Test of a hypothesis of realism in quantum theory using a Bayesian approach

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In this paper we propose a time-independent *equality* and time-dependent *inequality*, suitable for an experimental test of the hypothesis of realism. The derivation of these relations is based on the concept of conditional probability and on Bayes' theorem in the framework of Kolmogorov's axiomatics of probability theory. The equality obtained is intrinsically different from the well-known Greenberger-Horne-Zeilinger (GHZ) equality and its variants, because violation of the proposed equality might be tested in experiments with only two microsystems in a maximally entangled Bell state $|\Psi^-\rangle$, while a test of the GHZ equality requires at least three quantum systems in a special state $|\Psi^{\text{GHZ}}\rangle$. The obtained inequality differs from Bell's, Wigner's, and Leggett-Garg inequalities, because it deals with spin $s = 1/2$ projections onto only two nonparallel directions at two different moments of time, while a test of the Bell and Wigner inequalities requires at least three nonparallel directions, and a test of the Leggett-Garg inequalities requires at least three distinct moments of time. Hence, the proposed inequality seems to open an additional experimental possibility to avoid the "contextuality loophole." Violation of the proposed equality and inequality is illustrated with the behavior of a pair of anticorrelated spins in an external magnetic field and also with the oscillations of flavor-entangled pairs of neutral pseudoscalar mesons.

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I. INTRODUCTION

One of the main questions that arise when comparing predictions of quantum theory with experiment is to what extent the real physical properties of micro-objects correspond to the observed values, measured with macrodevices. Heisenberg noted in his book [1] that this essential question of quantum theory is close to the general analysis of our perception of the "phenomenon" of our world and the gist of the phenomenon, the "noumenon," according to Kant [2].

From the point of view of the simplest "orthodox" version of the quantum-mechanical formalism [3], a process of measurement corresponds to expansion of a microsystem state vector $|\psi\rangle$ into a superposition of a macroscopically definitive state $|a_\alpha\rangle$:

$$|\psi\rangle = \sum_\alpha C_\alpha |a_\alpha\rangle, \tag{1}$$

where, according to the law of Born, $|C_\alpha|^2$ defines the probability to find the system in state $|a_\alpha\rangle$ after measurement. Usually one takes $|a_\alpha\rangle$ as a set of eigenvectors of Hermitian operator $\hat{A}$ which corresponds to some physical characteristics (observables) A of the microsystem studied. Of course, the expansion (1) and law of Born may be generalized in terms of POVMs (positive operator-valued measures—a description of a measurement using positively defined operators) and the projection postulate of Dirac and von Neumann [4]. However it is not important for the consequent arguments which approach is used. We will use the simplest one, i.e., the superposition principle (1) and the law of Born.

Let a microsystem now have two distinct observables A and B which have spectra $\{a_\alpha\}$ and $\{b_\beta\}$ accordingly. If physical characteristics A and B may be simultaneously measured (i.e., may be measured with zero dispersion with a pair of macroscopic devices of the same type), then the vectors by which the state $|\psi\rangle$ is expanded must be the common eigenvectors of the operators $\hat{A}$ and $\hat{B}$, leading to the commutation condition $[\hat{A},\hat{B}] = 0$. If the operators $\hat{A}$ and $\hat{B}$ do not commute, then they do not have a common system of eigenvectors. In this case the observables A and B cannot be measured together by any macrodevice. The simplest example of the observables that cannot be measured together is the projection of a fermion spin onto two nonparallel directions, which are defined by unit vectors $\vec{a}$ and $\vec{b}$. Another example is the CP parity and flavor of a neutral pseudoscalar meson.

The following question may be posed: do the physical characteristics A and B exist simultaneously and independently without the assumption of the possibility to measure them by some macrodevices (this is the hypothesis of local realism). Usually the terms "hypothesis of local realism" and "concept of macroscopic realism" are understood as the possibility to describe a physical system in the classical paradigm using some assumptions about the nature of "classical reality." It might be for example locality or the negligible influence of a measurement device. All these assumptions we will call together the "hypothesis of realism" by Einstein [5]. It does not make sense to talk about the physical properties of a micro-object without making a statement about the macrodevices used to measure these properties (this is the Copenhagen interpretation of quantum mechanics and the principle of complementarity of Bohr [6]).

A natural (but probably not unique) way to write in mathematical terms the condition that a set of physical characteristics of a microsystem exists jointly regardless of the possibility to measure it with a macrodevice is that the joint probability of the set of observables under consideration is non-negative at any time. For example for the observables A and B that means that for any elements of the spectra $a_{\alpha'} \in \{a_\alpha\}$

and $b_{\beta'} \in \{b_\beta\}$ the probability of simultaneous existence of $a_{\alpha'}$ and $b_{\beta'}$—the joint probability $w(a_{\alpha'} \cap b_{\beta'})$—satisfies the following condition:

$$0 \leqslant w(a_{\alpha'} \cap b_{\beta'}) \leqslant 1. \tag{2}$$

The assumption of the existence of non-negative joint probabilities (2) was implicitly used by Bell in his pioneering works [7,8], as the density distribution $\rho(\lambda)$ of hidden variables λ is a direct corollary of (2). Later Bell's idea was developed by Clauser, Horne, Shimony, and Holt [9]. A historical review of Bell's inequalities may be found in [10–14], The idea of the existence of non-negative joint probabilities (2) was used by Wigner in [15]. In [16] the arguments of Bell were translated into non-negative joint probabilities.

In classical physics the joint probabilities $w(a_{\alpha'} \cap b_{\beta'}) = w(b_{\beta'} \cap a_{\alpha'})$ always exist and are well defined for any physical system. In quantum theory if $[\hat{A},\hat{B}] \neq 0$, the joint probabilities $w(a_{\alpha'} \cap b_{\beta'})$ cannot be directly measured by macrodevices. However in this case it is possible to use an indirect procedure based on specific properties of entangled states and the notion of an "element of physical reality" introduced by Einstein.

The element of physical reality is defined as follows [5]: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity" and "every element of the physical reality must have a counterpart in the physical theory." It is obvious that a given element may be identified with some property of a physical system (for example with a spin projection onto some direction) and that obtaining information about the element of physical reality differs from obtaining information about some observable only by measurement method. In the first case the measurement is indirect, in the second, direct, and accompanied by a reduction of the state vector or density matrix. Because of the above we will not make any distinction in the current paper between the observables and the elements of physical reality.

Let us show how the indirect procedure works using the decay of a pseudoscalar meson to a fermion-antifermion pair. If the decay happens at time t_0 through the strong or electromagnetic interaction (i.e., preserving P parity), then the pair will be in a spin-singlet Bell state $|\Psi^-\rangle$. This fact follows from the general structure of Hamiltonians,

$$\begin{aligned} \mathcal{H}^{(PS)}(x) &= g\,\varphi(x)\,[\bar{f}(x)\,\gamma^5\,f(x)]_N, \\ \mathcal{H}^{(A)}(x) &= g'[\partial_\mu\varphi(x)]\,[\bar{f}(x)\,\gamma^\mu\gamma^5\,f(x)]_N, \end{aligned} \tag{3}$$

which can be compared with similar decays in quantum field theory (QFT). Here $\varphi(x)$ is the field of pseudoscalar particles, $\bar{f}(x)$ and $f(x)$ are fermionic fields, $\partial_\mu = \partial/\partial x^\mu$ is divergence, g and g' are effective coupling constants. Let us denote the antifermion with index "1" and the fermion with index "2." Let the spin projections of the fermion and the antifermion onto two directions exist simultaneously or jointly. Note that the directions in space are defined by nonparallel unit vectors $\vec{a}$ and $\vec{b}$, such that the spin projection operators do not commute. For brevity let us denote the spin $1/2$ projection of fermion i onto any axis, specified by unit vector $\vec{n}$, as

$$s_{\vec{n}}^{(i)} = \pm\tfrac{1}{2} \equiv n_{\pm}^{(i)},$$

where $i = \{1,2\}$. Then the spin projections at the initial time t_0 onto each of the directions in the state $|\Psi^-\rangle$ satisfy the anticorrelation condition

$$n_{\pm}^{(2)}(t_0) = -\,n_{\mp}^{(1)}(t_0). \tag{4}$$

Let us denote the spin projection operators of the fermion and antifermion onto direction $\vec{a}$ as $\hat{A}^{(2)}$ and $\hat{A}^{(1)}$ accordingly. Similarly $\hat{B}^{(2)}$ and $\hat{B}^{(1)}$ are the spin projection operators onto direction $\vec{b}$. As the vectors $\vec{a}$ and $\vec{b}$ are nonparallel

$$[\hat{A}^{(1)},\hat{B}^{(1)}] \neq 0, \quad [\hat{A}^{(2)},\hat{B}^{(2)}] \neq 0. \tag{5}$$

At the same time, according to Eberhard's theorem [17],

$$[\hat{A}^{(1)},\hat{B}^{(2)}] = 0, \quad [\hat{A}^{(2)},\hat{B}^{(1)}] = 0. \tag{6}$$

Equalities (6) ensure locality of the quantum theory (even nonrelativistic) at the level of macrodevices (so-called "nonsignaling conditions").

