbf{O}_{i,j} = \mathrm{tr}(\hat{\rho}\hat{\alpha}_i^\dagger\hat{\alpha}_j)$ are then given by, using Eq. (43),

$$\mathbf{O}_{i,j} = -\Omega_{j,i} = -\sum_{m,n}\mathbf{W}_{Pj,m}\mathbf{Q}_{m,n}\mathbf{W}_{Pi,n}^* \tag{71}$$

Note that with this approach we only need to solve an eigenvalue decomposition problem, plus a few matrix multiplications. All the matrices involved in these procedures are of size $L \times L$. The complexity of this algorithm is thus $O(L^3)$ which is the complexity of solving a $L \times L$ non-Hermitian eigendecomposition problem.

It should also be noted that because of the structure of Eq. (70), this approach can become unstable if $\mathbf{P}$ has eigenvalues whose real part is very close to 0.

2. Scaling of the relaxation time across the phase transition

With the algorithm introduced by Eqs. (70) and (71), we can more effectively explore larger systems allowing us to easily study the scaling of the relaxation gap in various nonequilibrium phases. In the following we apply our method to study the boundary driven bosonic ladder when also a magnetic field is imposed on it. This system is depicted in Fig. 2(a). The Lindblad master equation for this bosonic ladder (BL) is

$$\mathcal{L}_{\mathrm{BL}}(\hat{\rho}) = -\frac{i}{\hbar}[\hat{H}_{\mathrm{BL}},\hat{\rho}] + \mathcal{D}_{\mathrm{BL}}(\hat{\rho}), \tag{72}$$

with Hamiltonian $\hat{H}_{\mathrm{BL}}$ given by

$$\hat{H}_{\mathrm{BL}} = -\left(J^\parallel\sum_{p,j}e^{i(-1)^{p+1}\phi/2}\hat{\alpha}_{j,p}^\dagger\hat{\alpha}_{j+1,p} + J^\perp\sum_j\hat{\alpha}_{j,1}^\dagger\hat{\alpha}_{j,2}\right) + \mathrm{H.c.} \tag{73}$$

Here $J^\parallel$ is the tunneling constant in the legs, and $J^\perp$ for the rungs. $\hat{\alpha}_{j,p}$ ($\hat{\alpha}_{j,p}^\dagger$) annihilates (creates) a boson in the upper (for $p = 1$) or lower (for $p = 2$) chain at the jth rung of the ladder. A particle tunneling around a plaquette would acquire a net phase of ϕ. We consider a dissipative coupling $\mathcal{D}_{\mathrm{BL}}$ on the two edges modelled by

$$\begin{aligned}\mathcal{D}_{\mathrm{BL}}(\hat{O}) = \sum_{j=1,L}\Gamma[&\bar{n}_{j,1}(\hat{\alpha}_{j,1}\hat{O}\hat{\alpha}_{j,1}^\dagger - \hat{\alpha}_{j,1}\hat{\alpha}_{j,1}^\dagger\hat{O}) \\ &+ (\bar{n}_{j,1}+1)(\hat{\alpha}_{j,1}^\dagger\hat{O}\hat{\alpha}_{j,1} - \hat{\alpha}_{j,1}^\dagger\hat{\alpha}_{j,1}\hat{O}) + \mathrm{H.c.}],\end{aligned} \tag{74}$$

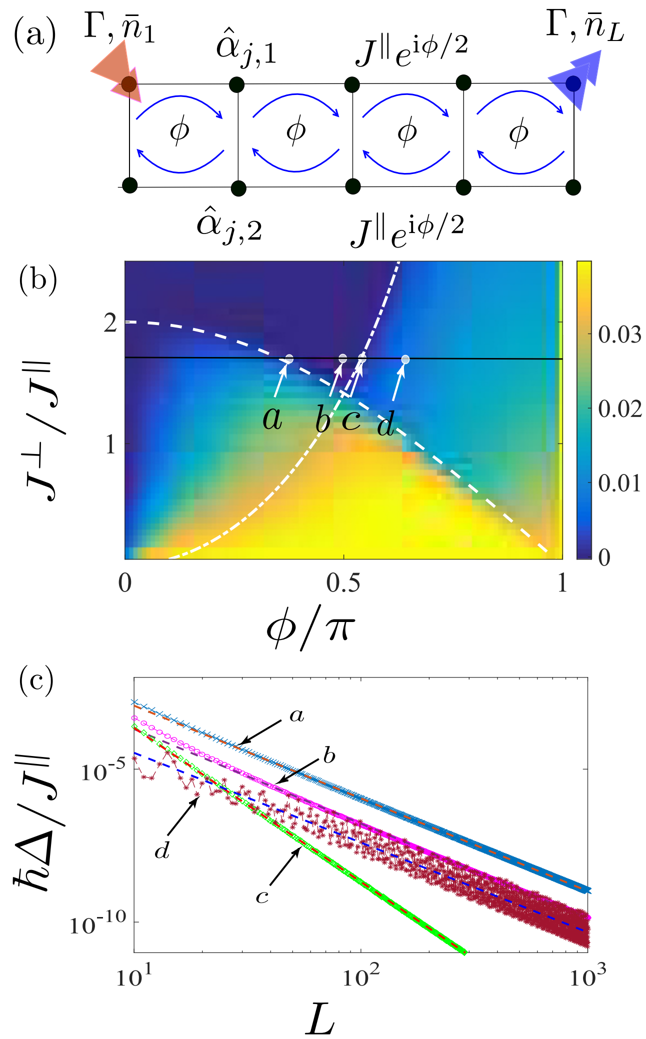

FIG. 2. (a) Ladder made of two coupled linear chains, with local bosonic excitations described by the annihilation operators at site j, $\hat{\alpha}_{j,p}$, where $p = 1,2$ for the upper and the lower leg respectively. $J^\perp$ is the tunneling between the legs, while $J^\parallel$ is the tunneling between sites in the legs. A gauge field imposes a phase ϕ. (b) Chiral current $\mathcal{J}_c$ as a function of $J^\perp$ and ϕ for $L = 500$. The white dashed and the white dot-dashed lines correspond respectively to the two phase transitions in Eqs. (79) and (80) respectively. The black horizonal line corresponds to the line $J^\perp = 1.7$, and the four white circles a, b, c, and d on this line correspond to $\phi = \phi_{c1}, 0.5, \phi_{c1}, 0.6$ respectively. Panels (a) and (b) are similar to [33]. (c) The relaxation time $\hbar\Delta/J^\parallel$ vs the length of the ladder L. Both $\hbar\Delta/J^\parallel$ and L are shown in log scale so that an algebraic decay is clearly represented by a straight line. The four lines correspond to the points a, b, c, and d in panel (b). The line marked with green diamonds (line c) shows the scaling at $\phi = 0.5398$ which corresponds to the nonequilibrium phase transition described by Eq. (80). The red dashed line is a linear fitting of line c, which has the exponent -5. The other three straight dashed lines are linear fittings of a, b, and d, all with the same exponent -3.

where Γ is the coupling constant of the bosons at sites $j = 1, L$, while $\bar{n}_{j,1}$ is the local particle density that the dissipator would impose to the bosonic site if the site was isolated. The dissipator is only coupled to the sites at the extremeties of the upper leg [see Fig. 2(a)].

The unitary counterpart of this system is known to exhibit a quantum phase transition from the Meissner to the vortex phase. The transition between the two phases is characterized by the chiral current $\mathcal{J}_c$, defined as the difference of the current between the upper and the lower leg,

$$\mathcal{J}_c = \sum_j (\mathcal{J}_{j,1} - \mathcal{J}_{j,2})/L, \tag{75}$$

where $\mathcal{J}_{j,p} = \langle i J^{\parallel} e^{i(-1)^{p+1}\phi/2} \hat{\alpha}^{\dagger}_{j,p}\hat{\alpha}_{j+1,p} + \text{H.c.}\rangle$ is the particle current out of site j on the pth leg. In the Meissner phase, $\mathcal{J}_c$ is nonzero, while in the vortex phase, $\mathcal{J}_c$ is greatly suppressed [34–36]. For an experimental realization with ultracold gases see [37].

For the open case this system was studied in detail in [33] where it was shown that two nonequilibrium phase transitions can emerge between phases with or without chiral current. Moreover, one of the two transitions would also be signalled by a sudden suppression of the current. The coupling to the baths studied here corresponds to the R configuration of [33] for which these two transitions can occur.

Translating this model to the elements of $\mathbf{P}$ we get

$$\mathbf{P}_{(1,1),(1,1)} = \mathbf{P}_{(L,1),(L,1)} = -\Gamma, \tag{76}$$

$$\mathbf{P}_{(j,1),(j,2)} = \mathbf{P}_{(j,2),(j,1)} = i\frac{J^{\perp}}{2\hbar}, \tag{77}$$

$$\mathbf{P}_{(j,p),(j+1,p)} = \mathbf{P}^{*}_{(j+1,p),(j,p)} = i\frac{J^{\parallel}}{2\hbar}e^{i(-1)^{p+1}\phi/2} \tag{78}$$

for $1 \leqslant j \leqslant L$ [except in Eq. (78) for which $j < L$]. All the other elements of $\mathbf{P}$ are zero. We note that in this case $\mathbf{P}$ is a block bordered Toeplitz matrix for which, to the best of our knowledge, the analytical eigendecomposition is not known [17]. The two phase transitions occur, respectively, for $\phi = \tilde{\phi}$, $\bar{\phi}$ and $J^{\perp} = \tilde{J}^{\perp}$, $\bar{J}^{\perp}$ given by

$$\bar{J}_{\perp} = 2J^{\parallel}\cos(\bar{\phi}/2), \tag{79}$$

$$\tilde{J}_{\perp} = 2J^{\parallel}\tan(\tilde{\phi}/2)\sin(\tilde{\phi}/2) \tag{80}$$

as shown in Fig. 2(b). The transition line Eq. (79) is depicted by a white dashed line, while the other transition line, Eq. (80), is represented by a white dot-dashed line.

