# Quantum dynamical speedup in a nonequilibrium environment

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We theoretically study the dynamical speedup of a quantum system in a nonequilibrium environment. Based on the trace distance, we derive the generalized Margolus-Levitin and Mandelstam-Tamm types of bounds on the quantum speed limit time of a quantum system evolving from an arbitrary initial state. We demonstrate that the mechanism for the speedup of dynamical evolution is closely associated with both the energy of the system and exchange of information between the system and its environment. It is shown that the nonequilibrium feature of the environment can speed up the quantum evolution in both Markovian and non-Markovian dynamics regions. We emphasize that the non-Markovian effect of the system dynamics is neither necessary nor sufficient to speed up the quantum evolution in a nonequilibrium environment.

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## I. INTRODUCTION

How to achieve the dynamical speedup of evolution of a quantum system and how to preserve quantum coherence are the main challenges in quantum computation and quantum information science [1–11]. Quantum speed limits (QSLs) are the lower bounds on the minimal evolution time between two distinguishable states of a quantum system, which would have potential applications in understanding how to quantify and manipulate quantum coherence and in protecting quantum information against decoherence induced by environmental noise [12–20]. For an isolated quantum system under unitary dynamics, two independent bounds on this minimum time were established by Mandelstam and Tamm (MT) [21] and Margolus and Levitin (ML) [22]. And then, the bounds were subsequently unified [23]. Also, QSLs for an open quantum system were investigated based on different geometric measures for the distinguishability between the initial and final states of the system [24–32]. Generalized ML and MT types of QSL bounds for an open quantum system under nonunitary dynamical evolution have been derived based on Bures angle by employing the von Neumann and Cauchy-Schwarz inequalities, and it has been shown that the ML type of bound is not only sharper than the MT type but also tight [26]. Actually, there are many QSLs since many distance metrics can be employed to measure the distinguishability of two quantum states [18]. The trace distance, as a good distance metric in the state space, is easy to calculate and measure and has the properties of contractivity and convexity. Also, it has a well-motivated physical interpretation as a measure for the distinguishability of two quantum states and possesses the properties of stability and chaining for estimating the error in a complex quantum information process [33–36].

The ratio between the QSL time and driving time estimates the potential capacity of speeding up quantum dynamical evolution. When the ratio equals one, there is no potential capacity for quantum dynamical evolution speedup, while for other cases the smaller the ratio is, the greater the potential capacity will be. There has been well-established investigations on the dynamical speedup of evolution of a quantum system with energy dissipation in an equilibrium environment theoretically [37–39], and the environment-assisted speedup of dynamical evolution is realized by controlling the environment experimentally [40]. The connection has been investigated between QSLs and entanglement and non-Markovianity [2–5,37–39]. It was shown that entanglement can speed up the dynamical evolution of a closed composite system. Also, it was confirmed that the non-Markovian effect of the open system dynamics is the unique condition for speeding up quantum dynamical evolution in the long driving time limit while it is a necessary but not sufficient condition to speed up the dynamical evolution of the quantum system within a given driving time [37,40].

Environmental effects on an open quantum system can be classified into two aspects: transition between quantum states and loss of phase coherence induced by dissipative and pure decoherence environments [19,41–44]. However, there are physical situations where nonequilibrium environmental effects could be important. For example, in transient and ultrafast processes in physical systems, the nonequilibrium feature of the environment has dominant influence on the dynamical evolution and decoherence of the quantum system [45–52], and it has been shown that the nonequilibrium decoherence process gives rise to a time-dependent frequency shift and results in the suppression of the non-Markovian effect of the system dynamics [52]. On the other hand, the mechanism for the speedup of quantum evolution in a pure decoherence environment is unknown. Therefore, several important questions need to be discussed: How can we relate the QSLs with dynamical decoherence, what is the mechanism for the dynamical speedup under nonequilibrium decoherence processes, and how can we speed up the quantum dynamical evolution in a nonequilibrium environment?