The commutation conditions (6) allow joint measurement for example of the projection of the fermion spin onto direction $\vec{a}$ and the projection of the antifermion spin onto direction $\vec{b}$. Hence the joint probability $w(a_\alpha^{(2)},\, b_\beta^{(1)},\, t)$ at any time is a well-defined value and it is possible to use for it probability theory based on Kolmogorov's axiomatics. Here $\{\alpha,\beta,\gamma\} = \{+,-\}$. Let us apply to this probability the concept of the elements of physics reality and the anticorrelation condition (4). Then for the time t_0 it is possible to formally introduce the joint probability

$$w\big(a_\alpha^{(2)},\, b_\beta^{(2)},\, t_0\big) \equiv w\big(a_\alpha^{(2)}, -b_{-\beta}^{(1)},\, t_0\big) \tag{7}$$

of the existence of physical characteristics of a microsystem (in our case—the projection of fermionic spins onto two nonparallel directions $\vec{a}$ and $\vec{b}$), corresponding to simultaneously nonmeasurable observables $A^{(2)}$ and $B^{(2)}$. So, despite condition (5), the definition of the element of physical reality allows us to give operational meaning to the joint probability $w(a_\alpha^{(2)},\, b_\beta^{(2)},\, t_0)$ and its analogs. I.e., formula (7) might be considered as a possible expansion of the definition of the joint probability concept to the area where Heisenberg's uncertainty principle prevents us from defining such a probability for direct measurements. It seems logical to assume that Bayes' theorem can be applied to probabilities like (7).

Using the concept of local realism, it is possible to derive not only Bell- or Wigner-like inequalities, but also equalities. Such equalities, often called Greenberger-Horne-Zeilinger (GHZ) equalities, were introduced in [18]. Proofs of GHZ equalities may be found in [19,20]. Using the concept of local realism and some additional assumptions, a system of equations has been obtained for distinct spin projections of three fermions in the GHZ state $|\Psi^{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}\,(|n_+^{(1)}\,n_+^{(2)}\,n_+^{(3)}\rangle - |n_-^{(1)}\,n_-^{(2)}\,n_-^{(3)}\rangle)$. This system is incompatible with calculations in nonrelativistic quantum mechanics.

Beginning in 1972 [21] there has been much experimental evidence of violation of Bell's and Wigner's inequalities, i.e., evidence of unsoundness of the hypothesis of local realism and/or the concept of elements of physical reality. However until recently these experiments were not free from some loopholes, which cast doubt on the connection between the violations of Bell's or Wigner's inequalities and the soundness of the quantum-mechanical description of the world. They are the locality loophole (or communication loophole), fair-sampling loophole (or detection loophole), and contextuality loophole.

Let us consider the first loophole using the example with fermion-antifermion pairs. In this case the locality loophole appears due to the fact that during the measurement of the spin projections of each of the particles, the space-time interval between them is timelike. Hence in the process of measurement it is not possible to exclude a hypothetical exchange of information between macrodevices, which enables a quasinonlocal strong correlation between the two measurements. Such a correlation may lead to the violation of Bell's or Wigner's inequalities. The "locality loophole" was overcome in experiments by Aspect [22] using the idea by Wheeler [23] of delayed choice for a pair of two-channel polarizers. In quantum optics this loophole is closed by "brute force" [24], when pairs of correlated photons are separated by a significant distance. For example in photon experiments with fiber cables this distance is greater than a hundred kilometers [25]. For spin-correlated fermions this distance is about 1.3 km [26]. When analyzing experiments [25,26] one should by definition take into account the limit of the speed of signal exchange between spatially separated subsystems of a correlated quantum system. Hence one should use QFT where the signal exchange speed is finite, instead of nonrelativistic quantum mechanics where this speed is formally infinite [27].

The second loophole arises due to the fact that all the detectors have certain registration efficiencies, and hence there is a gedanken possibility to select from the whole set the pairs of correlated particles which lead to the violation of Bell's or Wigner's inequalities, ignoring the others. For exclusion of the detection loophole in Bell's inequalities, one needs to have efficiency of the detector of the level $2/3$ [28]. In quantum optics this barrier was overcome only in 2013 [29,30]. For Wigner's inequalities the authors are not aware of any papers studying that efficiency value. The detection loophole should play an important role in particle physics, because typical efficiencies of detectors (taking into account selection criteria) do not exceed percents.

Finally we consider the "contextuality loophole" [31,32]. Let us define "context" as the aggregate of all the experimental conditions. Then, in order to obtain the values of correlators in Bell's inequalities or joint probabilities in Wigner's inequalities, it is necessary to conduct four or three independent experiments. There is no guarantee that the measurements are conducted under the same conditions. Even if one uses the same experimental devices, it is still impossible to repeat exactly all the internal parameters of macrodevices, because, for example, in each of experiments one should select different pairs of spin directions. I.e., it is impossible to make sure that quantum ensembles of correlated particles are identical in all the experiments. All the more, if the experiments were conducted in different and noncontrollable conditions, their results should not be summed up, subtracted, or compared with each other. If we suppose that every experiment has its own distribution of probabilities of the observable spectra, then it is possible to obtain a generalized Bell's inequality which is never violated in experiment [33].

One possible way to exclude the contextuality loophole is to conduct the measurements with an invariable state of the macrodevices, i.e., with only two nonparallel directions $\vec{a}$ and $\vec{b}$. One can use time as an additional degree of freedom. The same inequality, which potentially allows the experimenters to avoid the contextuality loophole, will be introduced in Sec. II of this paper.

Up to now, in each of the experiments for testing Bell's inequalities, it was possible to close only one or two loopholes [13,34]. However in 2015 three successful experiments were conducted (two with photon pairs [35,36], and one with correlated spins $s = 1/2$ [26]) which managed to avoid all three loopholes.

Also, as will be shown below, the experimental situation chosen to avoid the locality loophole [25,26,35,36] raises questions about applying these experiments to time-independent Bell's inequalities [7–9] and static Wigner's inequalities [15].

Among the rest of the loopholes, the most quoted in the literature is the freedom-of-choice loophole [37], where due to hidden interactions and unknown parameters an experiment itself causes the observer to select events with stronger correlations. In our opinion such a loophole is unfalsifiable and hence should not be considered scientifically according to Popper's refutability criterion [38].

Another loophole is so-called "memory loophole," which assumes that the preceding $n-1$ measurements may somehow (for example via some hidden variables) affect the consequent measurement [39,40]. Note that the memory loophole sometimes is mixed with the contextuality loophole in the literature, and hence the methods to avoid both of them might be similar. However in the current paper we adhere to the original definition from [39,40], where these loopholes are implicitly distinct. Note that experiments [26,35,36] seem to close this loophole.

Time-dependent generalizations of static Bell's and Wigner's inequalities may be justified in another way. The derivation of these inequalities strongly depends not only on the simultaneous existence of all of the physical characteristics of a microsystem, but also on the assumption of locality both on the macroscopic measurement level (i.e., Eberhard's theorem) and on the microscopic level (the hypothesis of local realism). Local realism contradicts the mathematical structure of nonrelativistic quantum mechanics (NRQM). Because of that, the violation of Bell's inequalities is often considered to be experimental proof of the nonlocality of quantum theory. However this is not true, as there are two inseparable potential causes of violation of the static Bell's and Wigner's inequalities: the absence of joint Kolmogorov's probabilities for observables, and nonlocality on the microscopic level. To exclude the second possibility, it is necessary to switch to calculations of probabilities and correlators in the framework of QFT, which is local on the microlevel by definition, for example because of Bogolyubov's principle of microcausality [41].

However in QFT it is not possible to use static Bell's or Wigner's inequalities, because it is not possible to exclude interactions of quantum fields with each other and with vacuum fluctuations [42]. I.e., in QFT any particle or system of particles is an open system. Also the description of an entangled quantum system at different space-time points should take into account relativistic effects, when the finite time of signal propagation between the two parts of the entangled microsystem is beyond the duration of macrodevice response. This is undoubtedly true for [26,35,36]. Hence to have the possibility of a theoretical description of locality loophole-free spatially separated experiments it is necessary to include time evolution into any Bell-like inequalities to assure their compatibility with the theory of relativity.

The Wigner's inequalities are more suited for relativistic generalization, because their intrinsic joint probabilities are well defined both in NRQM and QFT. The correlators in the Bell's inequalities are calculated from loop diagrams, part of which calculation depends on the renormalization procedure at each order of perturbation theory. It is not possible to get a definitive answer which does not depend on a renormalization technique [43].

This paper is a logical continuation of a series [27,44–46] in which we have studied possible relativistic corrections for static Wigner's inequalities, and then introduced inequalities that generalized static Wigner's inequalities for time-dependent ones required in QFT. The main goal of the papers above was testing Bohr's complementarity principle in the relativistic domain. The complementarity principle is directly related to the concept of realism. We believe that the statement of "testing the concept of realism" most correctly reflects the gist and the results of [44–46]. Also the connection between the violation of Leggett-Garg inequalities and the complementarity principle does not appear to be that obvious. In the current paper we will talk about realism, not complementarity, yet this approach is fully compatible with Kolmogorov's axiomatics of probability theory and with the concept of local realism; it is a particular realization of the supposition of the independent time evolution of every physical (micro)system. It would be nice to find a time evolution description that does not require the supposition of independence. In Sec. II of the current paper we will propose such a description, which may be obtained in the framework of Kolmogorov's probability theory using a Bayesian approach and operates only with (conditional) probabilities.

In addition to the works cited above, there are many proposals to test Bell's inequalities in particle physics. Most of these tests use oscillations of neutral pseudoscalar mesons. They were discussed in the work [45]. Note also the analysis of the test of static Bell's inequalities in neutrino oscillations [47,48].