Here we focus on the scaling of the relaxation gap of the Lindbladian (72) across the two open quantum phase transitions. In Fig. 2(b), we have chosen $J^{\perp}/J^{\parallel} = 1.7$, where the system exhibits two phase transitions at $\phi = \phi_{c1} \approx 0.3532$ and $\phi = \phi_{c2} \approx 0.5398$, calculated from Eqs. (79) and (80). In Fig. 2(c) we show the scaling of the relaxation gap as the size of the system increases (we consider $L = 10 \to 1000$). The gap is analyzed in four distinct points a, b, c, and d for $\phi = \phi_{c1}$, 0.5, ϕ_{c2}, and 0.6 as shown by the white dots in Fig. 2(b). For the parameters corresponding to points a, b, and d the scaling of the gap is proportional to L^{-3} as shown by, respectively, the blue crosses, the pink circles, and the red stars in Fig. 2(c). All the fits are represented by dashed lines. For the transition point c, green diamonds, the scaling is instead L^{-5} as predicted in [17,30]. It is here important to discuss the difference in scaling of the relaxation gap in the two transition lines. For the line given by Eq. (80), and hence also point c, the Hamiltonian of the bulk system presents a quantum phase transition, and the energy spectrum goes from one to two minima. At the transition point the spectrum is not quadratic but quartic thus affecting the scaling of the relaxation gap. Instead, for the line given by Eq. (79) the low-energy spectrum is not qualitatively changed; instead a gap opens (see [33]). While the opening of the gap affects the total and chiral currents, it does not change the scaling of the relaxation gap. At the intersection point between the two transition lines the scaling is indeed L^{-5}.

D. Computing the steady state

From Eq. (30) we understand that the steady state of the system is the vacuum of the operators $\hat{b}_j$, that is $|\rho_{ss}\rangle = |\mathbf{0}\rangle_b$. This is related to the vacuum of the $\hat{a}_j$, $|\mathbf{0}\rangle_a$, by a linear transformation. We can then write

$$|\rho_{ss}\rangle = \hat{S}^{-1}|\mathbf{0}\rangle_a. \tag{81}$$

In the following we show how to compute $\hat{S}$ from $\mathbf{W}_1$. First we write $\hat{S} = e^{\hat{T}}$, where $\hat{T}$ is

$$\begin{aligned}\hat{T} =& \frac{1}{2}\begin{pmatrix}\mathbf{a}^{\dagger}_{1\to L}\\ \mathbf{a}_{L+1\to 2L}\end{pmatrix}^{t}\begin{pmatrix}\mathbf{U} & \mathbf{V}\\ \mathbf{I} & \mathbf{J}\end{pmatrix}\begin{pmatrix}\mathbf{a}_{1\to L}\\ \mathbf{a}^{\dagger}_{L+1\to 2L}\end{pmatrix}\\ &+\frac{1}{2}\begin{pmatrix}\mathbf{a}_{1\to L}\\ \mathbf{a}^{\dagger}_{L+1\to 2L}\end{pmatrix}^{t}\begin{pmatrix}\mathbf{U}^{t} & \mathbf{I}^{t}\\ \mathbf{V}^{t} & \mathbf{J}^{t}\end{pmatrix}\begin{pmatrix}\mathbf{a}^{\dagger}_{1\to L}\\ \mathbf{a}_{L+1\to 2L}\end{pmatrix},\end{aligned} \tag{82}$$

where $\mathbf{U},\mathbf{V},\mathbf{I},\mathbf{J}$ are $L\times L$ matrix. Hereafter we will write

$$\mathbf{W} = \begin{pmatrix}\mathbf{U} & \mathbf{V}\\ \mathbf{I} & \mathbf{J}\end{pmatrix}. \tag{83}$$

To calculate $e^{\hat{T}}\hat{a}^{\dagger}_j e^{-\hat{T}}$ and $e^{\hat{T}}\hat{a}_{L+j}e^{-\hat{T}}$, we use the relations

$$\hat{E} := e^{\hat{T}}\hat{a}^{\dagger}_j e^{-\hat{T}} = \sum_{m=1}^{\infty}\frac{1}{m!}[\hat{T},\hat{a}^{\dagger}_j]_m,$$

$$\hat{F} := e^{\hat{T}}\hat{a}_{L+j}e^{-\hat{T}} = \sum_{m=1}^{\infty}\frac{1}{m!}[\hat{T},\hat{a}_{L+j}]_m,$$

where the nested commutator is defined recursively as $[\hat{A},\hat{B}]_{m+1} \equiv [\hat{A},[\hat{A},\hat{B}]_m]$ with $[\hat{A},\hat{B}]_0 \equiv \hat{B}$.

After a little algebra it is possible to show that

$$\begin{aligned}&\hat{S}\begin{pmatrix}\mathbf{a}^{\dagger}_{1\to L}\\ \mathbf{a}_{L+1\to 2L}\end{pmatrix}^{t}\hat{S}^{-1}\\ &=\begin{pmatrix}\mathbf{a}^{\dagger}_{1\to L}\\ \mathbf{a}_{L+1\to 2L}\end{pmatrix}^{t}e^{\mathbf{W}\mathbf{Z}_L}.\end{aligned} \tag{84}$$

Similarly, we have

$$\hat{S}\begin{pmatrix}\mathbf{a}_{1\to L}\\ \mathbf{a}^{\dagger}_{L+1\to 2L}\end{pmatrix}\hat{S}^{-1} = e^{-\mathbf{Z}_L\mathbf{W}}\begin{pmatrix}\mathbf{a}_{1\to L}\\ \mathbf{a}^{\dagger}_{L+1\to 2L}\end{pmatrix}. \tag{85}$$

We now denote $\tilde{\mathbf{W}}_1 = e^{-\mathbf{Z}_L\mathbf{W}}, \tilde{\mathbf{W}}_2 = e^{\mathbf{W}\mathbf{Z}_L}$, and we see that

$$\ln(\tilde{\mathbf{W}}_1)\mathbf{Z}_L + \mathbf{Z}_L\ln(\tilde{\mathbf{W}}_2) = 0, \tag{86}$$

which allows us to write

$$\tilde{\mathbf{W}}_2 = \mathbf{Z}_L \tilde{\mathbf{W}}_1^{-1} \mathbf{Z}_L. \quad (87)$$

Therefore we have that

$$\begin{aligned} \hat{S}\mathcal{L}\hat{S}^{-1} = & \begin{pmatrix} \mathbf{a}^\dagger_{1\to L} \\ \mathbf{a}_{L+1\to 2L} \end{pmatrix}^t \tilde{\mathbf{W}}_2 \mathbf{M} \tilde{\mathbf{W}}_1 \begin{pmatrix} \mathbf{a}_{1\to L} \\ \mathbf{a}^\dagger_{L+1\to 2L} \end{pmatrix} \\ & + \begin{pmatrix} \mathbf{a}_{1\to L} \\ \mathbf{a}^\dagger_{L+1\to 2L} \end{pmatrix}^t \tilde{\mathbf{W}}_1^t \mathbf{M}^t \tilde{\mathbf{W}}_2^t \begin{pmatrix} \mathbf{a}^\dagger_{1\to L} \\ \mathbf{a}_{L+1\to 2L} \end{pmatrix} \\ & + \mathrm{tr}(\mathbf{\Lambda}^{-t} - \mathbf{\Lambda}^+) \\ = & \begin{pmatrix} \mathbf{a}^\dagger_{1\to L} \\ \mathbf{a}_{L+1\to 2L} \end{pmatrix}^t \mathbf{Z}_L \tilde{\mathbf{W}}_1^{-1} \mathbf{Z}_L \mathbf{M} \tilde{\mathbf{W}}_1 \begin{pmatrix} \mathbf{a}_{1\to L} \\ \mathbf{a}^\dagger_{L+1\to 2L} \end{pmatrix} \\ & + \begin{pmatrix} \mathbf{a}_{1\to L} \\ \mathbf{a}^\dagger_{L+1\to 2L} \end{pmatrix}^t \tilde{\mathbf{W}}_1^t \mathbf{M}^t \mathbf{Z}_L (\tilde{\mathbf{W}}_1^t)^{-1} \mathbf{Z}_L \begin{pmatrix} \mathbf{a}^\dagger_{1\to L} \\ \mathbf{a}_{L+1\to 2L} \end{pmatrix} \\ & + \mathrm{tr}(\mathbf{\Lambda}^{-t} - \mathbf{\Lambda}^+). \end{aligned} \quad (88)$$

Thus we see that if we set $\tilde{\mathbf{W}}_1 = \mathbf{W}_1$ which means

$$\mathbf{W} = -\mathbf{Z}_L \ln \mathbf{W}_1, \quad (89)$$

then following from Eq. (17) and the explicit construction of $\mathbf{W}_1$ in Sec. II B, Eq. (88) can be simply written as

$$\hat{S}\mathcal{L}\hat{S}^{-1} = 2\sum_{i=1}^{L} \lambda_{P,i} \hat{a}_i^\dagger \hat{a}_i + 2\sum_{i=1}^{L} \lambda^*_{P,i} \hat{a}^\dagger_{L+i} \hat{a}_{L+i}. \quad (90)$$

It follows that the vacuum $|\mathbf{0}\rangle_a$ is the steady state of $\hat{S}\mathcal{L}\hat{S}^{-1}$ which implies that the steady state of $\mathcal{L}$ is given by Eq. (81). Since $\mathbf{W}_1$ is given by Eq. (89), we can also reconstruct $\hat{T}$ from Eq. (82).

III. FERMIONIC PARTICLE

A. Model

We consider an open quantum system of L sites with fermionic particles. Its dynamics is described by the quantum Lindblad master equation, Hamiltonian, and dissipators given, respectively, by Eqs. (1)–(3), however now the operators $\hat{\alpha}_j$ and $\hat{\alpha}_j^\dagger$ are fermionic.

B. Solving the master equation

1. Mapping the density operator into new representations

In order to be able to treat the open fermionic system, we proceed similarly to the bosonic case, however, in order to preserve anticommutation relations between the operators, we will have to use an ulterior transformation.