In this paper, we study the dynamical speedup of a quantum system under a nonequilibrium decoherence process with both nonstationary and non-Markovian statistical properties. Based on the trace distance, we derive the generalized ML and MT types of QSL bounds of the system evolving from an arbitrary initial state. We demonstrate that the generalized sharper QSL bound is the ML type and it is tight since it can be attained if the environment is in equilibrium and the decoherence dynamics of the system is Markovian. Under a nonequilibrium decoherence process, the QSL time is closely

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associated with both the energy of the system and information exchange between the system and the environment. We show that the environmental nonequilibrium feature can speed up the dynamical evolution of the quantum system in both Markovian and non-Markovian dynamics regions, and emphasize that the non-Markovian effect of the system dynamics is neither a necessary nor sufficient condition for the quantum speedup in a nonequilibrium environment.

This paper is organized as follows. In Sec. II we derive the generalized ML and MT types of QSL bounds of a quantum system evolving from an arbitrary initial state based on the trace distance and then analyze the speedup of dynamical evolution in a nonequilibrium environment. In Sec. III we discuss the influence of the environmental nonequilibrium feature on the dynamical speedup of the quantum evolution. In Sec. IV we give the conclusions drawn from the present study.

## II. THEORETICAL FRAMEWORK

### A. Derivation of generalized ML and MT types of QSL bounds

For an open quantum system, the reduced density matrix of the system is governed by

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t), \tag{1}$$

where $\mathcal{L}$ is a time-local superoperator. Since the trace distance $D(\rho_1,\rho_2) = \frac{1}{2}\mathrm{tr}|\rho_1 - \rho_2|$ can be employed as a distance metric to measure the distance between two distinguishable quantum states $\rho_1$ and $\rho_2$ [33–36], we quantify the distance between an initial state $\rho(0)$ and a final state $\rho(t)$ by

$$\mathcal{L}_D(\rho(0),\rho(t)) = \tfrac{1}{2}||\rho(t) - \rho(0)||_{\mathrm{tr}} = \tfrac{1}{2}\mathrm{tr}|\rho(t) - \rho(0)|, \tag{2}$$

where $||A||_{\mathrm{tr}} = \mathrm{tr}|A| = \mathrm{tr}\sqrt{A^\dagger A}$ is the trace norm of an operator $A$ and $0 \leqslant \mathcal{L}_D(\rho(0),\rho(t)) \leqslant 1$. To obtain the QSL time, we consider the dynamical velocity [53] with which the reduced density matrix evolves by differentiating Eq. (2) with respect to time

$$\frac{d}{dt}\mathcal{L}_D(\rho(0),\rho(t)) = \frac{1}{2}\mathrm{tr}\frac{[\rho(t) - \rho(0)]\mathcal{L}\rho(t)}{|\rho(t) - \rho(0)|}. \tag{3}$$

By employing the von-Neumann trace inequality for operators [54–56], we have

$$\begin{aligned}\left|\mathrm{tr}\frac{[\rho(t) - \rho(0)]\mathcal{L}\rho(t)}{|\rho(t) - \rho(0)|}\right| &\leqslant \sum_{i=1}^{n}\varrho_i\sigma_i \\ &= \sum_{i=1}^{n}\sigma_i = ||\mathcal{L}\rho(t)||_{\mathrm{tr}},\end{aligned} \tag{4}$$

where $\varrho_i = 1$ and $\sigma_1 \geqslant \cdots \geqslant \sigma_n$ are the singular values of the matrices $\frac{\rho(t)-\rho(0)}{|\rho(t)-\rho(0)|}$ and $\mathcal{L}\rho(t)$, respectively. For the case of unitary dynamical evolution of a quantum system with the Hamiltonian $H(t)$, the trace norm $||\mathcal{L}\rho(t)||_{\mathrm{tr}}$ is bounded by the average energy $\langle H\rangle$ of the system, namely,

$$||\mathcal{L}\rho(t)||_{\mathrm{tr}} = \left|\left|-\frac{i}{\hbar}[H(t),\rho(t)]\right|\right|_{\mathrm{tr}} \leqslant \frac{2}{\hbar}\langle H\rangle, \tag{5}$$

where the triangle inequality $|X \pm Y| \leqslant |X| + |Y|$ is used and the ground-state energy of the system is supposed to be zero [22,23]. Integrating Eq. (3), using the fact $|\int dt\dot{\mathcal{L}}_D(\rho(0),\rho(t))| \leqslant \int dt|\dot{\mathcal{L}}_D(\rho(0),\rho(t))|$, over time from $t = 0$ to $t = \tau$ leads to the generalized ML type of QSL bound:

$$\tau \geqslant \frac{\mathcal{L}_D(\rho(0),\rho(\tau))}{\frac{1}{2}\overline{\sum_{i=1}^{n}\sigma_i}}, \tag{6}$$

where the time average is denoted by $\overline{X} = \tau^{-1}\int_0^\tau dtX$.