Time-dependent inequalities were proposed by Leggett and Garg in their pioneering work [49]. The initial goal of this work was for testing whether quantum mechanics may be applied at the macroscopic scale for many-particle quantum systems in a coherent state. The inequality, which is satisfied by two-times correlators of one observable, assumes that this observable obeys the laws of classical physics (this is the concept of "macroscopic realism per se." The expression of the Leggett-Garg inequalities is similar to the Bell's inequalities in form [9]. Because of that the Leggett-Garg inequalities are sometimes called "temporal Bell inequalities" [50]. This name is not precise, because Bell's inequalities involve correlators of various observables at one moment of time, while the Leggett-Garg inequalities should involve different-time correlators of one observable. The name "time-dependent Bell's inequalities" should be attributed to inequalities that contain probabilities or correlators of various observables at different moments of time. Following this logic, the "time-dependent Wigner's inequalities" were introduced [44]. References to all the key works related to Leggett-Garg inequalities may be found in the review [51].

In 2015 a successful experimental test of violation of the Leggett-Garg inequalities were conducted: first in an experiment with nonclassical movement of a massive quantum particle over a lattice [52], and then in a violation of quantum coherence in macroscopic crystals [53].

The Leggett-Garg inequalities may also be used to test for the existence of joint probabilities of observables which have noncommuting operators in NRQM, i.e., to test the hypothesis of realism. Let us note that in the recent publication [54] authors contested the well-known statement that the Leggett-Garg inequalities might be applicable to test the principle of macroscopic realism. But their applicability to test the principle of (local) realism is not disputed. The Leggett-Garg inequalities may be reproduced if one supposes that some hidden parameters λ whose probability density $\rho(\lambda)$ depends on time in Markov's way, exist in a quantum system [55]. Such hidden parameters automatically lead to non-negative joint probabilities. While a test of macroscopic coherence requires soft measurements, a test of the existence of the joint probabilities (i.e., the "hypothesis of realism" without the term "macroscopic") requires the use of sets of parallel measurements, each conducted at only two fixed moments of time [56]. This approach is related to the methodology of the measurement of four distinct correlators in Bell's inequalities, so one might expect that this approach would not be free of the contextuality loophole. The use of projection measurements and sets of parallel experiments for testing the hypothesis of realism in particle physics is considered in [57], where entangled pairs of pseudoscalar mesons are used. In 2016 the neutrino experiment MINOS reported a test of the Leggett-Garg inequalities in neutrino oscillations [58].

The current paper considers some generic experimental situations where one measures some properties of a physical system, then constructs a Kolmogorov's space of elementary outcomes and introduces into this space events, corresponding to these experimental situations; we write some relations in terms of conditional probabilities, because they are well defined in both classical and quantum physics (in contrast to joint probabilities), and finally we show that corollaries of Bayes' theorem are violated for correlated quantum systems.

There are many works dedicated to studies of interconnections between the conditional probabilities in quantum and classical theories, starting from fundamental monography by von Neumann [59] and a paper by Lüders [60], where rules for calculating conditional probabilities were introduced. Important generalizations of the notion of the conditional probability on a generalized probability space of quantum mechanics were presented in [61,62]. In the current paper the calculation of conditional probabilities will be based on [61,62]. Generalizations of Lüders' rule for non-Hermitian projection operators for entangled and open quantum systems were proposed in [63,64]. Based on this generalization, a quantum-Bayesian interpretation of quantum mechanics was developed (so-called QBism) [65,66], which is now, we believe, one of the most elegant interpretations of quantum theory. It provides a unified approach to classical and quantum phenomena. QBism is criticized in [67].

A versatile analysis of quantum conditional probability and its relation to classics may be found in [68]. The main conclusion of these works is that von Neumann's formula can not be considered a good generalization of classical conditional probability for quantum phenomena, however there is no doubt in Lüder's rule [60] and its extension [61,62].

This paper consists of the following sections. In Sec. II, using Kolmogorov's approach, conditional probabilities, and Bayes' theorem, we obtain a static equality and a time-dependent inequality, which allow us to test the hypothesis of realism in time-dependent and open quantum systems. In Sec. III we use an example of correlated spin-1/2 particles in an external stationary and homogeneous magnetic field to demonstrate that in the framework of NRQM for open quantum systems the relations obtained in Sec. II are violated. Section IV is devoted to the study of corollaries to violation of Bayes' theorem for systems of pseudoscalar neutral mesons. Some experiments for testing the concept of realism are proposed. They might be applicable to experiments at the Large Hadron Collider [69–72] and Belle II [73]. Appendices A–D contain all the auxiliary formulas that are required for derivation of the results of Secs. III and IV.

II. TESTING THE REALISM HYPOTHESIS USING BAYES' THEOREM

Consider conditional probabilities in classical and quantum theories using the observables A and B. In contrast to joint probabilities (2), conditional probabilities like $w(b_{\beta'}|a_{\alpha'})$ are well defined in both classical and quantum theories.

In the classical case the probability to measure value $b_{\beta'}$ of the spectrum of observable B assuming that value $a_{\alpha'}$ of observable A was measured can be written as follows:

$$w(b_{\beta'}|a_{\alpha'}) = \frac{w(b_{\beta'} \cap a_{\alpha'})}{w(a_{\alpha'})}. \tag{8}$$

As was noted above, the joint probability $w(b_{\beta'} \cap a_{\alpha'}) = w(a_{\alpha'} \cap b_{\beta'})$ always exists. Value $w(a_{\alpha'}) \neq 0$ is the probability to measure value $a_{\alpha'}$ from the spectrum of observable A. From (8) a simplest case of Bayes' theorem can be derived:

$$w(b_{\beta'}|a_{\alpha'})\, w(a_{\alpha'}) = w(a_{\alpha'}|b_{\beta'})\, w(b_{\beta'}), \tag{9}$$

where $w(b_{\beta'}) \neq 0$: this is the probability to measure value $b_{\beta'}$ from the spectrum of observable B.

In the quantum case for the calculation of the conditional probability, one should use von Neumann's formula [59],

$$w(b_{\beta'}|a_{\alpha'}) = \frac{\mathrm{Tr}\big(\hat{P}^{(B)}_{\beta'}\, \hat{P}^{(A)}_{\alpha'}\, \hat{\rho}_0\, \hat{P}^{(A)}_{\alpha'}\, \hat{P}^{(B)}_{\beta'}\big)}{\mathrm{Tr}\big(\hat{P}^{(A)}_{\alpha'}\, \hat{\rho}_0\big)}, \tag{10}$$

where $\hat{\rho}_0$ is the density matrix of the quantum system in the initial state, $\hat{P}^{(A)}_{\alpha'}$ is the projector onto the state related to the value $a_{\alpha'}$ of the spectrum of the observable A, and $\hat{P}^{(B)}_{\beta'}$ is the analogous projector for the value $b_{\beta'}$ of the spectrum of the observable B. Applicability of formula (10) does not require commutation of $\hat{A}$ and $\hat{B}$. It does not matter whether the state is entangled or not (and hence to which subsystems the observables A and B correspond). It also does not matter whether the quantum system is open or not. It is obvious that if $[\hat{A},\hat{B}] \neq 0$, then

$$\mathrm{Tr}\big(\hat{P}^{(B)}_{\beta'}\, \hat{P}^{(A)}_{\alpha'}\, \hat{\rho}_0\, \hat{P}^{(A)}_{\alpha'}\, \hat{P}^{(B)}_{\beta'}\big) \neq \mathrm{Tr}\big(\hat{P}^{(A)}_{\alpha'}\, \hat{P}^{(B)}_{\beta'}\, \hat{\rho}_0\, \hat{P}^{(B)}_{\beta'}\, \hat{P}^{(A)}_{\alpha'}\big).$$

Hence in the quantum case it is not always possible to obtain an analog of Bayes' theorem (9). Moreover in the framework of QFT and for open quantum systems, field operators and observable operators, which consist of fields operators, do not commute at distinct moments of time, i.e., $[\hat{A}(t_1),\, \hat{A}(t_2)] \neq 0$. Because of that the time can be treated as an additional degree of freedom together with spatial directions $\vec{a}$, $\vec{b}$ and so on (if we are talking about spins, for example).

Although we will refer to spin projections onto various directions in space, the static equality and time-dependent inequality obtained below are true for any set of dichotomic observables.

Select three space directions, which are defined by nonparallel unit vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. Let the system of antifermion "1" and fermion "2" be in a singlet spin state at time t_0. Suppose that the concept of realism is true, i.e., spin projections $a^{(i)}_{\pm}$, $b^{(i)}_{\pm}$, and $c^{(i)}_{\pm}$ of the antifermion and fermion onto all three directions exist simultaneously at any time t, despite the fact that they cannot be measured by any macrodevice. At the time t_0 these projections obey the anticorrelation condition (4).

For this hypothetical situation it is easy to introduce a classical probability model based on Kolmogorov's axiomatics. Let us define space Ω of elementary outcomes ω_k. Each of them is one of the possible sets $\{a^{(1)}_{\alpha} b^{(1)}_{\beta} c^{(1)}_{\gamma}\, a^{(2)}_{\alpha'} b^{(2)}_{\beta'} c^{(2)}_{\gamma'}\}$ of spin projections onto all three chosen directions $\vec{a}$, $\vec{b}$, and $\vec{c}$, where indices $\{\alpha,\alpha',\beta,\beta',\gamma,\gamma'\} = \{+,-\}$. The set of the elements of the space Ω does not depend on time.