First we perform a one-to-one mapping from the density operator basis elements $|n_1,n_2,\dots n_L\rangle\langle n'_1,n'_2,\dots n'_L|$ to a state vector basis (with $2L$ sites) which we denote as $|n_1,\dots n_L,n'_1,\dots n'_L\rangle_{\mathcal{A}}$ (see for example [18,23–25]). As a result, the operator $\hat{\alpha}_i$ acting on site i to the left of the density matrix is mapped to $\hat{a}_i$ acting on the state vector on the ith site too, while the operator $\hat{\alpha}_i$ acting on the right of the density matrix is mapped to $\hat{a}^\dagger_{L+i}$ acting on the state vector. We refer to this new representation defined by the $2L$ modes $\hat{a}$ as $\mathcal{A}$. The $2L$ modes $\hat{a}_i$ satisfy the following relations:

$$\{\hat{a}_i,\hat{a}_j\} = 0, \quad \{\hat{a}_i^\dagger,\hat{a}_j^\dagger\} = 0, \quad (91a)$$

$$\{\hat{a}_{L+i},\hat{a}_{L+j}\} = 0, \quad \{\hat{a}^\dagger_{L+i},\hat{a}^\dagger_{L+j}\} = 0, \quad (91b)$$

$$\{\hat{a}_i,\hat{a}_j^\dagger\} = \delta_{ij}, \quad \{\hat{a}_{L+i},\hat{a}^\dagger_{L+j}\} = \delta_{ij}, \quad (91c)$$

$$[\hat{a}_i,\hat{a}_{L+j}] = 0, \quad [\hat{a}_i^\dagger,\hat{a}^\dagger_{L+j}] = 0, \quad (91d)$$

$$[\hat{a}_i,\hat{a}^\dagger_{L+j}] = 0, \quad [\hat{a}_i^\dagger,\hat{a}_{L+j}] = 0. \quad (91e)$$

The operators acting on the group of sites $1 \to L$ and the group of sites $L+1 \to 2L$ satisfy fermionic anticommutation relations among themselves separately. However, the operators between these two groups commute with each other.

To enforce the fermionic anticommutation relations over all the sites, we perform a second mapping from $2L$ modes $\hat{a}_i$ to another set of $2L$ modes $\hat{b}_i$, which we refer to as the $\mathcal{B}$ representation:

$$\hat{b}_i = \hat{a}_i, \quad \hat{b}_i^\dagger = \hat{a}_i^\dagger, \quad (92a)$$

$$\hat{b}_{L+i} = \mathcal{P}\hat{a}_{L+i}, \quad \hat{b}^\dagger_{L+i} = \hat{a}^\dagger_{L+i}\mathcal{P}, \quad (92b)$$

where $\mathcal{P}$ is the parity operator defined as (following [17])

$$\mathcal{P} = e^{i\pi\mathcal{N}}, \quad \mathcal{N} = \sum_{j=1}^{2L} \hat{b}_j^\dagger \hat{b}_j. \quad (93)$$

We note that $\mathcal{P}$ anticommutes with all the operators $\hat{b},\hat{b}^\dagger$, which means

$$\{\mathcal{P},\hat{b}_i\} = 0, \quad \{\mathcal{P},\hat{b}_i^\dagger\} = 0, \quad (94)$$

for $1 \leqslant i \leqslant 2L$.

Now it is straightforward to verify that the $2L$ modes $\hat{b}$ satisfy the fermionic anticommutation relations

$$\{\hat{b}_i,\hat{b}_{L+j}\} = \{\hat{a}_i,\mathcal{P}\hat{a}_{L+j}\} = 0, \quad (95a)$$

$$\{\hat{b}_i^\dagger,\hat{b}^\dagger_{L+j}\} = \{\hat{a}_i^\dagger,\hat{a}^\dagger_{L+j}\mathcal{P}\} = 0, \quad (95b)$$

$$\{\hat{b}_i,\hat{b}^\dagger_{L+j}\} = \{\hat{a}_i,\hat{a}^\dagger_{L+j}\mathcal{P}\} = 0, \quad (95c)$$

$$\{\hat{b}_i^\dagger,\hat{b}_{L+j}\} = \{\hat{a}_i^\dagger,\mathcal{P}\hat{a}_{L+j}\} = 0, \quad (95d)$$

$$\{\hat{b}_{L+i},\hat{b}_{L+j}\} = \{\mathcal{P}\hat{a}_{L+i},\mathcal{P}\hat{a}_{L+j}\} = 0, \quad (95e)$$

$$\{\hat{b}^\dagger_{L+i},\hat{b}^\dagger_{L+j}\} = \{\mathcal{P}\hat{a}^\dagger_{L+i},\hat{a}^\dagger_{L+j}\mathcal{P}\} = 0, \quad (95f)$$

$$\{\hat{b}_i,\hat{b}_j^\dagger\} = \{\mathcal{P}\hat{a}_i,\hat{a}_j^\dagger\mathcal{P}\} = \delta_{ij}, \quad (95g)$$

$$\{\hat{b}_{L+i},\hat{b}^\dagger_{L+j}\} = \{\mathcal{P}\hat{a}_{L+i},\hat{a}^\dagger_{L+j}\mathcal{P}\} = \delta_{ij}. \quad (95h)$$

The unitary part of Eq. (1) can be written in the $\mathcal{B}$ representation as

$$[\hat{H},\hat{\rho}]_{\mathcal{B}} = \sum_{i,j=1}^{L} (\mathbf{h}_{ij}\hat{b}_i^\dagger\hat{b}_j - \mathbf{h}_{ji}\hat{b}^\dagger_{L+i}\hat{b}_{L+j})|\rho\rangle_{\mathcal{B}}, \quad (96)$$

and the dissipative part of Eq. (1) can be written in the $\mathcal{B}$ representation as

$$\mathcal{D}^{\mathcal{B}}|\rho\rangle_{\mathcal{B}} = \sum_{i,j=1}^{L}(\mathbf{\Lambda}^{+}_{ij}\hat{b}^{\dagger}_{i}\hat{b}^{\dagger}_{L+j}\mathcal{P} - \mathbf{\Lambda}^{+}_{ji}\hat{b}_{i}\hat{b}^{\dagger}_{j} + \mathbf{\Lambda}^{-}_{ji}\hat{b}_{L+i}\hat{b}_{j}\mathcal{P} - \mathbf{\Lambda}^{-}_{ji}\hat{b}^{\dagger}_{i}\hat{b}_{j} - \mathbf{\Lambda}^{+*}_{ij}\hat{b}^{\dagger}_{L+i}\hat{b}^{\dagger}_{j}\mathcal{P} - \mathbf{\Lambda}^{+*}_{ji}\hat{b}_{L+i}\hat{b}^{\dagger}_{L+j} - \mathbf{\Lambda}^{-*}_{ji}\hat{b}_{i}\hat{b}_{L+j}\mathcal{P} - \mathbf{\Lambda}^{-*}_{ji}\hat{b}^{\dagger}_{L+i}\hat{b}_{L+j})|\rho\rangle_{\mathcal{B}}, \quad (97)$$

where $\mathcal{D}^{\mathcal{B}}$ is the dissipator $\mathcal{D}$ in the $\mathcal{B}$ representation while $|\rho\rangle_{\mathcal{B}}$ the density operator $\hat{\rho}$ in $\mathcal{B}$.

Now the system is almost in quadratic form of operators $\hat{b}_i$, $\hat{b}^{\dagger}_i$ except for the presence of the parity operator $\mathcal{P}$. To remove the $\mathcal{P}$ operators, we first note that in general we can write $|\rho\rangle_{\mathcal{B}}$ as

$$|\rho\rangle_{\mathcal{B}} = \sum_{n_1,\ldots,n_L,n'_1,\ldots,n'_L} \hat{b}^{\dagger,n_1}_{1}\ldots\hat{b}^{\dagger,n_L}_{L}\hat{b}^{\dagger,n'_1}_{L+1}\ldots\hat{b}^{\dagger,n'_L}_{2L}|\mathbf{0}\rangle_{\mathcal{B}}, \quad (98)$$

where $n_i,n'_i = 0,1$ for $1 \leqslant i \leqslant L$, and $|\mathbf{0}\rangle_{\mathcal{B}}$ is the vacuum state $|\mathbf{0}\rangle\langle\mathbf{0}|$ in the $\mathcal{B}$ representation. We can see that $\mathcal{P}$ conserves the parity of the number of operators of each term in Eq. (98), which is $N = \sum_{i=1}^{L}(n_i + n'_i)$. Therefore, the even sector, defined as the group of terms for which N is even, and the odd sector, defined as the group of terms for which N is odd, of $|\rho\rangle_{\mathcal{B}}$, when acted on by $\mathcal{P}$, will obtain opposite signs. Moreover, we can see that each term in Eqs. (96) and (97) does not change the parity of N, which means the even sector and the odd sector of $|\rho\rangle_b$ are decoupled under the evolution of Eq. (1). Thus we can treat them separately, and in the following we only consider the even sector for which we can just set $\mathcal{P} = 1$ [38]. Hence we have

$$\mathcal{D}^{\mathcal{B}}|\rho\rangle_{\mathcal{B}} = \sum_{i,j=1}^{L}(\mathbf{\Lambda}^{+}_{ij}\hat{b}^{\dagger}_{i}\hat{b}^{\dagger}_{L+j} - \mathbf{\Lambda}^{+}_{ji}\hat{b}_{i}\hat{b}^{\dagger}_{j} + \mathbf{\Lambda}^{-}_{ji}\hat{b}_{L+i}\hat{b}_{j} - \mathbf{\Lambda}^{-}_{ji}\hat{b}^{\dagger}_{i}\hat{b}_{j} - \mathbf{\Lambda}^{+*}_{ij}\hat{b}^{\dagger}_{L+i}\hat{b}^{\dagger}_{j} - \mathbf{\Lambda}^{+*}_{ji}\hat{b}_{L+i}\hat{b}^{\dagger}_{L+j} - \mathbf{\Lambda}^{-*}_{ji}\hat{b}_{i}\hat{b}_{L+j} - \mathbf{\Lambda}^{-*}_{ji}\hat{b}^{\dagger}_{L+i}\hat{b}_{L+j})|\rho\rangle_{\mathcal{B}}. \quad (99)$$

2. *Master equation in the new representation*

Combining Eqs. (96) and (99), the Lindbladian $\mathcal{L}$ of Eq. (1) can be written in the $\mathcal{B}$ representation as

$$\mathcal{L}^{\mathcal{B}} = \begin{pmatrix}\mathbf{b}^{\dagger}_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^{t}\mathbf{M}\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}^{\dagger}_{L+1\to 2L}\end{pmatrix} - \begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}^{\dagger}_{L+1\to 2L}\end{pmatrix}^{t}\mathbf{M}^{t}\begin{pmatrix}\mathbf{b}^{\dagger}_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix} - \mathrm{tr}(\mathbf{\Lambda}^{-t} + \mathbf{\Lambda}^{+}), \quad (100)$$

where $\mathbf{M}$ is a $2L \times 2L$ matrix,

$$\mathbf{M} = \begin{pmatrix}\mathbf{K} & \mathbf{\Lambda}^{+}\\ \mathbf{\Lambda}^{-t} & -\mathbf{K}^{\dagger}\end{pmatrix}, \quad (101)$$

and $\mathbf{K} = (-i\mathbf{h}/\hbar + \mathbf{\Lambda}^{+} - \mathbf{\Lambda}^{-t})/2$, where with $\mathbf{A}^{t}$ we indicate the transpose of the matrix $\mathbf{A}$. We have also used the notation $\mathbf{b}_{1\to L}$ to mean a column vector with elements $\hat{b}_1,\hat{b}_2,\ldots,\hat{b}_L$ and $\mathbf{b}^{\dagger}_{1\to L}$ a column vector with elements $\hat{b}^{\dagger}_1,\hat{b}^{\dagger}_2,\ldots,\hat{b}^{\dagger}_L$ (and similarly for both $\mathbf{b}_{L+1\to 2L}$ and $\mathbf{b}^{\dagger}_{L+1\to 2L}$). We should note here the difference of the last term of Eq. (100), $-\mathrm{tr}(\mathbf{\Lambda}^{-t} + \mathbf{\Lambda}^{+})$, compared to the bosonic case, $\mathrm{tr}(\mathbf{\Lambda}^{-t} - \mathbf{\Lambda}^{+})$. As for the bosonic case in Sec. II, we stress here that the simple form of Eq. (100) is due to our choice of a number conserving Hamiltonian.