By employing the Cauchy-Schwarz inequality for operators, we have

$$\begin{aligned}\left|\mathrm{tr}\frac{[\rho(t) - \rho(0)]\mathcal{L}\rho(t)}{|\rho(t) - \rho(0)|}\right| &\leqslant \sqrt{\sum_{i=1}^{n}\varrho_i^2\sum_{i=1}^{n}\sigma_i^2} \\ &= \sqrt{n\sum_{i=1}^{n}\sigma_i^2} = \sqrt{n}||\mathcal{L}\rho(t)||_{\mathrm{hs}},\end{aligned} \tag{7}$$

where $||A||_{\mathrm{hs}} = \sqrt{\mathrm{tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm of an operator $A$. In the case of unitary dynamical evolution, the Hilbert-Schmidt norm $||\mathcal{L}\rho(t)||_{\mathrm{hs}}$ is bounded by the variance of the energy $\Delta H$ of the system

$$||\mathcal{L}\rho(t)||_{\mathrm{hs}} = \left|\left|-\frac{i}{\hbar}[H(t),\rho(t)]\right|\right|_{\mathrm{hs}} \leqslant \frac{\sqrt{2}}{\hbar}\Delta H, \tag{8}$$

where the equality sign holds only if the state of the system is pure. Similarly, we obtain the generalized MT type of QSL bound

$$\tau \geqslant \frac{\mathcal{L}_D(\rho(0),\rho(\tau))}{\frac{1}{2}\overline{\sqrt{n\sum_{i=1}^{n}\sigma_i^2}}}. \tag{9}$$

By combining Eqs. (6) and (9), we obtain the unified expression for the generalized ML and MT types of QSL bounds for the quantum system evolving initially from an arbitrary state as follows:

$$\tau_{\mathrm{QSL}} = \max\left\{\frac{1}{\frac{1}{2}\overline{\sum_{i=1}^{n}\sigma_i}}, \frac{1}{\frac{1}{2}\overline{\sqrt{n\sum_{i=1}^{n}\sigma_i^2}}}\right\}\mathcal{L}_D(\rho(0),\rho(\tau)). \tag{10}$$

Since the inequality $\frac{\sum_{i=1}^{n}\sigma_i}{n} \leqslant \sqrt{\frac{\sum_{i=1}^{n}\sigma_i^2}{n}}$ between the arithmetic and quadratic means holds, the generalized ML type of QSL bound is sharper than the MT type. In the following, we demonstrate the generalized ML type bound is tight; that is, the bound can be exactly attained.

### B. Dynamical speedup under nonequilibrium decoherence processes

We consider a two-level quantum system coupled to a nonequilibrium environment. The environmental effect on the quantum system could be described by means of stochastic fluctuations in some system observable based on the Kubo-Anderson spectral diffusion process [57–59]. We assume that the energy of the system is conserved and the stochastic fluctuations only cause the decoherence of the quantum system, and the pure decoherence Hamiltonian of the system is written as [57–61]

$$H(t) = \frac{\hbar}{2}[\omega_0 + \xi(t)]\sigma_z, \tag{11}$$

where $\sigma_z$ is the Pauli matrix, $\omega_0$ is the intrinsic transition frequency between the excited state $|e\rangle$ and ground state $|g\rangle$, and $\xi(t)$ denotes the environmental noise which is subject to a nonstationary and non-Markovian stochastic process.