Denote "elementary event" $\mathcal{K}_{a^{(1)}_{\alpha} b^{(1)}_{\beta} c^{(1)}_{\gamma}\, a^{(2)}_{\alpha'} b^{(2)}_{\beta'} c^{(2)}_{\gamma'}}$ as a subset of all elementary outcomes ω_k of the set Ω (i.e., $\mathcal{K}_{a^{(1)}_{\alpha} b^{(1)}_{\beta} c^{(1)}_{\gamma}\, a^{(2)}_{\alpha'} b^{(2)}_{\beta'} c^{(2)}_{\gamma'}} \subseteq \Omega$ and $\omega_k \in \mathcal{K}_{a^{(1)}_{\alpha} b^{(1)}_{\beta} c^{(1)}_{\gamma}\, a^{(2)}_{\alpha'} b^{(2)}_{\beta'} c^{(2)}_{\gamma'}}$). Realization of any of these events gives rise to an element of physical reality—a concrete set of spin projections $\{a^{(1)}_{\alpha} b^{(1)}_{\beta} c^{(1)}_{\gamma}\, a^{(2)}_{\alpha'} b^{(2)}_{\beta'} c^{(2)}_{\gamma'}\}$. The aggregate of the considered events forms an algebra (σ algebra) $\mathcal{F}$. More complicated events may be constructed by merging elementary events. It is possible to introduce a probability measure w on $(\Omega,\mathcal{F})$, which is always real and non-negative. It is additive (σ additive) for nonintersecting events. Using this measure we can define probabilities of joint and conditional events on Ω.

In order to derive a static equality let us consider three events $\mathcal{S}_1(t_0) = \{a^{(2)}_{+}, b^{(1)}_{+}\}$, $\mathcal{S}_2(t_0) = \{c^{(2)}_{+}, b^{(1)}_{+}\}$, and event $\mathcal{S}_3(t_0)$, when the fermion-antifermion pair is in the spin singlet state at t_0. Events $\mathcal{S}_1(t_0)$ and $\mathcal{S}_2(t_0)$ can be easily constructed in Ω using the elementary events and condition (4):

$$\begin{aligned}
\mathcal{S}_1(t_0) &= \mathcal{K}_{a^{(1)}_{-} b^{(1)}_{+} c^{(1)}_{+}\, a^{(2)}_{+} b^{(2)}_{-} c^{(2)}_{-}}(t_0) \cup \mathcal{K}_{a^{(1)}_{-} b^{(1)}_{+} c^{(1)}_{-}\, a^{(2)}_{+} b^{(2)}_{-} c^{(2)}_{+}}(t_0),\\
\mathcal{S}_2(t_0) &= \mathcal{K}_{a^{(1)}_{-} b^{(1)}_{+} c^{(1)}_{-}\, a^{(2)}_{+} b^{(2)}_{-} c^{(2)}_{+}}(t_0) \cup \mathcal{K}_{a^{(1)}_{+} b^{(1)}_{+} c^{(1)}_{-}\, a^{(2)}_{-} b^{(2)}_{-} c^{(2)}_{+}}(t_0).
\end{aligned}$$

In Ω space, event $\mathcal{S}_3$ is defined as follows:

$$\begin{aligned}
\mathcal{S}_3 = \{&(a^{(2)}_{+}, a^{(1)}_{-} \cup a^{(2)}_{-}, a^{(1)}_{+}) \cup (b^{(2)}_{+}, b^{(1)}_{-} \cup b^{(2)}_{-}, b^{(1)}_{+})\\
&\cup (c^{(2)}_{+}, c^{(1)}_{-} \cup c^{(2)}_{-}, c^{(1)}_{+})\}.
\end{aligned} \tag{11}$$

This notation corresponds to the classical approach, which in this case is identical to the concept of local realism, and in essence differs from a description of event $\mathcal{S}_3(t_0)$ in NRQM using a maximally entangled Bell state,

$$|\Psi^-(t_0)\rangle = \frac{1}{\sqrt{2}}\left(|n_+^{(2)}\rangle|n_-^{(1)}\rangle - |n_-^{(2)}\rangle|n_+^{(1)}\rangle\right), \qquad (12)$$

where $\vec{n}$ is any of the directions $\vec{a}$, $\vec{b}$, or $\vec{c}$. In QFT the initial state is defined using a Hamiltonian (3), which creates a corresponding entangled state when calculating the evolution operator matrix element.

If the concept of realism is true, then we can consider non-negative joint and conditional probabilities for the events $\mathcal{S}_1(t_0)$, $\mathcal{S}_2(t_0)$, and $\mathcal{S}_3(t_0)$. It is possible to apply to them a multiplication theorem on $(\Omega, \mathcal{F})$. Then

$$\begin{aligned} w(\mathcal{S}_1 \cap \mathcal{S}_2|\mathcal{S}_3) &= \frac{w(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3)}{w(\mathcal{S}_3)} = \frac{w(\mathcal{S}_3 \cap \mathcal{S}_1 \cap \mathcal{S}_2)}{w(\mathcal{S}_3)} \\ &= \frac{w(\mathcal{S}_3)\, w(\mathcal{S}_1|\mathcal{S}_3)\, w(\mathcal{S}_2|\mathcal{S}_1 \cap \mathcal{S}_3)}{w(\mathcal{S}_3)} \\ &= w(\mathcal{S}_1|\mathcal{S}_3)\, w(\mathcal{S}_2|\mathcal{S}_1 \cap \mathcal{S}_3). \end{aligned}$$

In analogy

$$w(\mathcal{S}_1 \cap \mathcal{S}_2|\mathcal{S}_3) = w(\mathcal{S}_2|\mathcal{S}_3)\, w(\mathcal{S}_1|\mathcal{S}_2 \cap \mathcal{S}_3).$$

Equating these results with each other, we obtain the following variant of Bayes' theorem:

$$w(\mathcal{S}_1|\mathcal{S}_3)\, w(\mathcal{S}_2|\mathcal{S}_1 \cap \mathcal{S}_3) = w(\mathcal{S}_2|\mathcal{S}_3)\, w(\mathcal{S}_1|\mathcal{S}_2 \cap \mathcal{S}_3). \qquad (13)$$

Experiments that can test the hypothesis of realism using the static equality (13) are fully identical to those that test static Bell's [7–9] and Wigner's [15] inequalities. However it is easier to check the violation of (13) than the violation of Bell's inequalities. Also, (13) has an advantage over the GHZ equality [18–20], because it allows the experimenter to check in a system of only two (entangled) subsystems, while an experimental check of violation of the GHZ equality requires at least three subsystems in an entangled state. The last condition makes it almost impossible to study the GHZ equality in particle physics.

The static equality (13) is one of two main results of this paper. In Secs. III and IV it will be shown that this equality is violated in the framework of quantum theory.

We now derive a time-dependent inequality which follows from the hypothesis of realism and Bayes' theorem. In [44–46] the authors have suggested a variant of the time-dependent Wigner's inequality, based on Kolmogorov's axiomatics and an assumption about the statistical independence of processes of evolution of dichotomic variables of each of two subsystems, which satisfy the condition (4) at the time t_0 (and only at that time). For $t_1 \neq t_2$ this inequality may be written as follows:

$$\begin{aligned} w(a_+^{(2)}(t_2) \cap b_+^{(1)}(t_1)) \leqslant\ & w(a_+^{(2)}(t_0) \to a_+^{(2)}(t_2))\,[w(b_+^{(1)}(t_0) \to b_+^{(1)}(t_1)) + w(b_-^{(1)}(t_0) \to b_+^{(1)}(t_1))]\, w(a_+^{(2)}(t_0) \cap c_+^{(1)}(t_0)) \\ &+ w(a_-^{(2)}(t_0) \to a_+^{(2)}(t_2))\,[w(b_+^{(1)}(t_0) \to b_+^{(1)}(t_1)) + w(b_-^{(1)}(t_0) \to b_+^{(1)}(t_1))]\, w(a_-^{(2)}(t_0) \cap c_+^{(1)}(t_0)) \\ &+ w(b_+^{(1)}(t_0) \to b_+^{(1)}(t_1))\,[w(a_+^{(2)}(t_0) \to a_+^{(2)}(t_2)) + w(a_-^{(2)}(t_0) \to a_+^{(2)}(t_2))]\, w(c_+^{(2)}(t_0) \cap b_+^{(1)}(t_0)) \\ &+ w(b_-^{(1)}(t_0) \to b_+^{(1)}(t_1))\,[w(a_+^{(2)}(t_0) \to a_+^{(2)}(t_2)) + w(a_-^{(2)}(t_0) \to a_+^{(2)}(t_2))]\, w(c_+^{(2)}(t_0) \cap b_-^{(1)}(t_0)), \end{aligned} \qquad (14)$$

where the dichotomic variable of the first subsystem is measured at the time $t_1 > t_0$, while the dichotomic variable of the second subsystem is measured at the time $t_2 > t_0$. Here the following notation is used: $w(n_\pm^{(i)}(t_0) \to n_\pm^{(i)}(t_{1,2}))$ is the probability of spin projection of particle i onto $\vec{n}$ at $t_{1,2}$ to be $\pm 1/2$ if at t_0 the same projection to the same $\vec{n}$ was $\pm 1/2$; $w(a_+^{(2)}(t_2) \cap b_+^{(1)}(t_1))$ is the joint probability of the projection of antifermion spin to $\vec{b}$ to be $+1/2$ at t_1, and the projection of fermion spin to $\vec{a}$ is $+1/2$ at t_2. Although the assumption of statistical independence of the evolution of classical observables is almost obvious in the framework of the hypothesis of local realism, it is quite hard to prove in some cases. For this reason we would like to write an inequality in which the time evolution is a consequence of a more common property of classical objects rather than the property of statistical independence, which is used in the derivation of (14). That more common property might be Bayes' theorem.