3. *Normal master modes of the master equation*

In general $\mathbf{M}$ is not Hermitian and it cannot always be diagonalized, however in the following we start from the assumption that we know a transformation which can diagonalize $\mathbf{M}$ and preserves fermionic anticommutation relations. This assumption is *a posteriori* verified in all the cases we considered. This transformation is given by the matrices $\mathbf{W}_1$ and $\mathbf{W}_2$ as follows:

$$\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}^{\dagger}_{L+1\to 2L}\end{pmatrix} = \mathbf{W}_1\begin{pmatrix}\mathbf{c}_{1\to L}\\ \mathbf{c}'_{L+1\to 2L}\end{pmatrix}, \quad (102)$$

$$\begin{pmatrix}\mathbf{b}^{\dagger}_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix} = \mathbf{W}_2\begin{pmatrix}\mathbf{c}'_{1\to L}\\ \mathbf{c}_{L+1\to 2L}\end{pmatrix}, \quad (103)$$

where as for $\mathbf{b}_{1\to L}$ and $\mathbf{b}^{\dagger}_{L\to L}$, $\mathbf{c}_{1\to L}$ means the column vector made of operators $\hat{c}_1,\hat{c}_2,\ldots,\hat{c}_L$ and $\mathbf{c}'_{1\to L}$ means the column vector made of $\hat{c}'_1,\hat{c}'_2,\ldots,\hat{c}'_L$, and similarly for $\mathbf{c}_{L+1\to 2L}$ and $\mathbf{c}'_{L+1\to 2L}$. In the following we refer to the new representation defined by $\hat{c}$ as the $\mathcal{C}$ representation. Using this transformation we get

$$\mathcal{L}^{\mathcal{C}} = \begin{pmatrix}\mathbf{c}'_{1\to L}\\ \mathbf{c}_{L+1\to 2L}\end{pmatrix}^{t}\mathbf{W}_2^{t}\mathbf{M}\mathbf{W}_1\begin{pmatrix}\mathbf{c}_{1\to L}\\ \mathbf{c}'_{L+1\to 2L}\end{pmatrix} - \begin{pmatrix}\mathbf{c}_{1\to L}\\ \mathbf{c}'_{L+1\to 2L}\end{pmatrix}^{t}\mathbf{W}_1^{t}\mathbf{M}^{t}\mathbf{W}_2\begin{pmatrix}\mathbf{c}'_{1\to L}\\ \mathbf{c}_{L+1\to 2L}\end{pmatrix} - \mathrm{tr}(\mathbf{\Lambda}^{-t} + \mathbf{\Lambda}^{+}), \quad (104)$$

where $\mathcal{L}^{\mathcal{C}}$ denotes the Lindbladian $\mathcal{L}$ in the $\mathcal{C}$ representation. The fermionic anticommutation relation can be written as

$$\left\{\begin{pmatrix}\mathbf{b}_{1\to L}\\ \mathbf{b}^{\dagger}_{L+1\to 2L}\end{pmatrix},\begin{pmatrix}\mathbf{b}^{\dagger}_{1\to L}\\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^{t}\right\} = \mathbf{1}_{2L},$$

and requiring for the fermionic anticommutation relation to apply also to the $\hat{c}$ we get

$$\left\{\begin{pmatrix}\mathbf{c}_{1\to L}\\ \mathbf{c}'_{L+1\to 2L}\end{pmatrix},\begin{pmatrix}\mathbf{c}'_{1\to L}\\ \mathbf{c}_{L+1\to 2L}\end{pmatrix}^{t}\right\} = \mathbf{1}_{2L}$$

and hence

$$\mathbf{W}_2 = \mathbf{W}_1^{t\,-1}. \quad (105)$$

Using the properties of the $\mathbf{Z}_L$, $\mathbf{X}_L$, and $\mathbf{Y}_L$ matrices, Eqs. (10)–(14), it again follows that

$$\mathcal{L}^{\mathcal{C}} = \begin{pmatrix}\mathbf{c}'_{1\to L}\\ \mathbf{c}_{L+1\to 2L}\end{pmatrix}^{t}\mathbf{W}_1^{-1}\mathbf{M}\mathbf{W}_1\begin{pmatrix}\mathbf{c}_{1\to L}\\ \mathbf{c}'_{L+1\to 2L}\end{pmatrix} - \begin{pmatrix}\mathbf{c}_{1\to L}\\ \mathbf{c}'_{L+1\to 2L}\end{pmatrix}^{t}\mathbf{W}_1^{t}\mathbf{M}^{t}\mathbf{W}_1^{t\,-1}\begin{pmatrix}\mathbf{c}'_{1\to L}\\ \mathbf{c}_{L+1\to 2L}\end{pmatrix} - \mathrm{tr}(\mathbf{\Lambda}^{-t} + \mathbf{\Lambda}^{-}). \quad (106)$$

This implies that the problem of finding the normal modes of the system reduces to finding a $\mathbf{W}_1$ such that $\mathbf{M}$ can be diagonalized, that is,

$$\mathbf{W}_1^{-1}\mathbf{M}\mathbf{W}_1 = \mathrm{diag}(\beta_1,\beta_2,\ldots,\beta_{2L}), \tag{107}$$

where $\mathrm{diag}(\vec{v})$ is a diagonal matrix with the elements of the vector $\vec{v}$ on its diagonal. It is then possible to write the following compact form for $\mathcal{L}^{\mathcal{C}}$:

$$\mathcal{L}^{\mathcal{C}} = 2\sum_{i=1}^{L}(\beta_i\hat{c}_i'\hat{c}_i - \beta_{L+i}\hat{c}_{L+i}'\hat{c}_{L+i}) - \sum_{i=1}^{L}(\beta_i - \beta_{L+i}) - \mathrm{tr}(\mathbf{\Lambda}^{-^t} + \mathbf{\Lambda}^{+}). \tag{108}$$

4. *Diagonalizing* M

As we have seen until now, due to the different statistics of fermions and bosons, in order to preserve their commutation or anticommutation, for the bosonic case we diagonalize the matrix $\mathbf{Z}_L\mathbf{M}$ (see Sec. II), while for the fermionic case the relevant matrix to be diagonalized is $\mathbf{M}$. Now we explicitly construct the eigenvalues and eigenvectors of the matrix $\mathbf{M}$. Noticing the relation

$$\mathbf{Y}_L\mathbf{M}\mathbf{Y}_L = -\mathbf{M}^\dagger \tag{109}$$

we find that if

$$x = \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}$$

is a right eigenvector of $\mathbf{M}$ with eigenvalue ω, then $x^\dagger\mathbf{Y}$ is a left eigenvector of $\mathbf{M}$ with eigenvalue $-\omega^*$. In fact

$$\begin{aligned}\mathbf{M}x &= \omega x \to x^\dagger\mathbf{M}^\dagger = \omega^* x^\dagger,\\ &\to x^\dagger\mathbf{Y}_L\mathbf{Y}_L\mathbf{M}^\dagger\mathbf{Y}_L = \omega^* x^\dagger\mathbf{Y}_L,\\ &\to x^\dagger\mathbf{Y}_L\mathbf{M} = -\omega^* x^\dagger\mathbf{Y}_L.\end{aligned} \tag{110}$$

Moreover if x_1 is a right eigenvector of $\mathbf{M}$ with eigenvalue ω_1, and x_2 is a right eigenvector of $\mathbf{M}$ with eigenvalue ω_2, then if $\omega_1 + \omega_2^* \neq 0$ then $x_1^\dagger\mathbf{Y}_L x_2 = 0$. In fact

$$\mathbf{M}x_1 = \omega_1 x_1;$$
$$\mathbf{M}x_2 = \omega_2 x_2,$$

then

$$\begin{aligned}x_1^\dagger\mathbf{Y}_L\mathbf{M} &= -\omega_1^* x_1^\dagger\mathbf{Y}_L;\\ \mathbf{M}x_2 &= \omega_2 x_2;\\ &\to (\omega_1^* + \omega_2)x_1^\dagger\mathbf{Y}_L x_2 = 0.\end{aligned}$$