The dynamical evolution for the total density matrix yields the Liouville equation

$$\frac{\partial}{\partial t}\rho(t;\xi(t)) = -\frac{i}{\hbar}[H(t),\rho(t;\xi(t))], \tag{12}$$

where the notation $\rho(t;\xi(t))$ is used to indicate that the total density matrix depends on the environmental noise $\xi(t)$, and the reduced density matrix of the system can be derived by taking an average over the environmental noise as $\rho(t) = \langle\rho(t;\xi(t))\rangle$. The total density matrix elements in the basis $\{|e\rangle,|g\rangle\}$ satisfy the stochastic differential equations

$$\begin{aligned}\dot{\rho}_{ee}(t;\xi(t)) =&0,\\ \dot{\rho}_{ge}(t;\xi(t)) =&i[\omega_0 + \xi(t)]\rho_{ge}(t;\xi(t)),\end{aligned} \tag{13}$$

where $\rho_{gg}(t;\xi(t)) = 1 - \rho_{ee}(t;\xi(t))$ and $\rho_{eg}(t;\xi(t)) = \rho^*_{ge}(t;\xi(t))$. Transforming Eq. (13) into the integral form gives

$$\begin{aligned}\rho_{ee}(t;\xi(t)) =&\rho_{ee}(0;\xi(0)),\\ \rho_{ge}(t;\xi(t)) =&e^{i\omega_0 t + i\int_0^t \xi(t')dt'}\rho_{ge}(0;\xi(0)).\end{aligned} \tag{14}$$

Assuming there is no initial correlation between the system and its surroundings

$$\rho(0) = \langle\rho(0;\xi(0))\rangle = \rho(0;\xi(0)), \tag{15}$$

and taking the average over the environmental noise $\xi(t)$ gives the reduced density matrix elements

$$\begin{aligned}\rho_{ee}(t) =&\rho_{ee}(0),\\ \rho_{ge}(t) =&e^{i\omega_0 t}F(t)\rho_{ge}(0),\end{aligned} \tag{16}$$

where $\rho_{gg}(t) = 1 - \rho_{ee}(t)$, $\rho_{eg}(t) = \rho^*_{ge}(t)$, and $F(t) = \langle e^{i\int_0^t dt'\xi(t')}\rangle$ is the decoherence factor quantifying the system coherence evolution for the system initially in the coherent superposition of the two states $|e\rangle$ and $|g\rangle$ [59,62].

Then we obtain the reduced density matrix of the quantum system in matrix notation as

$$\rho(t) = \begin{pmatrix} \rho_{ee}(0) & \rho_{eg}(0)e^{-i\omega_0 t}F^*(t)\\ \rho_{ge}(0)e^{i\omega_0 t}F(t) & \rho_{gg}(0)\end{pmatrix}. \tag{17}$$

Taking a derivative of Eq. (17) with respect to time, as long as $F(t) \neq 0$, the evolution equation for the reduced density matrix elements yields

$$\begin{aligned}\dot{\rho}_{ee}(t) =&0,\\ \dot{\rho}_{ge}(t) =&\left[i\omega_0 + \frac{\dot{F}(t)}{F(t)}\right]\rho_{ge}(t).\end{aligned} \tag{18}$$

In view of $F(t)$ being a complex time-dependent function [52], we can write a time-local master equation for the reduced density matrix as

$$\begin{aligned}\frac{d}{dt}\rho(t) =& \mathcal{L}\rho(t)\\ =& -\frac{i}{2}[\omega_0 - s(t)][\sigma_z,\rho(t)] + \frac{1}{2}\gamma(t)[\sigma_z\rho(t)\sigma_z - \rho(t)],\end{aligned} \tag{19}$$

where $s(t) = -\mathrm{Im}[\dot{F}(t)/F(t)]$ and $\gamma(t) = -\mathrm{Re}[\dot{F}(t)/F(t)]$ represent the frequency shift and decoherence rate, respectively.

Under a nonequilibrium decoherence process, the environmental noise $\xi(t)$ is assumed to be subject to the nonstationary and non-Markovian dichotomic process. This noise process can be characterized by the nonequilibrium parameter $a$ and the memory decay rate $\kappa$, and its amplitude switches randomly with the jumping rate $\lambda$ between the values $\pm\nu$ [52]. Under nonequilibrium dynamical decoherence, the decoherence factor for the quantum system can be analytically solved as [52]

$$\begin{aligned}F(t) =& \mathscr{L}^{-1}[\mathcal{F}(p)],\\ \mathcal{F}(p) =& \frac{p^2 + (\kappa + ia\nu)p + \kappa(2\lambda + ia\nu)}{p^3 + \kappa p^2 + (2\kappa\lambda + \nu^2)p + \kappa\nu^2},\end{aligned} \tag{20}$$

where $\mathscr{L}^{-1}$ indicates the inverse Laplace transform and the initial condition is $F(0) = 1$.