Consider two moments in time: the initial t_0, and some $t > t_0$. The anticorrelation condition (4) is supposed to be true only at the time t_0. At any other moment of time it might not be satisfied. As the space of elementary outcomes Ω does not depend on time, it is possible to select the following events: the event $\mathcal{S}_1(t_0) = \{a_+^{(2)}, b_+^{(1)}, t_0\}$, the event $S_2(t) = \{a_{\alpha'}^{(2)}, b_{\beta'}^{(1)}, t\}$, where $\{\alpha', \beta'\} = \{+, -\}$, and the event S_3 for which at time t_0 the fermion-antifermion pair were in a singlet spin state. Under the hypothesis of realism we again use Bayes' theorem, but now for the two moments of time,

$$\begin{aligned} &w(\mathcal{S}_1(t_0)|\mathcal{S}_3(t_0))\, w(\mathcal{S}_2(t)|\mathcal{S}_1(t_0) \cap \mathcal{S}_3(t_0)) \\ &\quad = w(\mathcal{S}_2(t)|\mathcal{S}_3(t_0))\, w(\mathcal{S}_1(t_0)|\mathcal{S}_2(t) \cap \mathcal{S}_3(t_0)). \end{aligned}$$

In this formula the conditional probability $w(\mathcal{S}_1(t_0)|\mathcal{S}_2(t) \cap \mathcal{S}_3(t_0))$ is badly defined mathematically in both NRQM and QFT. However if we suppose that the hypothesis of realism is true then this conditional probability must satisfy the following conditions:

$$0 \leqslant w(\mathcal{S}_1(t_0)|\mathcal{S}_2(t) \cap \mathcal{S}_3(t_0)) \leqslant 1.$$

We obtain the time-dependent inequality

$$w(\mathcal{S}_1(t_0)|\mathcal{S}_3(t_0))\, w(\mathcal{S}_2(t)|\mathcal{S}_1(t_0) \cap \mathcal{S}_3(t_0)) \leqslant w(\mathcal{S}_2(t)|\mathcal{S}_3(t_0)). \qquad (15)$$

The time-dependent inequality (15) is the second main result of the paper. Only two directions, $\vec{a}$ and $\vec{b}$, were used in the derivation of this inequality, not three or more as in Bell's and Wigner's inequalities. Potentially that allows the experimenter to test inequality (15) using only one series of experiments and thus evade the contextuality loophole.

It is suitable to write inequality (15) using the spin-1/2 projections onto directions $\vec{a}$ and $\vec{b}$:

$$w(\{a_+^{(2)},b_+^{(1)},t_0\}|\mathcal{S}_3(t_0))\, w(\{a_{\alpha'}^{(2)},b_{\beta'}^{(1)},t\}|\{a_+^{(2)},b_+^{(1)},t_0\}\cap\mathcal{S}_3(t_0)) \leqslant w(\{a_{n'}^{(2)},b_{m'}^{(1)},t\}|\mathcal{S}_3(t_0)), \quad (16)$$

where, let us emphasize again, we do not suppose any specific dependence of the observables on time, and the event $\mathcal{S}_3(t_0)$ may correspond, in principle, to any initial condition of a system, not only the condition which satisfies (4). Event $\mathcal{S}_1(t_0)$ also might be selected in a generic way as $\mathcal{S}_1(t_0) = \{a_{\alpha''}^{(2)},b_{\beta''}^{(1)},t_0\}$; however that generalization does not lead to any new types of violation of the inequality (16).

III. COROLLARIES OF BAYES' THEOREM AND ANTICORRELATED SPINS IN AN EXTERNAL MAGNETIC FIELD

We first show that the static equation (13) is violated in NRQM if we consider a positron and electron in a spin singlet Bell state. The density matrix corresponding to this state is $\hat{\rho}_0 = |\Psi^-(t_0)\rangle\langle\Psi^-(t_0)|$. For tests of static inequality the time is not important, so let $t_0 = 0$. Main formulas required for derivation of conditional probabilities in static equation (13) are given in Appendix A. State vector $|\Psi^-(t_0)\rangle$ is defined as (A2).

In plane (x,z), define three directions $\vec{a}$, $\vec{b}$, and $\vec{c}$. Projectors onto the events $\mathcal{S}_1(t_0) = \{a_+^{(2)},b_+^{(1)}\}$, $\mathcal{S}_2(t_0) = \{c_+^{(2)},b_+^{(1)}\}$, and spin singlet $\mathcal{S}_3(t_0)$ are

$$\hat{P}_{\mathcal{S}_1} = |a_+^{(2)}\rangle|b_+^{(1)}\rangle\langle b_+^{(1)}|\langle a_+^{(2)}|,$$
$$\hat{P}_{\mathcal{S}_2} = |c_+^{(2)}\rangle|b_+^{(1)}\rangle\langle b_+^{(1)}|\langle c_+^{(2)}|, \quad \text{and}$$
$$\hat{P}_{\mathcal{S}_3} = \hat{\rho}_0. \quad (17)$$

Using (17), (A1), and (10) we obtain

$$w(\mathcal{S}_1|\mathcal{S}_3) = |\langle\Psi^-(t_0)|a_+^{(2)}\rangle|b_+^{(1)}\rangle|^2 = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2},$$
$$w(\mathcal{S}_2|\mathcal{S}_3) = |\langle\Psi^-(t_0)|c_+^{(2)}\rangle|b_+^{(1)}\rangle|^2 = \frac{1}{2}\sin^2\frac{\theta_{bc}}{2}, \quad (18)$$

where $\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$.

For calculation of the conditional probabilities $w(\mathcal{S}_2|\mathcal{S}_1\cap\mathcal{S}_3)$ and $w(\mathcal{S}_1|\mathcal{S}_2\cap\mathcal{S}_3)$ it is necessary to define projectors onto states $|\Psi_{\mathcal{S}_1\cap\mathcal{S}_3}\rangle$ and $|\Psi_{\mathcal{S}_2\cap\mathcal{S}_3}\rangle$. In the general case such a procedure might be nontrivial. However the isotropy of Bell state $|\Psi^-(t_0)\rangle$ allows us to obtain a simple calculation algorithm. Let us find for example $|\Psi_{\mathcal{S}_1\cap\mathcal{S}_3}\rangle$. Rewrite (12) in terms of the projection onto the direction $\vec{a}$:

$$|\Psi^-(t_0)\rangle = \frac{1}{\sqrt{2}}(|a_+^{(2)}\rangle|a_-^{(1)}\rangle - |a_-^{(2)}\rangle|a_+^{(1)}\rangle)$$
$$= \frac{1}{\sqrt{2}}\left(|a_+^{(2)}\rangle\left[-\sin\frac{\theta_{ab}}{2}|b_+^{(1)}\rangle + \cos\frac{\theta_{ab}}{2}|b_-^{(1)}\rangle\right] - |a_-^{(2)}\rangle|a_+^{(1)}\rangle\right)$$
$$= \cdots + |\Psi_{\mathcal{S}_1}\rangle + \cdots. \quad (19)$$

From this, according to the superposition principle and the results of [61,62], we obtain the non-normalized state vector of event $\mathcal{S}_1\cap\mathcal{S}_3$:

$$|\Psi_{\mathcal{S}_1\cap\mathcal{S}_3}\rangle = -\frac{1}{\sqrt{2}}\sin\frac{\theta_{ab}}{2}|a_+^{(2)}\rangle|b_+^{(1)}\rangle = -\frac{1}{\sqrt{2}}\sin\frac{\theta_{ab}}{2}|\Psi_{\mathcal{S}_1}\rangle.$$

Hence

$$\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3} = |\Psi_{\mathcal{S}_1\cap\mathcal{S}_3}\rangle\langle\Psi_{\mathcal{S}_1\cap\mathcal{S}_3}| = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}\hat{P}_{\mathcal{S}_1}.$$

In analogy

$$\hat{P}_{\mathcal{S}_2\cap\mathcal{S}_3} = \frac{1}{2}\sin^2\frac{\theta_{bc}}{2}\hat{P}_{\mathcal{S}_2}.$$

Using the von Neumann rule (10), we obtain the following result. If the e^+e^- pair is in a spin singlet state,

$$w(\mathcal{S}_2|\mathcal{S}_1\cap\mathcal{S}_3) = \frac{\mathrm{Tr}(\hat{P}_{\mathcal{S}_2}\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}\hat{\rho}_0\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}\hat{P}_{\mathcal{S}_2})}{\mathrm{Tr}(\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}\hat{\rho}_0\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3})}$$
$$= \frac{\mathrm{Tr}(\hat{P}_{\mathcal{S}_2}\hat{P}_{\mathcal{S}_1}\hat{\rho}_0\hat{P}_{\mathcal{S}_1}\hat{P}_{\mathcal{S}_2})}{\mathrm{Tr}(\hat{P}_{\mathcal{S}_1}\hat{\rho}_0\hat{P}_{\mathcal{S}_1})}$$
$$= \frac{\mathrm{Tr}(\hat{P}_{\mathcal{S}_2}\hat{P}_{\mathcal{S}_1}\hat{P}_{\mathcal{S}_2})}{\mathrm{Tr}(\hat{P}_{\mathcal{S}_1})} = \mathrm{Tr}(\hat{P}_{\mathcal{S}_1}\hat{P}_{\mathcal{S}_2})$$
$$= w(\mathcal{S}_1|\mathcal{S}_2\cap\mathcal{S}_3). \quad (20)$$

We used the fact that for a Bell state $\hat{\rho}_0$,

$$\hat{P}_{\mathcal{S}_1}\hat{\rho}_0\hat{P}_{\mathcal{S}_1} \sim \hat{P}_{\mathcal{S}_1} \quad \text{and} \quad \hat{P}_{\mathcal{S}_2}\hat{\rho}_0\hat{P}_{\mathcal{S}_2} \sim \hat{P}_{\mathcal{S}_2}. \quad (21)$$

The equality of the conditional probabilities (20) is not general and is only related to the special choice of the initial state $|\Psi^-(t_0)\rangle$.