Since the eigenvalues of $\mathbf{M}$ always appear in pairs, we could list the eigenvalues and the corresponding eigenvectors of $\mathbf{M}$ as $\omega_1,\omega_2,\ldots,\omega_L, -\omega_1^*,\ldots,\omega_L^*$, with the matrix $\mathbf{W}_1$ composed in each column by the right eigenvectors $\mathbf{W}_1 = (\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_{2L})$. Then following Eq. (110) we know that $\vec{x}_{L+j}^\dagger\mathbf{Y}_L$ is the left eigenvector of $\mathbf{M}$ corresponding to ω_j, and $\vec{x}_j^\dagger\mathbf{Y}_L$ is the left eigenvector corresponding to $-\omega_j^*$, for $1 \leqslant j \leqslant L$. Therefore the left eigenvectors of $\mathbf{M}$ constitute the matrix $\mathbf{X}_L\mathbf{W}_1^\dagger\mathbf{Y}_L$. We can now choose to renormalize the right eigenvectors as

$$i\mathbf{X}_L\mathbf{W}_1^\dagger\mathbf{Y}_L\mathbf{W}_1 = \mathbf{Z}_L \Leftrightarrow \mathbf{Y}_L\mathbf{W}_1^\dagger\mathbf{Y}_L\mathbf{W}_1 = -\mathbf{1}_{2L} \tag{111}$$

so that we have

$$\mathbf{W}_1^{-1} = -\mathbf{Y}_L\mathbf{W}_1^\dagger\mathbf{Y}_L, \tag{112}$$

$$\mathbf{W}_2 = -\mathbf{Y}_L\mathbf{W}_1^*\mathbf{Y}_L. \tag{113}$$

At this point we define a new $L\times L$ matrix $\mathbf{P}$, which satisfies

$$\mathbf{P} = \mathbf{K} - \mathbf{\Lambda}^{+} = (-i\mathbf{h}/\hbar - \mathbf{\Lambda}^{+} - \mathbf{\Lambda}^{-^t})/2, \tag{114}$$

for which we assume to have the eigendecomposition

$$\mathbf{P}\mathbf{W}_P = \mathbf{W}_P\boldsymbol{\lambda}_P, \tag{115}$$

where $\mathbf{W}_P$ and $\boldsymbol{\lambda}_P$ are eigenvectors and eigenvalues. Then we find that the $2L\times L$ matrix formed by $(\frac{\mathbf{W}_P}{-\mathbf{W}_P})$ constitutes L right eigenvectors of $\mathbf{M}$, corresponding to $\boldsymbol{\lambda}_P$, and the $L\times 2L$ matrix $(\mathbf{W}_P^\dagger \;\; \mathbf{W}_P^\dagger)$ constitutes L left eigenvectors of $\mathbf{M}$, corresponding to $-\boldsymbol{\lambda}_P^*$. This can be shown from

$$\mathbf{M}\begin{pmatrix}\mathbf{W}_P\\ -\mathbf{W}_P\end{pmatrix} = \begin{pmatrix}\mathbf{P}\mathbf{W}_P\\ -\mathbf{P}\mathbf{W}_P\end{pmatrix} = \begin{pmatrix}\mathbf{W}_P\\ -\mathbf{W}_P\end{pmatrix}\boldsymbol{\lambda}_P$$

and

$$\begin{aligned}(\mathbf{W}_P^\dagger,\mathbf{W}_P^\dagger)\mathbf{M} &= (-\mathbf{W}_P^\dagger\mathbf{P}^\dagger, -\mathbf{W}_P^\dagger\mathbf{P}^\dagger)\\ &= -\boldsymbol{\lambda}_P^*(\mathbf{W}_P^\dagger,\mathbf{W}_P^\dagger).\end{aligned}$$

By denoting the remaining L right eigenvectors of $\mathbf{M}$ as $(\frac{\mathbf{C}}{\mathbf{D}})$, where $\mathbf{C}$, $\mathbf{D}$ are $L\times L$ matrices, we know that they form the right eigenvectors with eigenvalues $-\boldsymbol{\lambda}_P^*$, which are paired with the left eigenvectors $(\mathbf{W}_P^\dagger\;,\;\mathbf{W}_P^\dagger)$. Also $(-\mathbf{D}^\dagger\;,\;\mathbf{C}^\dagger)$ will be the left eigenvectors corresponding the eigenvalues $\boldsymbol{\lambda}_P$, which are paired with the right eigenvectors $(\frac{\mathbf{W}_P}{-\mathbf{W}_P})$.

Therefore $\mathbf{W}_1$ and $\mathbf{W}_2$ can be written more explicitly as

$$\mathbf{W}_1 = \begin{pmatrix}\mathbf{W}_P & \mathbf{C}\\ -\mathbf{W}_P & \mathbf{D}\end{pmatrix},\; \mathbf{W}_1^{-1} = \begin{pmatrix}-\mathbf{D}^\dagger & \mathbf{C}^\dagger\\ -\mathbf{W}_P^\dagger & -\mathbf{W}_P^\dagger\end{pmatrix}, \tag{116}$$

$$\mathbf{W}_2 = \begin{pmatrix}-\mathbf{D}^* & -\mathbf{W}_P^*\\ \mathbf{C}^* & -\mathbf{W}_P^*\end{pmatrix},\; \mathbf{W}_2^{-1} = \begin{pmatrix}\mathbf{W}_P^t & -\mathbf{W}_P^t\\ \mathbf{C}^t & \mathbf{D}^t\end{pmatrix}, \tag{117}$$

which means

$$\mathbf{b}_{1\to L} = \mathbf{W}_P\mathbf{c}_{1\to L} + \mathbf{C}\mathbf{c}'_{L+1\to 2L}; \tag{118a}$$

$$\mathbf{b}^\dagger_{L+1\to 2L} = -\mathbf{W}_P\mathbf{c}_{1\to L} + \mathbf{D}\mathbf{c}'_{L+1\to 2L}; \tag{118b}$$

$$\mathbf{b}^\dagger_{1\to L} = -\mathbf{D}^*\mathbf{c}'_{1\to L} - \mathbf{W}_P^*\mathbf{c}_{L+1\to 2L}; \tag{118c}$$

$$\mathbf{b}_{L+1\to 2L} = \mathbf{C}^*\mathbf{c}'_{1\to L} - \mathbf{W}_P^*\mathbf{c}_{L+1\to 2L} \tag{118d}$$

and the inverse equation

$$\mathbf{c}_{1\to L} = -\mathbf{D}^\dagger\mathbf{b}_{1\to L} + \mathbf{C}^\dagger\mathbf{b}^\dagger_{L+1\to 2L}; \tag{119a}$$

$$\mathbf{c}'_{L+1\to 2L} = -\mathbf{W}_P^\dagger\mathbf{b}_{1\to L} - \mathbf{W}_P^\dagger\mathbf{b}^\dagger_{L+1\to 2L}; \tag{119b}$$

$$\mathbf{c}'_{1\to L} = \mathbf{W}_P^t\mathbf{b}^\dagger_{1\to L} - \mathbf{W}_P^t\mathbf{b}_{L+1\to 2L}; \tag{119c}$$

$$\mathbf{c}_{L+1\to 2L} = \mathbf{C}^t\mathbf{b}^\dagger_{1\to L} + \mathbf{D}^t\mathbf{b}_{L+1\to 2L}. \tag{119d}$$

Noticing that $\sum\lambda_{P,i} = \mathrm{tr}(\mathbf{P}) = [-i\;\mathrm{tr}(\mathbf{h}/\hbar) + \mathrm{tr}(\mathbf{\Lambda}^{-^t} - \mathbf{\Lambda}^{+})]/2$ and since the $(\lambda_{P,1},\ldots\lambda_{P,L}, -\lambda_{P,1}^*,\ldots -\lambda_{P,L}^*)$

correspond to the eigenvalues of $\mathbf{M}$, $(\beta_1, \dots \beta_{2L})$, we get the following identity:

$$\sum_{i=1}^{L}(\beta_i - \beta_{L+i}) = \sum_{i=1}^{L}(\lambda_{P,i} + \lambda^*_{P,i}) = -\mathrm{tr}(\mathbf{\Lambda}^+ + \mathbf{\Lambda}^{-t}), \tag{120}$$

which exactly cancels the last term in the expression of $\mathcal{L}^{\mathcal{C}}$ in Eq. (108).

We can then write $\mathcal{L}^{\mathcal{C}}$ as

$$\mathcal{L}^{\mathcal{C}} = 2\sum_{i=1}^{L}\lambda_{P,i}\hat{c}'_i\hat{c}_i + 2\sum_{i=1}^{L}\lambda^*_{P,i}\hat{c}'_{L+i}\hat{c}_{L+i}. \tag{121}$$

The state $|\rho_{ss}\rangle_{\mathcal{C}}$ which annihilates all the operator $\mathbf{c}_{1\to 2L}$ is the steady state because $\mathcal{L}^c|\rho_{ss}\rangle_{\mathcal{C}} = 0$. The $\hat{c}_i$ are the normal master modes of the Lindblad master equation and the $\lambda_{P,i}$ the rapidities.

5. Computing the expectation value $\langle\hat{\alpha}_i^\dagger\hat{\alpha}_j\rangle$

The expectation value $\langle\hat{\alpha}_i^\dagger\hat{\alpha}_j\rangle$ is given by $\mathrm{tr}(\hat{\alpha}_i^\dagger\hat{\alpha}_j\hat{\rho}_{ss})$. In the $\mathcal{A}$ representation this is written as ${}_{\mathcal{A}}\langle\mathbf{1}|\hat{a}_i^\dagger\hat{a}_j|\rho_{ss}\rangle_{\mathcal{A}}$ where ${}_{\mathcal{A}}\langle\mathbf{1}|$ is the transpose of the identity operator in the $\mathcal{A}$ representation, $|\mathbf{1}\rangle_{\mathcal{A}} = \sum_{i_1,i_2,\dots,i_L}|i_1,i_2,\dots,i_L,i_1,i_2,\dots,i_L\rangle_{\mathcal{A}}$. In the following we will compute this quantity by transforming the $\hat{a}_i^\dagger$ and $\hat{a}_j$ in the $\mathcal{C}$ representation, and using the fact that the steady state is the vacuum of the $\hat{c}_i$, i.e., $|\rho_{ss}\rangle_{\mathcal{A}} = |\mathbf{0}\rangle_{\mathcal{C}}$.