Based on the master equation for the reduced density matrix in Eq. (19), the distance from an initial state $\rho(0)$ within a given driving time $\tau$ can be expressed as

$$\begin{aligned}\mathcal{L}_D(\rho(0),\rho(\tau)) =& \frac{1}{2}\mathrm{tr}|\rho(\tau) - \rho(0)|\\ =& \frac{1}{2}\sqrt{r_x^2 + r_y^2}\left|\int_0^\tau \dot{F}(t)dt\right|\\ =& \frac{1}{2}\sqrt{r_x^2 + r_y^2}\,|1 - F(\tau)|.\end{aligned} \tag{21}$$

Here we have set $\omega_0 = 0$ and chosen an arbitrary initial state of the quantum system as [33]

$$\rho(0) = \tfrac{1}{2}(I + \boldsymbol{r}\cdot\boldsymbol{\sigma}), \quad |\boldsymbol{r}| \leqslant 1, \tag{22}$$

where $I$ is the identity matrix, $\boldsymbol{r} = (r_x,r_y,r_z)$ is a real vector, and $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ denotes the vector of Pauli matrices. If and only if $|\boldsymbol{r}| = 1$, the initial state is pure, otherwise, mixed. And the singular values of $\mathcal{L}\rho(t)$ satisfy

$$\begin{aligned}\sigma_1 = \sigma_2 =& \tfrac{1}{2}\sqrt{r_x^2 + r_y^2}\,|\dot{F}(t)|\\ =& \tfrac{1}{2}\sqrt{r_x^2 + r_y^2}\sqrt{s(t)^2 + \gamma^2(t)}|F(t)|.\end{aligned} \tag{23}$$

Thus the path distance can be expressed as

$$\begin{aligned}\ell_D^p(\rho(0),\rho(\tau)) =& \frac{1}{2}\overline{\sum_{i=1}^n \sigma_i}\,\tau\\ =& \frac{1}{2}\sqrt{r_x^2 + r_y^2}\int_0^\tau dt\sqrt{s(t)^2 + \gamma^2(t)}|F(t)|.\end{aligned} \tag{24}$$

Based on the generalized ML type of QSL bound in Eq. (10), the ratio between the QSL time and the driving time can be expressed as

$$\frac{\tau_{\mathrm{QSL}}}{\tau} = \frac{\left|\int_0^\tau dt\dot{F}(t)\right|}{\int_0^\tau dt|\dot{F}(t)|} = \frac{|1 - F(\tau)|}{\int_0^\tau dt\sqrt{s^2(t) + \gamma^2(t)}|F(t)|}. \tag{25}$$

The ratio is independent of the initial state due to the fact that there is no change of the population of the quantum system and the energy of the system is conserved under pure decoherence processes. Consequently, the QSL time under nonequilibrium decoherence is closely associated with the decoherence dynamics of the system, and the dynamical speedup of evolution contributes from both the unitary part of the evolution and the flow of information exchange between the system and the environment. The numerator and denominator in the right-hand side of Eq. (25) stand for the scaled distance $\mathcal{L}_D(\tau)$ and path distance $\ell_D^p(\tau)$,

$$\mathcal{L}_D(\tau) = \frac{\mathcal{L}_D(\rho(0),\rho(\tau))}{\frac{1}{2}\sqrt{r_x^2+r_y^2}} = |1-F(\tau)|,$$
$$\ell_D^p(\tau) = \frac{\ell_D^p(\rho(0),\rho(\tau))}{\frac{1}{2}\sqrt{r_x^2+r_y^2}} = \int_0^\tau dt\sqrt{s^2(t)+\gamma^2(t)}|F(t)|, \tag{26}$$

with a scaled constant $\frac{1}{2}\sqrt{r_x^2+r_y^2}$. Then the speed of dynamical evolution is proportional to

$$\begin{aligned} v(t) &= \sqrt{v_s^2(t)+v_\gamma^2(t)} = \sqrt{s^2(t)+\gamma^2(t)}|F(t)| \\ &\leqslant \sqrt{s^2(t)+\gamma^2(t)} = \langle H_{\text{eff}}\rangle. \end{aligned} \tag{27}$$