Substituting (18) and (20) into (13), if NRQM is compatible with the hypothesis of local realism (and with probability theory in Kolmogorov's axiomatics), then for any three directions $\vec{a}$, $\vec{b}$, and $\vec{c}$ in plane (x,z), the following equation should be always satisfied:

$$\sin^2\frac{\theta_{ab}}{2} = \sin^2\frac{\theta_{bc}}{2}. \quad (22)$$

Obviously this is not true. If the vector $\vec{a}$ is perpendicular to the vector $\vec{b}$, while the vector $\vec{c}$ is the bisector of the angle between $\vec{a}$ and $\vec{b}$, then (22) is violated.

We now show that the time-dependent inequality (15) may also be violated in NRQM. Again consider an e^+e^- pair, which at the time $t_o = 0$ is described by the density matrix $\hat{\rho}_0 = |\Psi^-(t_0)\rangle\langle\Psi^-(t_0)|$. Put the system into an external constant and homogeneous magnetic field $\vec{\mathcal{H}}$ aligned along the y axis. From all possible decays, select only those where the leptons are propagated along the magnetic field. This is assumed for simplification of calculation of probabilities. Choose two space directions $\vec{a}$ and $\vec{b}$ lying in the plane (x,z); then it is more suitable to test the violation of (16) than of (15). Projectors onto the events $\mathcal{S}_1(t_0) = \{a_+^{(2)},b_+^{(1)}\}$, and $\mathcal{S}_3(t_0)$ may be written as

$$\hat{P}_{\mathcal{S}_1} = |a_+^{(2)}\rangle|b_+^{(1)}\rangle\langle b_+^{(1)}|\langle a_+^{(2)}|, \quad \text{and} \quad \hat{P}_{\mathcal{S}_3} = \hat{\rho}_0, \quad (23)$$

The expression for the conditional probability $w(\mathcal{S}_1(t_0)|\mathcal{S}_3(t_0)) = w(\{a_+^{(2)},b_+^{(1)},t_0\}|\mathcal{S}_3(t_0))$ is calculated in (18). In order to obtain two other conditional probabilities, which are included in the formulas (15) and (16), it is necessary to consider four cases for different values of α' and β' for the event $\mathcal{S}_2(t)$.

(a) Let at time $t > t_0$ the indices $\alpha' = +$ and $\beta' = +$, i.e., $\mathcal{S}_2(t) = \{a_+^{(2)}, b_+^{(1)}\}$. Considering that $\hat{\rho}_0^2 = \hat{\rho}_0$ and that $\mathrm{Tr}\hat{\rho}_0 = 1$, using the von Neumann rule (10), we find that

$$\begin{aligned} w(\mathcal{S}_2(t)|\mathcal{S}_3(t_0)) &= w(\{a_+^{(2)}, b_+^{(1)}, t\}|\mathcal{S}_3(t_0)) \\ &= \frac{\mathrm{Tr}\big(\hat{P}_{\mathcal{S}_2}\hat{U}(t,t_0)\hat{P}_{\mathcal{S}_3}\hat{\rho}_0\hat{P}_{\mathcal{S}_3}\hat{U}^\dagger(t,t_0)\hat{P}_{\mathcal{S}_2}\big)}{\mathrm{Tr}\big(\hat{P}_{\mathcal{S}_3}\hat{\rho}_0\hat{P}_{\mathcal{S}_3}\big)} \\ &= \mathrm{Tr}\big[\hat{P}_{\mathcal{S}_2}\hat{U}(t,t_0)\hat{\rho}_0\hat{U}^\dagger(t,t_0)\hat{P}_{\mathcal{S}_2}\big] \\ &= |\langle\Psi^-(t)|a_+^{(2)}\rangle|b_+^{(1)}\rangle|^2. \end{aligned}$$

The square of the modulus of the corresponding matrix element is calculated using expressions (A3)–(A5). Finally,

$$\begin{aligned} w(\mathcal{S}_2(t)|\mathcal{S}_3(t_0)) &= w(\{a_+^{(2)}, b_+^{(1)}, t\}|\mathcal{S}_3(t_0)) \\ &= \frac{1}{2}\sin^2\left(\frac{\theta_{ba}}{2} + 2\omega t\right). \end{aligned} \tag{24}$$

Using the von Neumann rule, property (21), and formulas (A3) and (A4), one can obtain

$$\begin{aligned} &w(\mathcal{S}_2(t)|\mathcal{S}_1(t_0) \cap \mathcal{S}_3(t_0)) \\ &= w(\{a_+^{(2)}, b_+^{(1)}, t\}|\{a_+^{(2)}, b_+^{(1)}, t_0\} \cap \mathcal{S}_3(t_0)) \\ &= \frac{\mathrm{Tr}\big(\hat{P}_{\mathcal{S}_2}\hat{U}(t,t_0)\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}\hat{\rho}_0\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}\hat{U}^\dagger(t,t_0)\hat{P}_{\mathcal{S}_2}\big)}{\mathrm{Tr}\big(\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}\hat{\rho}_0\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}\big)} \\ &= \frac{\mathrm{Tr}\big(\hat{P}_{\mathcal{S}_2}\hat{U}(t,t_0)\hat{P}_{\mathcal{S}_1}\hat{\rho}_0\hat{P}_{\mathcal{S}_1}\hat{U}^\dagger(t,t_0)\hat{P}_{\mathcal{S}_2}\big)}{\mathrm{Tr}\big(\hat{P}_{\mathcal{S}_1}\hat{\rho}_0\hat{P}_{\mathcal{S}_1}\big)} \\ &= \frac{\mathrm{Tr}\big(\hat{P}_{\mathcal{S}_2}\hat{U}(t,t_0)\hat{P}_{\mathcal{S}_1}\hat{U}^\dagger(t,t_0)\hat{P}_{\mathcal{S}_2}\big)}{\mathrm{Tr}\big(\hat{P}_{\mathcal{S}_1}\big)} \\ &= |\langle a_+^{(2)}|a_+^{(2)}(t)\rangle|^2|\langle b_+^{(1)}|b_+^{(1)}(t)\rangle|^2 = \cos^4(\omega t). \end{aligned} \tag{25}$$

We have used the standard properties of projection operators: $\hat{P}_{\mathcal{S}_i}^2 = \hat{P}_{\mathcal{S}_i}$ and $\mathrm{Tr}\hat{P}_{\mathcal{S}_i} = 1$, where $i = \{1,2\}$.

Combine results (18), (24), and (25), and substitute them into the inequality (16), to get

$$\sin^2\left(\frac{\theta_{ba}}{2}\right)\cos^4(\omega t) \leqslant \sin^2\left(\frac{\theta_{ba}}{2} + 2\omega t\right). \tag{26}$$

If the concept of realism is true then the inequality (26) should never be violated. However if we choose t such that $\omega t = -\theta_{ba}/4$, then (26) becomes

$$\sin^2\left(\frac{\theta_{ba}}{2}\right)\cos^4\left(\frac{\theta_{ba}}{4}\right) \leqslant 0, \tag{27}$$

which is violated for most angles θ_{ba}.

The inequality (26) may be tested selecting events from one experiment without changing the internal state of a macrodevice, to avoid the contextuality loophole. However the detection loophole in this case might still be open.