Since for $1 \leqslant j \leqslant L$, $\hat{a}_j = \hat{b}_j, \hat{a}_j^\dagger = \hat{b}_j^\dagger$, we have $\mathrm{tr}(\hat{\rho}\hat{\alpha}_i^\dagger\hat{\alpha}_j) = {}_{\mathcal{A}}\langle\mathbf{1}|\hat{b}_i^\dagger\hat{b}_j|\mathbf{0}\rangle_{\mathcal{C}}$. Using Eqs. (118) we get

$$\hat{b}_i^\dagger = -\sum_{k=1}^{L}\mathbf{D}^*_{i,k}\hat{c}'_k - \sum_{k=1}^{L}\mathbf{W}_P{}^*_{i,k}\hat{c}_{L+k}, \tag{122}$$

$$\hat{b}_j = \sum_{k=1}^{L}\mathbf{W}_{P\,j,k}\hat{c}_k + \sum_{k=1}^{L}\mathbf{C}_{j,k}\hat{c}'_{L+k}. \tag{123}$$

Using this we can write

$$\begin{aligned}\hat{b}_i^\dagger\hat{b}_j = &-\sum_{k,m=1}^{L}\mathbf{D}^*_{i,k}\mathbf{W}_{P\ j,m}\hat{c}'_k\hat{c}_m - \sum_{k,m=1}^{L}\mathbf{D}^*_{i,k}\mathbf{C}_{j,m}\hat{c}'_k\hat{c}'_{L+m}\\ &-\sum_{k,m=1}^{L}\mathbf{W}_P{}^*_{i,k}\mathbf{W}_{P\ j,m}\hat{c}_{L+k}\hat{c}_m\\ &-\sum_{k,m=1}^{L}\mathbf{W}_P{}^*_{i,k}\mathbf{C}_{j,m}\hat{c}_{L+k}\hat{c}'_{L+m}.\end{aligned} \tag{124}$$

We then show that ${}_{\mathcal{A}}\langle\mathbf{1}|$ is annihilated by all the operators $\mathbf{c}'_{1\to 2L}$. From Eq. (119) we note that

$${}_{\mathcal{A}}\langle\mathbf{1}|\hat{c}'_i = \sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|\sum_{k=1}^{L}\mathbf{W}^t_{P\,i,k}(\hat{b}_k^\dagger - \hat{b}_{L+k}), \tag{125}$$

$${}_{\mathcal{A}}\langle\mathbf{1}|\hat{c}'_{L+i} = -\sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|\sum_{k=1}^{L}\mathbf{W}^\dagger_{P\,i,k}(\hat{b}_k + \hat{b}_{L+k}^\dagger), \tag{126}$$

for $1 \leqslant i \leqslant L$ and $n_i = 0,1$. It is thus sufficient to prove that

$$\sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|(\hat{b}_k^\dagger - \hat{b}_{L+k}) = 0 \tag{127}$$

and

$$\sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|(\hat{b}_k + \hat{b}_{L+k}^\dagger) = 0, \tag{128}$$

for $1 \leqslant k \leqslant L$. We have, using the operator $\mathcal{N}_k = \sum_{j=1}^{k}\hat{n}_j$ and considering that $\mathcal{P}|\mathbf{1}\rangle_{\mathcal{A}} = |\mathbf{1}\rangle_{\mathcal{A}}$ (there is always an even number of particles in the identity of the $\mathcal{A}$ representation),

$$\begin{aligned}&\sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|(\hat{b}_k^\dagger - \hat{b}_{L+k})\\
&= \sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|(\hat{a}_k^\dagger - \mathcal{P}\hat{a}_{L+k})\\
&= \sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|\hat{a}_k^\dagger - \sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|\hat{a}_{L+k}\\
&= \sum_{n_1,\dots,n_L}(-1)^{\mathcal{N}_{k-1}}n_k{}_{\mathcal{A}}\langle\dots,1-n_k,\dots,n_k,\dots|\\
&\quad - \sum_{n_1,\dots,n_L}(-1)^{\mathcal{N}_{k-1}}(1-n_k)_{\mathcal{A}}\langle\dots,n_k,\dots,1-n_k,\dots|\\
&= \sum_{n_1,\dots,n_L}(-1)^{\mathcal{N}_{k-1}}(1-n_k)_{\mathcal{A}}\langle\dots,n_k,\dots,1-n_k,\dots|\\
&\quad - \sum_{n_1,\dots,n_L}(-1)^{\mathcal{N}_{k-1}}(1-n_k)_{\mathcal{A}}\langle\dots,n_k,\dots,1-n_k,\dots|\\
&= 0,\end{aligned} \tag{129}$$

and

$$\begin{aligned}&\sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|(\hat{b}_k + \hat{b}_{L+k}^\dagger)\\
&= \sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|(\hat{a}_k + \hat{a}_{L+k}^\dagger\mathcal{P})\\
&= \sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|\hat{a}_k - \sum_{n_1,\dots,n_L}{}_{\mathcal{A}}\langle\mathbf{1}|\mathcal{P}\hat{a}_{L+k}^\dagger\\
&= \sum_{n_1,\dots,n_L}(-1)^{\mathcal{N}_{k-1}}(1-n_k)\ {}_{\mathcal{A}}\langle\dots,1-n_k,\dots,n_k,\dots|\\
&\quad - \sum_{n_1,\dots,n_L}(-1)^{\mathcal{N}_{k-1}}n_k\ {}_{\mathcal{A}}\langle\dots,n_k,\dots,1-n_k,\dots|\\
&= \sum_{n_1,\dots,n_L}(-1)^{\mathcal{N}_{k-1}}n_k\ {}_{\mathcal{A}}\langle\dots,n_k,\dots,1-n_k,\dots|\\
&\quad - \sum_{n_1,\dots,n_L}(-1)^{\mathcal{N}_{k-1}}n_k\ {}_{\mathcal{A}}\langle\dots,n_k,\dots,1-n_k,\dots|\\
&= 0.\end{aligned} \tag{130}$$

In both cases we have changed $n_k \to 1 - n_k$ to get the terms to cancel. Using $\mathbf{W}_1^{-1}\ \mathbf{W}_1 = \mathbf{1}_{2L}$ and Eq. (116) we can write

$$\mathbf{D} = -\mathbf{C} - \mathbf{W}_P^{\dagger\ -1}, \tag{131}$$

$$\mathbf{C} = \mathbf{W}_P\mathbf{Q}, \tag{132}$$

where $\mathbf{Q}$ is a $L \times L$ Hermitian matrix. This allows us to write

$$\mathbf{W}_1 = \begin{pmatrix} \mathbf{W}_P & \mathbf{W}_P\mathbf{Q} \\ -\mathbf{W}_P & -\mathbf{W}_P\mathbf{Q} - \mathbf{W}_P^{\dagger\,-1} \end{pmatrix} \tag{133}$$

and, from Eq. (133), together with the definition $\Omega = \mathbf{W}_P\mathbf{Q}\mathbf{W}_P^\dagger$, we have

$$\mathbf{P}\Omega + \Omega\mathbf{P}^\dagger = \mathbf{\Lambda}^+. \tag{134}$$

Hence, we find that only the last term of Eq. (124) does not vanish and gives

$$\begin{aligned} {}_\mathcal{A}\langle\mathbf{1}|\hat{a}_i^\dagger\hat{a}_j|\rho_{ss}\rangle_\mathcal{A} &= -{}_\mathcal{A}\langle\mathbf{1}|\sum_{k,m=1}^{L}{\mathbf{W}_P}^*_{i,k}\mathbf{C}_{j,m}\hat{c}_{L+k}\hat{c}'_{L+m}|\mathbf{0}\rangle_\mathcal{C} \\ &= -\sum_{k,m=1}^{L}{\mathbf{W}_P}^*_{i,k}\mathbf{C}_{j,m}\delta_{k,l} = -(\mathbf{C}{\mathbf{W}_P}^\dagger)_{j,i} \\ &= -(\mathbf{W}_P\mathbf{Q}\mathbf{W}_P^\dagger)_{j,i} = -\Omega_{j,i}. \end{aligned} \tag{135}$$

In complete analogy to the bosonic case, the observable matrix $\mathbf{O}_{i,j} = \mathrm{tr}(\hat{\rho}\hat{\alpha}_i^\dagger\hat{\alpha}_j)$ is then given by

$$\mathbf{O} = -\Omega^t. \tag{136}$$

C. Exact solution of a boundary driven XX model

Here we apply our method to directly obtain the spectrum of the Eq. (100) for the boundary driven XX model, which can then be solved analytically in the limit of a long chain. We note that an approximate steady-state solution, and exact one- and two-point correlations of the XX chain, including also local dephasing, were computed in [15]. The Lindblad equation we consider is

$$\mathcal{L}_{\mathrm{XX}}(\hat{\rho}) = -\frac{i}{\hbar}[\hat{H}_{\mathrm{XX}},\hat{\rho}] + \mathcal{D}_{\mathrm{XX}}(\hat{\rho}) \tag{137}$$

with

$$\hat{H}_{\mathrm{XX}} = J\sum_{l=1}^{L-1}(\hat{\sigma}_l^+\hat{\sigma}_{l+1}^- + \hat{\sigma}_l^-\hat{\sigma}_{l+1}^+) + h_z\sum_{l=1}^{L}\hat{\sigma}_l^z \tag{138}$$

and

$$\mathcal{D}_{\mathrm{XX}}(\hat{\rho}) = \sum_{l=1,L}[\Lambda_l^+(2\hat{\sigma}_l^+\hat{\rho}\hat{\sigma}_l^- - \{\hat{\sigma}_l^-\hat{\sigma}_l^+,\hat{\rho}\}) \tag{139}$$

$$+\ \Lambda_l^-(2\hat{\sigma}_l^-\hat{\rho}\hat{\sigma}_l^+ - \{\hat{\sigma}_l^+\hat{\sigma}_l^-,\hat{\rho}\})]. \tag{140}$$

First we apply the Jordan-Wigner transformation [39,40] to make it a fermionic chain,

$$\hat{\sigma}_j^+ = e^{-i\pi\sum_{k=1}^{j-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_j^\dagger, \tag{141}$$

$$\hat{\sigma}_j^- = e^{i\pi\sum_{k=1}^{j-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_j, \tag{142}$$

$$\hat{\sigma}_j^z = 2\hat{\alpha}_j^\dagger\hat{\alpha}_j - 1, \tag{143}$$

with Hamiltonian

$$\hat{H}_F = J\sum_{m=1}^{L-1}(\hat{\alpha}_m^\dagger\hat{\alpha}_{m+1} + \hat{\alpha}_{m+1}^\dagger\hat{\alpha}_m) + h_z\sum_{m=1}^{L}(2\hat{\alpha}_m^\dagger\hat{\alpha}_m - 1). \tag{144}$$