Here the effective Hamiltonian of the quantum system satisfies $H_{\text{eff}} = -\frac{\hbar}{2}s(t)\sigma_z - \frac{i\hbar}{2}\gamma(t)I$ based on Eq. (19), the ground-state energy of the system is taken to be zero, and the quantum average of a non-Hermitian operator is given by its absolute value [63]. As a consequence, the dynamical speed of evolution is bounded by the average energy of the system. To further understand the physical meaning of $v_s(t) = |s(t)F(t)|$, taking the integration of $v_s(t)$ over time from 0 to $\tau$, we have

$$\begin{aligned} \int_0^\tau dt\, v_s(t) &= \int_0^\tau dt|s(t)|\left[1-\int_0^t dt'\gamma(t')|F(t')|\right] \\ &= \int_0^\tau dt|s(t)| - \int_0^\tau dt|s(t)|\int_0^t dt'\gamma(t')|F(t')|. \end{aligned} \tag{28}$$

In the case of the unitary dynamical evolution, namely, $\gamma(t)=0$, the integration in Eq. (28) is the Anandan-Aharonov geometric distance [53] since $|s(t)|$ is the Lamb frequency shift. Here we note $\int_0^\tau dt\, v_s(t)$ as the generalized Anandan-Aharonov geometric distance to describe the contribution from the unitary part of the evolution under nonequilibrium dynamical decoherence. The physical meaning of $v_\gamma(t) = |\gamma(t)F(t)|$ is obvious since

$$\int_0^\tau dt\, v_\gamma(t) = 2\mathcal{N}(\tau) + 1 - |F(\tau)|, \tag{29}$$

where $\mathcal{N}(\tau) = -\int_{0\,\gamma(t)<0}^\tau dt\gamma(t)|F(t)|$ is the non-Markovianity characterizing the backflow of information from the environment to the system within a given driving time [42–44]. Thus, $\int_0^\tau dt\, v_\gamma(t)$ is related to the contribution from the nonunitary part of dynamical evolution in terms of the information exchange between the system and its environment.

It is worth noting the situation where the system undergoes a equilibrium decoherence process. In this case, the decoherence factor $F(t)$ is real and the frequency shift $s(t)=0$. Thus, the ratio in Eq. (25) can be reduced to

$$\frac{\tau_{\text{QSL}}}{\tau} = \frac{|1-F(t)|}{1-|F(t)|+2\mathcal{N}(\tau)}. \tag{30}$$

In this situation, the ratio $\tau_{\text{QSL}}/\tau$ is associated with the non-Markovianity and change of coherence in a limited driving time, whereas it depends only on the non-Markovianity in the long time limit [37–40]. When the decoherence dynamics of the quantum system is Markovian, the decoherence factor $0 \leqslant F(t) \leqslant 1$ and the non-Markovianity $\mathcal{N}(\tau)=0$. Then the QSL time is equal to the driving time and consequently the bound is attained, which indicates that the generalized ML type of QSL bound is also tight.

## III. RESULTS AND DISCUSSION

In the following, we illustrate the influence of nonequilibrium and non-Markovian dynamics on the quantum speed limit time.

Figure 1 shows the ratio $\tau_{\text{QSL}}/\tau$ for different values of $a$ as a function of the driving time $\tau$. In both Markovian and non-Markovian dynamics regions, when the environment is out of equilibrium, the ratio $\tau_{\text{QSL}}/\tau < 1$ indicates the potential capacity for quantum dynamical speedup and that the dynamical evolution departs from the geodesic distance. Furthermore, as the environment is away from equilibrium, for a given driving time, the decreases of the ratio $\tau_{\text{QSL}}/\tau$ imply that the potential capacity for the dynamical speedup enhances.