(b) Let at the time $t > t_0$ the indices $\alpha' = -$ and $\beta' = -$, i.e., let us consider the event $\mathcal{S}_2(t) = \{a_-^{(2)}, b_-^{(1)}\}$. Performing calculations analogous to the above, we find the following values for the conditional probabilities:

$$\begin{aligned} &w(\mathcal{S}_2(t)|\mathcal{S}_3(t_0)) \\ &= w(\{a_-^{(2)}, b_-^{(1)}, t\}|\mathcal{S}_3(t_0)) \\ &= \frac{1}{2}\sin^2\left(\frac{\theta_{ba}}{2} + 2\omega t\right); \\ &w(\mathcal{S}_2(t)\big|\mathcal{S}_1(t_0) \cap \mathcal{S}_3(t_0)) \\ &= w(\{a_-^{(2)}, b_-^{(1)}, t\}|\{a_+^{(2)}, b_+^{(1)}, t_0\} \cap \mathcal{S}_3(t_0)) \\ &= \sin^4(\omega t). \end{aligned} \tag{28}$$

Substituting probabilities from (18) and (28) into inequality (16), we find that

$$\sin^2\left(\frac{\theta_{ba}}{2}\right)\sin^4(\omega t) \leqslant \sin^2\left(\frac{\theta_{ba}}{2} + 2\omega t\right). \tag{29}$$

If we choose $\omega t = -\theta_{ba}/4$, inequality (29) is almost always violated:

$$\sin^2\left(\frac{\theta_{ba}}{2}\right)\sin^4\left(\frac{\theta_{ba}}{4}\right) \leqslant 0. \tag{30}$$

(c) Finally let us consider the situation when $\alpha' = \mp$ and $\beta' = \pm$. Then $\mathcal{S}_2(t) = \{a_\mp^{(2)}, b_\pm^{(1)}\}$. The corresponding conditional probabilities are equal to

$$\begin{aligned} &w(\mathcal{S}_2(t)\big|\mathcal{S}_3(t_0)) \\ &= w(\{a_\mp^{(2)}, b_\pm^{(1)}, t\}|\mathcal{S}_3(t_0)) \\ &= \frac{1}{2}\cos^2\left(\frac{\theta_{ba}}{2} + 2\omega t\right); \\ &w(\mathcal{S}_2(t)\big|\mathcal{S}_1(t_0) \cap \mathcal{S}_3(t_0)) \\ &= w(\{a_\mp^{(2)}, b_\pm^{(1)}, t\}|\{a_+^{(2)}, b_+^{(1)}, t_0\} \cap \mathcal{S}_3(t_0)) \\ &= \sin^2(\omega t)\cos^2(\omega t). \end{aligned} \tag{31}$$

Inequality (16) turns into

$$\sin^2\left(\frac{\theta_{ba}}{2}\right)\sin^2(\omega t)\cos^2(\omega t) \leqslant \cos^2\left(\frac{\theta_{ba}}{2} + 2\omega t\right). \tag{32}$$

If we set $\omega t = \pi/4 - \theta_{ba}/4$, we obtain

$$\sin^2\theta_{ba} \leqslant 0, \tag{33}$$

which, like (27) and (30), is wrong for almost all choices of directions $\vec{a}$ and $\vec{b}$.

Note that in real experimental situation Eq. (22) and the inequalities (26), (29), and (32) may be violated less significantly than in the simplest case of ideal anticorrelation, which is considered in the current paper. It might be hard to prepare the pure Bell state $|\Psi^-(t_0)\rangle$. Also in real experiments one should account for a noise. In the future it would be interesting to study the violation of (13) and (15) for more complicated states, for example for Werner states [74] and other states defined by a density matrix obtained from experiment.

IV. COROLLARIES OF BAYES' THEOREM FOR SYSTEMS OF NEUTRAL PSEUDOSCALAR MESONS

In Sec. II, equality (13) and inequality (15) were obtained in terms of spin $1/2$ projections onto various directions in three-dimensional space. Actually, relations (13) and (15) are true for any dichotomic observables of any nature in any space. In the case of neutral pseudoscalar mesons $M = \{K, D, B_d, B_s\}$, the dichotomic variables can be the flavor of the meson, its CP parity, and the lifetime (or mass). Pseudoscalar mesons are unstable particles, hence they may serve as a simple model of an open quantum system. Formulas (13) and (15), in principle, may be tested in experiments at the Large Hadron Collider [69–72], at B factory Belle II [73] and at ϕ factories.

When decaying a neutral vector meson with the quantum numbers $J^{PC} = 1^{--}$ of a photon into a $M\bar{M}$ pair, the latter rests in the Bell state $|\Psi^-\rangle$ by flavor, CP parity, or lifetime (H/L). In Appendix B the main properties of pseudoscalar mesons are presented as well as the formula for evolution of an entangled Bell state (here and below we use $\hbar = c = 1$).

As it is impossible to unambiguously relate the spin projections to the directions $\vec{a}$, $\vec{b}$, and $\vec{c}$ and projections of states of pseudoscalar mesons onto "directions" of flavor, CP parity and "directions" with a definitive mass or lifetime, we need to consider some variants of these correspondences. Note that $\langle M|\bar{M}\rangle = \langle M_1|M_2\rangle = 0$, but $\langle M_H|M_L\rangle \neq 0$. Then it is suitable to use the following: $a_+ \to M$, $a_- \to \bar{M}$, $b_+ \to M_L$, $b_- \to M_H$, $c_+ \to M_2$, and $c_- \to M_1$, for the algorithms of calculation of projectors $\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}$ and $\hat{P}_{\mathcal{S}_2\cap\mathcal{S}_3}$ in Sec. III to be analogous to the ones from Sec. IV.

We now show that equality (13) is violated in systems of neutral pseudoscalar mesons. At $t_0 = 0$ the state of the $M\bar{M}$ system is defined by density matrix $\hat{\rho}_0 = |\Psi^-(t_0)\rangle\langle\Psi^-(t_0)|$, where the Bell state $|\Psi^-(t_0)\rangle$ is defined by formula (B2). Projection operators onto events $\mathcal{S}_1(t_0) = \{M^{(2)}, M_L^{(1)}\}$, $\mathcal{S}_2(t_0) = \{M_2^{(2)}, M_L^{(1)}\}$, and singlet state in the flavor space $\mathcal{S}_3(t_0)$ are

$$\begin{aligned} \hat{P}_{\mathcal{S}_1} &= |M^{(2)}\rangle\big|M_L^{(1)}\big\rangle\big\langle M_L^{(1)}\big|\langle M^{(2)}|,\\ \hat{P}_{\mathcal{S}_2} &= \big|M_2^{(2)}\big\rangle\big|M_L^{(1)}\big\rangle\big\langle M_L^{(1)}\big|\big\langle M_2^{(2)}\big|, \quad \text{and } \hat{P}_{\mathcal{S}_3} = \hat{\rho}_0. \end{aligned} \tag{34}$$

Using the von Neumann rule (10), formulas (34), and (B2) analogous to (18), one may write that [see formula (C1)]

$$\begin{aligned} w(\mathcal{S}_1|\mathcal{S}_3) &= w\big(M^{(2)}, M_L^{(1)}, t_0\big) = \tfrac{1}{2}|q|^2,\\ w(\mathcal{S}_2|\mathcal{S}_3) &= w\big(M_2^{(2)}, M_L^{(1)}, t_0\big) = \tfrac{1}{4}|p+q|^2. \end{aligned} \tag{35}$$

Using a condition of orthogonality analogous to calculations from Sec. III for non-normalized projection operators onto the states corresponding to events $\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}$ and $\hat{P}_{\mathcal{S}_2\cap\mathcal{S}_3}$, we have

$$\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3} = \frac{1}{8|q|^2}\hat{P}_{\mathcal{S}_1}, \quad \hat{P}_{\mathcal{S}_2\cap\mathcal{S}_3} = \frac{|p+q|^2}{16|pq|^2}\hat{P}_{\mathcal{S}_2}.$$

Then calculation of conditional probabilities according to von Neumann's rule (10) leads to the equality

$$w(\mathcal{S}_2|\mathcal{S}_1\cap\mathcal{S}_3) = w(\mathcal{S}_1|\mathcal{S}_2\cap\mathcal{S}_3) = \mathrm{Tr}\big(\hat{P}_{\mathcal{S}_1}\hat{P}_{\mathcal{S}_2}\big) = \tfrac{1}{2}, \tag{36}$$

which is analogous to formula (20). Substituting (35) and (36) into the equality (13), we have

$$2 = \left|1 + \frac{p}{q}\right|^2. \tag{37}$$

The equality should be satisfied if the hypothesis of realism is true. As shown in Appendix B for neutral K and D mesons, the ratio q/p is close to $+1$ while for B_d and B_s mesons it is almost always equals to -1. Hence (37) implies false relations like $2 \approx 4$ and $2 \approx 0$.

If we use the correspondence $a_+ \to M$, $a_- \to \bar{M}$, $b_+ \to M_H$, $b_- \to M_L$, $c_+ \to M_2$, and $c_- \to M_1$, which differs by interchanging M_L and M_H, then in the framework of the hypothesis of realism we come to the following equality:

$$2 = \left|1 - \frac{p}{q}\right|^2, \tag{38}$$

which, like (37), is not true in flavor-entangled systems of neutral pseudoscalar mesons. Examination of other correspondences leads to equalities which do not provide anything new.

Now consider the time-dependent inequality (15), and let us show that it is also violated in systems of neutral pseudoscalar mesons. The most natural choice of "directions" for neutral D and B_q mesons is related to states with a definite flavor and CP parity. For example at B factories the flavor of a neutral B_d meson is determined by a lepton sign in semileptonic decay, while CP parity is determined using the decay $B_d \to J/\psi K_S^0$. At hadron machines the task is much more complicated, as $B\bar{B}$ pairs are mainly produced not through the $\Upsilon(4S)$ decay but through the process of direct hadronization of $b\bar{b}$ quark pairs. In order to select states corresponding to $\Upsilon(4S)$, one needs to know the invariant mass of the $B\bar{B}$ pair, i.e., to fully reconstruct the energy and momentum of each B_d meson. So at hadron machines it is not possible to use semileptonic decays with branching ratios of the order of 10^{-1} for determination of the B-meson flavor. An alternative way to detect the flavor is to use the cascade decay $B_d^0 \to (D^- \to K^-K^+\pi^-)K^+$, with a branching ratio of about 10^{-5}. For B_s mesons at the LHC experiments, the flavor can be determined in the decay $B_s^0 \to (D_s^- \to K^-\pi^+\pi^-)\pi^+$. At hadron machines, the statistics required for testing for violation of (15) should be higher by a few orders of magnitude than at the B factories. For D mesons the situation is slightly better because the flavor of the D meson may be determined in the decay $D^0 \to K^-\pi^+$, which has a branching ratio of about 4%. In the case of K mesons, the hadron machines are not suitable at all, and the test for violation of the inequality (15) must be performed only at ϕ factories.