In this case, the nonzero elements of the matrix $\mathbf{h}$ from Eq. (2) are

$$\mathbf{h}_{j,j} = 2h_z, \tag{145}$$

$$\mathbf{h}_{j,j+1} = \mathbf{h}_{j+1,j} = J. \tag{146}$$

The dissipation instead becomes

$$\begin{aligned} \mathcal{D}(\hat{\rho}) &= \sum_{m=1,L}\Lambda_m^+\big(2e^{-i\pi\sum_{k=1}^{m-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_m^\dagger\hat{\rho}e^{i\pi\sum_{k=1}^{m-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_m \\ &\quad - \{\hat{\alpha}_m\hat{\alpha}_m^\dagger,\hat{\rho}\}\big) \\ &\quad + \sum_{m=1,L}\Lambda_m^-\big(2e^{i\pi\sum_{k=1}^{m-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_m\hat{\rho}e^{-i\pi\sum_{k=1}^{m-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_m^\dagger \\ &\quad - \{\hat{\alpha}_m^\dagger\hat{\alpha}_m,\hat{\rho}\}\big) \\ &= \Lambda_1^+(2\hat{\alpha}_1^\dagger\hat{\rho}\hat{\alpha}_1 - \{\hat{\alpha}_1\hat{\alpha}_1^\dagger,\hat{\rho}\}) + \Lambda_1^-(2\hat{\alpha}_1\hat{\rho}\hat{\alpha}_1^\dagger - \{\hat{\alpha}_1^\dagger\hat{\alpha}_1,\hat{\rho}\}) \\ &\quad + \Lambda_L^+\big(2e^{-i\pi\sum_{k=1}^{L-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_L^\dagger\hat{\rho}e^{i\pi\sum_{k=1}^{L-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_L - \{\hat{\alpha}_L\hat{\alpha}_L^\dagger,\hat{\rho}\}\big) \\ &\quad + \Lambda_L^-\big(2e^{i\pi\sum_{k=1}^{L-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_L\hat{\rho}e^{-i\pi\sum_{k=1}^{L-1}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_L^\dagger - \{\hat{\alpha}_L^\dagger\hat{\alpha}_L,\hat{\rho}\}\big) \\ &= \Lambda_1^+(2\hat{\alpha}_1^\dagger\hat{\rho}\hat{\alpha}_1 - \{\hat{\alpha}_1\hat{\alpha}_1^\dagger,\hat{\rho}\}) + \Lambda_1^-(2\hat{\alpha}_1\hat{\rho}\hat{\alpha}_1^\dagger - \{\hat{\alpha}_1^\dagger\hat{\alpha}_1,\hat{\rho}\}) \\ &\quad + \Lambda_L^+\big(2\hat{\alpha}_L^\dagger e^{-i\pi\sum_{k=1}^{L}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\rho}e^{i\pi\sum_{k=1}^{L}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_L - \{\hat{\alpha}_L\hat{\alpha}_L^\dagger,\hat{\rho}\}\big) \\ &\quad + \Lambda_L^-\big(2\hat{\alpha}_L e^{i\pi\sum_{k=1}^{L}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\rho}e^{-i\pi\sum_{k=1}^{L}\hat{\alpha}_k^\dagger\hat{\alpha}_k}\hat{\alpha}_L^\dagger - \{\hat{\alpha}_L^\dagger\hat{\alpha}_L,\hat{\rho}\}\big), \end{aligned} \tag{147}$$

where in the last lines we have included an extra term in the string operator and shifted its position. Therefore, in the $\mathcal{A}$ representation, we have

$$\begin{aligned} \mathcal{D}^\mathcal{A} &= \Lambda_1^+(2\hat{a}_1^\dagger\hat{a}_{L+1}^\dagger - \hat{a}_1\hat{a}_1^\dagger - \hat{a}_{L+1}\hat{a}_{L+1}^\dagger) \\ &\quad + \Lambda_1^-(2\hat{a}_1\hat{a}_{L+1} - \hat{a}_1^\dagger\hat{a}_1 - \hat{a}_{L+1}^\dagger\hat{a}_{L+1}) \\ &\quad + \Lambda_L^+(2\hat{a}_L^\dagger\hat{a}_{2L}^\dagger\mathcal{P} - \hat{a}_L\hat{a}_L^\dagger - \hat{a}_{2L}\hat{a}_{2L}^\dagger) \\ &\quad + \Lambda_L^-(2\hat{a}_L\hat{a}_{2L}\mathcal{P} - \hat{a}_L^\dagger\hat{a}_L - \hat{a}_{2L}^\dagger\hat{a}_{2L}), \end{aligned} \tag{148}$$

which is, in the $\mathcal{B}$ representation,

$$\begin{aligned} \mathcal{D}^\mathcal{B} &= \Lambda_1^+(2\hat{b}_1^\dagger\hat{b}_{L+1}^\dagger\mathcal{P} - \hat{b}_1\hat{b}_1^\dagger - \hat{b}_{L+1}\hat{b}_{L+1}^\dagger) \\ &\quad + \Lambda_1^-(-2\hat{b}_1\hat{b}_{L+1}\mathcal{P} - \hat{b}_1^\dagger\hat{b}_1 - \hat{b}_{L+1}^\dagger\hat{b}_{L+1}) \\ &\quad + \Lambda_L^+(2\hat{b}_L^\dagger\hat{b}_{2L}^\dagger - \hat{b}_L\hat{b}_L^\dagger - \hat{b}_{2L}\hat{b}_{2L}^\dagger) \\ &\quad + \Lambda_L^-(-2\hat{b}_L\hat{b}_{2L} - \hat{b}_L^\dagger\hat{b}_L - \hat{b}_{2L}^\dagger\hat{b}_{2L}). \end{aligned} \tag{149}$$

Then, restricting ourselves to the even sector as in Sec. II B, we can simply write $\mathcal{D}^\mathcal{B}$ as

$$\begin{aligned} \mathcal{D}^\mathcal{B} &= \Lambda_1^+(2\hat{b}_1^\dagger\hat{b}_{L+1}^\dagger - \hat{b}_1\hat{b}_1^\dagger - \hat{b}_{L+1}\hat{b}_{L+1}^\dagger) \\ &\quad + \Lambda_1^-(-2\hat{b}_1\hat{b}_{L+1} - \hat{b}_1^\dagger\hat{b}_1 - \hat{b}_{L+1}^\dagger\hat{b}_{L+1}) \\ &\quad + \Lambda_L^+(2\hat{b}_L^\dagger\hat{b}_{2L}^\dagger - \hat{b}_L\hat{b}_L^\dagger - \hat{b}_{2L}\hat{b}_{2L}^\dagger) \\ &\quad + \Lambda_L^-(-2\hat{b}_L\hat{b}_{2L} - \hat{b}_L^\dagger\hat{b}_L - \hat{b}_{2L}^\dagger\hat{b}_{2L}), \end{aligned} \tag{150}$$

which is exactly the same as Eq. (99). Hence we can follow the derivation in Sec. II B, and reduce the problem of diagonalizing $\mathcal{L}$ to the eigendecomposition of the matrix $\mathbf{P}$ defined in Eq. (23).

For convenience we rewrite the four coefficients Λ_l^a with four new parameters:

$$\Gamma_1 = \Lambda_1^- + \Lambda_1^+, \quad \bar{n}_1 = \frac{\Lambda_1^+}{\Gamma_1}, \tag{151}$$

$$\Gamma_L = \Lambda_L^- + \Lambda_L^+, \quad \bar{n}_L = \frac{\Lambda_L^+}{\Gamma_L} \tag{152}$$

and then all the nonzero elements of matrix $\mathbf{P}$ are

$$\mathbf{P}_{1,1} = -i\frac{h_z}{\hbar} - \frac{\Gamma_1}{2}, \quad \mathbf{P}_{L,L} = -i\frac{h_z}{\hbar} - \frac{\Gamma_L}{2}, \tag{153a}$$

$$\mathbf{P}_{m,m} = -i\frac{h_z}{\hbar}, \quad \text{for } 1 < m < L, \tag{153b}$$

$$\mathbf{P}_{m,m+1} = \mathbf{P}_{m+1,m} = -\frac{iJ}{2\hbar}, \quad \text{for } 1 \leqslant m < L. \tag{153c}$$

It thus results in that $\mathbf{P}$ is a bordered tridiagonal Toeplitz matrix, whose eigenvalues and eigenvectors can be analytically computed [27], and we can follow the derivation in Sec. II to get the explicit spectrum under the condition $J^2 = \hbar^2\Gamma_1\Gamma_L$. The only difference is that here there is a constant shift ih for the diagonal terms of matrix $\mathbf{P}$ which was not present in Sec. II. As a result, the eigenvalues of $\mathbf{P}$ are

$$\hbar\lambda = -J\sin(\alpha)\sinh(\beta) - i[h_z + J\cos(\alpha)\cosh(\beta)] \tag{154}$$

with

$$\alpha = \frac{k\pi}{L}, \tag{155}$$

$$\beta = \frac{1}{2L}\ln\left(\frac{1 + \frac{2\sqrt{\kappa}}{\kappa+1}\sin\frac{k\pi}{L}}{1 - \frac{2\sqrt{\kappa}}{\kappa+1}\sin\frac{k\pi}{L}}\right), \tag{156}$$

where k is an integer $1 \leqslant k < L$ and $\kappa = [J/(\hbar\Gamma_1)]^2$. For more details on the steps from Eq. (153) to Eqs. (154)–(156) see Sec. II.