The nonequilibrium feature of the environment makes small changes in the distance $\mathcal{L}_D(\tau)$ in both the dynamics regions. And the influences of the environmental nonequilibrium feature on the dynamical speedup are obvious in the path distance $\ell_D^p(\tau)$. To present the contributions from $v_s(t)$ and $v_\gamma(t)$ on the dynamical evolution, we plot $v_s(t)$ and $v_\gamma(t)$ as a function of the driving time $\tau$ for different values of $a$ in Fig. 2. In both the dynamical regions of Markovian and non-Markovian, when the environment is in equilibrium, $v_s(t)=0$ indicates that the contributions for the dynamical evolution entirely come from $v_\gamma(t)$. As the environment gradually departs from equilibrium, $v_s(t)$ increases significantly which indicates that the contribution from unitary part of the evolution enhances. It arises from the fact that the frequency shift $s(t)$ increases and the decoherence dynamics gets suppressed as the environment

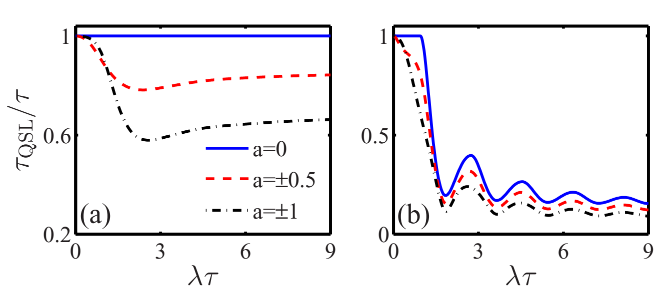


FIG. 1. Ratio $\tau_{\text{QSL}}/\tau$ as a function of the driving time $\tau$ for different values of $a$ in (a) Markovian dynamics region with $\nu = 0.8\lambda$ and (b) non-Markovian dynamics region with $\nu = 3\lambda$. The memory decay rate $\kappa = 2\lambda$.

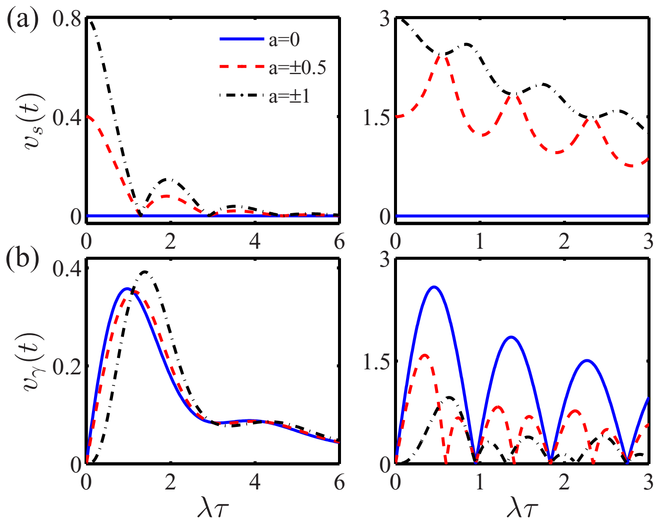


FIG. 2. Speed of the dynamical evolution as a function of the driving time $\tau$ for different values of $a$. The upper panels are for $v_s(t)$ and the bottom panels for $v_\gamma(t)$. Left and right columns are the cases of Markovian and non-Markovian dynamics regions with $\nu = 0.8\lambda$ and $\nu = 3\lambda$, respectively. The memory decay rate $\kappa = 2\lambda$.