D. Computing the steady state

From Eq. (30) we understand that the steady state of the system is the vacuum of the operators $\hat{c}_j$, that is $|\rho_{ss}\rangle = |\mathbf{0}\rangle_{\mathcal{C}}$. This is related to the vacuum of the $\hat{b}_j$, $|\mathbf{0}\rangle_{\mathcal{B}} = |\mathbf{0}\rangle_{\mathcal{A}}$, by a linear transformation. We can then write

$$|\rho_{ss}\rangle = |\mathbf{0}\rangle_{\mathcal{C}} = \hat{\mathcal{S}}^{-1}|\mathbf{0}\rangle_{\mathcal{B}}. \tag{157}$$

In the following we show how to compute $\hat{\mathcal{S}}$ from $\mathbf{W}_1$. First we write $\hat{\mathcal{S}} = e^{\hat{T}}$, where $\hat{T}$ is

$$\hat{T} = \frac{1}{2}\begin{pmatrix}\mathbf{b}_{1\to L}^\dagger \\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^t \begin{pmatrix}\mathbf{U} & \mathbf{V} \\ \mathbf{I} & \mathbf{J}\end{pmatrix}\begin{pmatrix}\mathbf{b}_{1\to L} \\ \mathbf{b}_{L+1\to 2L}^\dagger\end{pmatrix} - \frac{1}{2}\begin{pmatrix}\mathbf{b}_{1\to L} \\ \mathbf{b}_{L+1\to 2L}^\dagger\end{pmatrix}^t \begin{pmatrix}\mathbf{U}^t & \mathbf{I}^t \\ \mathbf{V}^t & \mathbf{J}^t\end{pmatrix}\begin{pmatrix}\mathbf{b}_{1\to L}^\dagger \\ \mathbf{b}_{L+1\to 2L}\end{pmatrix} \tag{158}$$

and where $\mathbf{U},\mathbf{V},\mathbf{I},\mathbf{J}$ are $L \times L$ matrices. It should be noted that the presence of a minus sign in the second line of Eq. (158) is different from the bosonic case. Hereafter we will write

$$\mathbf{W} = \begin{pmatrix}\mathbf{U} & \mathbf{V} \\ \mathbf{I} & \mathbf{J}\end{pmatrix}. \tag{159}$$

To calculate $e^{\hat{T}}\hat{b}_j^\dagger e^{-\hat{T}}$ and $e^{\hat{T}}\hat{b}_{L+j}e^{-\hat{T}}$, we use the relations

$$\hat{E} := e^{\hat{T}}\hat{b}_j^\dagger e^{-\hat{T}} = \sum_{m=1}^{\infty}\frac{1}{m!}[\hat{T},\hat{b}_j^\dagger]_m,$$

$$\hat{F} := e^{\hat{T}}\hat{b}_{L+j}e^{-\hat{T}} = \sum_{m=1}^{\infty}\frac{1}{m!}[\hat{T},\hat{b}_{L+j}]_m,$$

where the nested commutator is defined recursively as $[\hat{A},\hat{B}]_{m+1} \equiv [\hat{A},[\hat{A},\hat{B}]_m]$ with $[\hat{A},\hat{B}]_0 \equiv \hat{B}$. This results in

$$\hat{\mathcal{S}}\begin{pmatrix}\mathbf{b}_{1\to L}^\dagger \\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^t \hat{\mathcal{S}}^{-1} = \begin{pmatrix}\mathbf{b}_{1\to L}^\dagger \\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^t e^{\mathbf{W}}. \tag{160}$$

Similarly we can write

$$\hat{\mathcal{S}}\begin{pmatrix}\mathbf{b}_{1\to L} \\ \mathbf{b}_{L+1\to 2L}^\dagger\end{pmatrix}\hat{\mathcal{S}}^{-1} = e^{-\mathbf{W}}\begin{pmatrix}\mathbf{b}_{1\to L} \\ \mathbf{b}_{L+1\to 2L}^\dagger\end{pmatrix}, \tag{161}$$

which results in

$$\begin{aligned}\hat{\mathcal{S}}\mathcal{L}\hat{\mathcal{S}}^{-1} &= \begin{pmatrix}\mathbf{b}_{1\to L}^\dagger \\ \mathbf{b}_{L+1\to 2L}\end{pmatrix}^t e^{\mathbf{W}}\mathbf{M}e^{-\mathbf{W}}\begin{pmatrix}\mathbf{b}_{1\to L} \\ \mathbf{b}_{L+1\to 2L}^\dagger\end{pmatrix} \\ &\quad - \begin{pmatrix}\mathbf{b}_{1\to L} \\ \mathbf{b}_{L+1\to 2L}^\dagger\end{pmatrix}^t e^{-\mathbf{W}^t}\mathbf{M}^t e^{\mathbf{W}^t}\begin{pmatrix}\mathbf{b}_{1\to L}^\dagger \\ \mathbf{b}_{L+1\to 2L}\end{pmatrix} \\ &\quad - \mathrm{tr}(\mathbf{\Lambda}^{-t} + \mathbf{\Lambda}^+).\end{aligned} \tag{162}$$

Thus, as in the bosonic case by setting $\mathbf{W}_1 = e^{-\mathbf{W}}$, which means

$$\mathbf{W} = -\ln\mathbf{W}_1, \tag{163}$$

we can diagonalize $\hat{\mathcal{S}}\mathcal{L}\hat{\mathcal{S}}^{-1}$ whose steady state is $|\mathbf{0}\rangle_{\mathcal{B}}$. This then allows us to derive Eq. (157).

IV. CONCLUSIONS

We have shown how to map the problem of computing the relaxation rates and the normal master modes of a Lindblad master equation for the dissipatively boundary driven uniform noninteracting bosonic chain and the XX chain, to the diagonalization of a tridiagonal bordered Toeplitz matrix. This special structure of the matrix also allows us to find explicit analytical solutions and we have shown an approximate solution for a large system. With the approach presented, for a system of size L, the matrix to be diagonalized is only of size $L \times L$ (when considering Hamiltonians which conserve the total number of particles, i.e., there are no terms of the type $\hat{\alpha}_i\hat{\alpha}_j$ or $\hat{\alpha}_i^\dagger\hat{\alpha}_j^\dagger$). For more general Hamiltonians our approach can be readily extended, however the matrix to be diagonalized would be a $2L \times 2L$ block bordered Toeplitz matrix (for a uniform bulk Hamiltonian with boundary dissipative driving) which cannot be diagonalized with the same analytical formulas. The method here presented can be useful to study both the time evolution (since it gives access to all the normal master modes and rapidities) and steady states for open bosonic and fermionic systems far from equilibrium. Due to its simplicity, this method can allow one to find more analytically solvable solutions.

We have also proposed a numerical algorithm which can efficiently compute observables of the type $\langle\hat{\alpha}_i^\dagger\hat{\alpha}_j\rangle$. We have then used this method to compute the relaxation gap of a boundary driven bosonic quadratic system which presents two different nonequilibrium phase transitions. A scaling analysis of the gap shows that the gap scale as $1/L^3$ in all the parameter space except at one of the two phase transitions, for which the relaxation gap scales as $1/L^5$. This is due to the different behavior of the spectrum of the Hamiltonian of the bulk of the system at the two transition points.

ACKNOWLEDGMENTS

We acknowledge insightful discussions with U. Bissbort. D.P. acknowledges support from Singapore Ministry of Education, Singapore Academic Research Fund Tier-I (Project No. SUTDT12015005).

[1] S. Diehl, A. Micheli, A. Kantian, B. Kraus, H. P. Büchler, and P. Zoller, Nat. Phys. **4**, 878 (2008).
[2] S. Diehl, A. Tomadin, A. Micheli, R. Fazio, and P. Zoller, Phys. Rev. Lett. **105**, 015702 (2010).
[3] E. G. Dalla Torre, E. Demler, T. Giamarchi, and E. Altman, Nat. Phys. **6**, 806 (2010).
[4] D. Poletti, J. S. Bernier, A. Georges, and C. Kollath, Phys. Rev. Lett. **109**, 045302 (2012).
[5] Z. Cai and T. Barthel, Phys. Rev. Lett. **111**, 150403 (2013).
[6] D. Poletti, P. Barmettler, A. Georges, and C. Kollath, Phys. Rev. Lett. **111**, 195301 (2013).
[7] B. Sciolla, D. Poletti, and C. Kollath, Phys. Rev. Lett. **114**, 170401 (2015).
[8] T. Prosen, Phys. Rev. Lett. **106**, 217206 (2011).
[9] T. Prosen, Phys. Rev. Lett. **107**, 137201 (2011).
[10] T. Prosen, Phys. Rev. Lett. **112**, 030603 (2014).
[11] D. Karevski, V. Popkov, and G. M. Schutz, Phys. Rev. Lett. **110**, 047201 (2013).
[12] V. Popkov, D. Karevski, and G. M. Schutz, Phys. Rev. E **88**, 062118 (2013).
[13] M. V. Medvedyeva, F. H. L. Essler, and T. Prosen, Phys. Rev. Lett. **117**, 137202 (2016).
[14] T. Prosen, J. Phys. A: Math. Theor. **48**, 373001 (2015).
[15] M. Znidaric, J. Stat. Mech.: Theory Exp. (2010) L05002.
[16] A. Asadian, D. Manzano, M. Tiersch, and H. J. Briegel, Phys. Rev. E **87**, 012109 (2013).
[17] T. Prosen, New J. Phys. **10**, 043026 (2008).
[18] T. Prosen, J. Stat. Mech: Theory Exp. (2010) P07020.
[19] T. Prosen and T. H. Seligman, J. Phys. A **43**, 392004 (2010).
[20] This matrix is assumed to be diagonalizable and the steady state is assumed to be unique. While this restricts the type of systems which can be analyzed, it includes a large number of physically relevant systems as many boundary dissipatively driven systems.
[21] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
[22] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[23] T. Prosen and I. Pizorn, Phys. Rev. Lett. **101**, 105701 (2008).
[24] T. Prosen and B. Zunkovic, New J. Phys. **12**, 025016 (2010).
[25] I. Pizorn, V. Eisler, S. Andergassen, and M. Troyer, New J. Phys. **16**, 073007 (2014).
[26] A. Bermudez, M. Bruderer, and M. B. Plenio, Phys. Rev. Lett. **111**, 040601 (2013).
[27] W. C. Yueh, Appl. Math. E-Notes **5**, 66 (2005).
[28] S. Kouachi, Electron. J. Linear Algebra **15**, 115 (2006).
[29] A. R. Willms, SIAM J. Matrix Anal. Appl. **30**, 639 (2008).
[30] M. Znidaric, Phys. Rev. E **92**, 042143 (2015).
[31] R. H. Bartels and G. W. Stewart, Commun. ACM **15**, 820 (1972).
[32] G. H. Golub, S. Nash, and C. F. Van Loan, IEEE Trans. Autom. Control **24**, 909 (1979).
[33] C. Guo and D. Poletti, Phys. Rev. A **94**, 033610 (2016).
[34] M. Kardar, Phys. Rev. B **33**, 3125 (1986).
[35] E. Granato, Phys. Rev. B **42**, 4797(R) (1990).
[36] C. Denniston and C. Tang, Phys. Rev. Lett. **75**, 3930 (1995).
[37] M. Atala, M. Aidelsburger, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Nat. Phys. **10**, 588 (2014).
[38] The odd sector can be solved following similar steps however it should be noted that (i) the unique steady state is in the even sector and (ii) the odd sector is irrelevant in the computation of quadratic observables like currents and densities.
[39] P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928).
[40] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961).