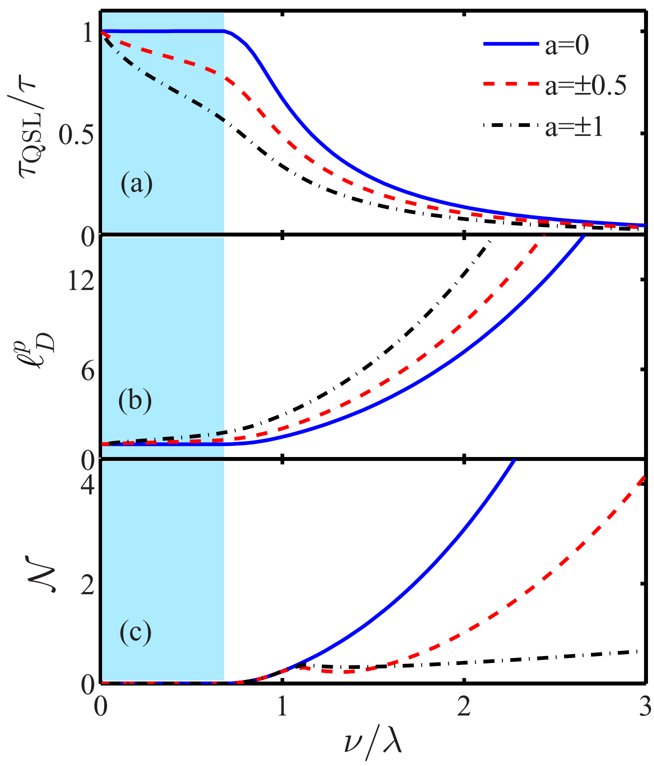


FIG. 3. (a) Ratio $\tau_{\rm QSL}/\tau$, (b) path distance $\ell_D^p$, and (c) non-Markovianity $\mathcal{N}$ as functions of the coupling $\nu$ for different values of $a$ in the long driving time limit ($\tau \to \infty$). The memory decay rate $\kappa = \lambda$. The colored region is Markovian dynamics region and the boundary of the dynamics regions from Markovian to non-Markovian is $\nu = 0.68\lambda$.

deviates from equilibrium [52]. However, for the contribution of $v_\gamma(t)$, there is no obvious change in the Markovian dynamical region (left panel, bottom); contrarily, its contribution weakens remarkably in the non-Markovian dynamical region (right panel, bottom). Furthermore, the contributions of $v_s(t)$ and $v_\gamma(t)$ in the non-Markovian region are much larger than that in the Markovian region due to the non-Markovianity induced by the strong coupling between the system and the environment.

As a comparison, we also show the results of the environment in equilibrium: In the Markovian dynamics region as shown in Fig. 1(a) the ratio $\tau_{\rm QSL}/\tau = 1$ at any time reveals that the dynamical evolution is always along the geodesic distance and there is no potential capacity to speed up the dynamical evolution [$v_s(t) = 0$ in Fig. 2, correspondingly]; whereas in the non-Markovian dynamics region as shown in Fig. 1(b), it exhibits a plateau ($\tau_{\rm QSL}/\tau = 1$) in short time and then decreases, which indicates that the dynamical evolution starts along and then gradually deviates from the geodesic distance.

Figure 3(a) shows the ratio $\tau_{\rm QSL}/\tau$ as a function of the coupling $\nu$ for different values of $a$ in the long driving time limit. As shown in the figure, the dynamical speedup occurs in both Markovian and non-Markovian dynamics regions, when the environment is in nonequilibrium. Physically, the environmental nonequilibrium feature makes the quantum dynamical speedup depart from the geodesic distance in the Markovian and non-Markovian dynamics regions as shown in Fig. 3(b). Furthermore, for a given coupling $\nu$, as the environment departs from equilibrium, the quantum dynamical speedup becomes obvious arising from the fact that the path distance diverges far from the geodesic distance. As shown in Fig. 3(c), there is no non-Markovianity in the Markovian dynamics region and the non-Markovianity $\mathcal{N}$ decreases in the non-Markovian dynamics region as the environment diverges from equilibrium. Also, the results in Fig. 3 indicate that the possible speedup capacity of a quantum system becomes strong when the environment is far from equilibrium. This indicates that the environmental nonequilibrium feature is the only reason for quantum evolution speedup in the Markovian dynamics region, and the environmental nonequilibrium feature and the non-Markovianity are both reasons for quantum evolution speedup in the non-Markovian dynamics region.

## IV. CONCLUSIONS

We derived the generalized ML and MT type bounds for QSLs of the system evolving from an arbitrary initial state based on the trace distance measure. We demonstrated that quantum dynamical speedup is related to the contributions from the unitary part of the evolution in terms of the energy of the system and from the nonunitary part of dynamical evolution based on the exchange of information between the system and its environment. We showed that in the Markovian dynamics region, the nonequilibrium feature of the environment is the only reason to speed up the dynamical evolution, whereas the environmental nonequilibrium feature and the non-Markovian effect of the system dynamics are both reasons for quantum evolution speedup in the non-Markovian dynamics region. The non-Markovian effect of the system dynamics is neither necessary nor sufficient to speed up the quantum evolution under nonequilibrium decoherence.

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