Consider events $\mathcal{S}_1(t_0) = \{M_1^{(2)}, M^{(1)}\}$, $\mathcal{S}_2(t) = \{M_1^{(2)}, M^{(1)}\}$, and event $\mathcal{S}_3(t_0)$, which corresponds to the singlet spin state of the $M\bar{M}$ pair. Using formulas (C1) and (C2) from Appendix C and the algorithm for calculation of projector $\hat{P}_{\mathcal{S}_1\cap\mathcal{S}_3}$ described in Sec. III, we find

$$w(\mathcal{S}_1(t_0)|\mathcal{S}_3(t_0)) = w\big(M_1^{(2)}, M^{(1)}, t_0\big) = \frac{1}{4},$$

$$w(\mathcal{S}_2(t)|\mathcal{S}_3(t_0)) = w\big(M_1^{(2)}, M^{(1)}, t\big) = \frac{1}{4}e^{-2\Gamma t},$$

TABLE I. Experimentally found parameters of oscillations and CP violation for systems of neutral pseudoscalar mesons. The table is modelled on one in [75]. The minus sign in the numerical values of $\Delta\Gamma$ reflects the difference in definitions between the current paper and [75]. Dimensionless variable $\lambda = {}^{\Delta M}/{}_{\Delta\Gamma}$.

Meson	$\Delta\Gamma$, MeV	ΔM, MeV	$\mathrm{tg}\,\beta = \lvert{}^{q}/{}_{p}\rvert_M^{\mathrm{expt}}$	λ
B_s^0	-6.0×10^{-11}	1.2×10^{-8}	1.0039 ± 0.0021	-0.2×10^3
K^0	-7.3×10^{-12}	3.5×10^{-12}	0.99668 ± 0.00004	-4.8×10^{-1}
D^0	-2.1×10^{-11}	-6.3×10^{-12}	$0.92^{+0.12}_{-0.09}$	0.3

$$w(\mathcal{S}_2(t)|\mathcal{S}_1(t_0) \cap \mathcal{S}_3(t_0)) = w(M_1(0) \to M_1(t))w(M(0) \to M(t)) = \left| g_+(t) - \frac{1}{2}\left(\frac{q}{p} + \frac{p}{q}\right) g_-(t)\right|^2 |g_+(t)|^2. \quad (39)$$

Using the set of probabilities (39), inequality (15) becomes

$$\left| g_+(t) - \frac{1}{2}\left(\frac{q}{p} + \frac{p}{q}\right) g_-(t)\right|^2 |g_+(t)|^2 e^{2\Gamma t} \leqslant 1. \quad (40)$$

To understand for which mesons inequality (40) may be violated, consider the case without CP violation, i.e., when $q/p = \pm 1$. Then for K and D mesons formula (40) becomes

$$e^{\Delta\Gamma t/2} + 2\cos(\Delta M t) \leqslant 3e^{-\Delta\Gamma t/2},$$

which is not violated for any values of $t \geqslant 0$, as $\Delta\Gamma < 0$ (see Table I). For B_q mesons, inequality (40) becomes

$$e^{-\Delta\Gamma t/2} + 2\cos(\Delta M t) \leqslant 3e^{\Delta\Gamma t/2}. \quad (41)$$

We estimate, for B_s^0 mesons, at which times $t \geqslant 0$ the inequality (41) should be violated. From Table I one can see that $\cos(\Delta M t) \approx \cos(200\Delta\Gamma t)$. Hence for small adjustments of parameter t, the argument of the cosine goes through its full period. Hence the condition of guaranteed violation of inequality (41) is given by

$$e^{-\Delta\Gamma t/2} \geqslant 3e^{\Delta\Gamma t/2} + 2 \quad \text{or} \quad t \geqslant \frac{2\ln 3}{|\Delta\Gamma|}.$$

$|\Delta\Gamma_{B_s}| = (0.091 \pm 0.008) \times 10^{12}\ \mathrm{s}^{-1}$ and $\tau_{B_s} = (1.512 \pm 0.007) \times 10^{-12}$ s [75], where τ_{B_s} is the average lifetime of the B_s^0 meson, so the condition for guaranteed violation of inequality (41) is $t \geqslant 16\tau_{B_s}$. Due to the small magnitude of CP-violation effects, the exact inequality (40) should be violated at times of the same order.

From the calculations above it is clear that for various choices of events $\mathcal{S}_1(t_0)$ and $\mathcal{S}_2(t)$, inequality (15) will always transform into the following expression:

$$F_N(x,r,\zeta,\lambda) \leqslant 1, \quad (42)$$

where functions F_N depend on dimensionless variables $x = \Delta\Gamma t$, $\lambda = \Delta M/\Delta\Gamma$, the modulus of r, and phase ζ of the ratio q/p (see Appendix B).

There are some experimental limits on the range of possible values of parameters r and ζ. In [45] and [46] concerning the modeling of the violation of time-dependent Wigner's inequalities in systems of B_s mesons the following values were selected: $r = 1.004$ and $\zeta = 185^\circ$. In the present paper we will also use these values.

If we introduce function

$$F_1(x,r,\zeta,\lambda) = \left| g_+(t) - \frac{1}{2}\left(\frac{q}{p} + \frac{p}{q}\right) g_-(t)\right|^2 |g_+(t)|^2 e^{2\Gamma t},$$

then inequality (40) transforms into (42). A plot of the function $F_1(x,r,\zeta,\lambda)$ is shown in Fig. 1 left. That taking into account the oscillations, the violation of inequality (40) holds for

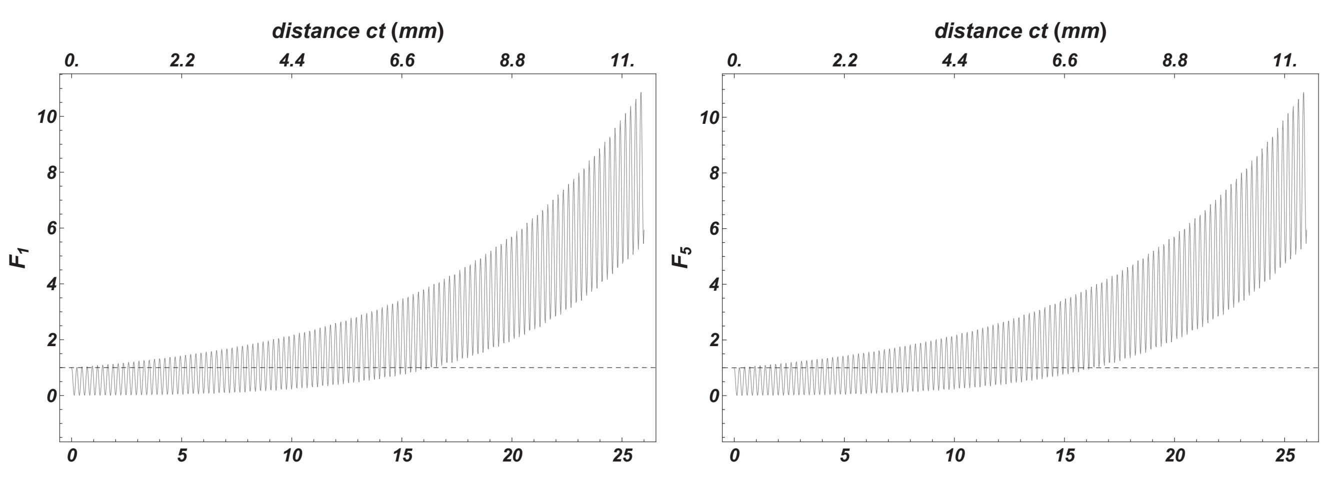

FIG. 1. Functions $F_{1,5}(x,r,\zeta,\lambda)$ for B_s mesons. The top axis corresponds to ct (in mm), the bottom axis to the time in units of lifetime $z = (\Gamma_H + \Gamma_L)t/2 = \Gamma t = t/\tau_{B_s}$, where time t is calculated in the B_s-meson rest frame. The plots are produced for $r = 1.004$ and $\zeta = 185^\circ$.

TABLE II. Table of correspondences between sets of events, functions F_N related to these sets, and conditions of violation of the inequality (42). The events $\mathcal{S}_1(t_0)$ and $\mathcal{S}_2(t)$ depend on the same directions of pseudoscalar mesons, so in the first column we provide only directions. The event $\mathcal{S}_3(t_0)$ is identical for all the sets of events and is not shown.

Set of events	Function	Conditions of violations of inequality (42)
$\{M_1^{(2)},M^{(1)}\}$	F_1	Violates for B_s mesons
$\{M_1^{(2)},\bar{M}^{(1)}\}$	F_1	Violates for B_s mesons
$\{M_1^{(2)},M_H^{(1)}\}$	F_2	Violates for B_s mesons
$\{M_2^{(2)},M^{(1)}\}$	F_3	Violates for K and D mesons
$\{M_2^{(2)},\bar{M}^{(1)}\}$	F_3	Violates for K and D mesons
$\{M_2^{(2)},M_H^{(1)}\}$	F_4	Violates for K and D mesons
$\{M^{(2)},M_H^{(1)}\}$	F_5	Violates for K, D, and B_s mesons
$\{\bar{M}^{(2)},M_H^{(1)}\}$	F_5	Violates for K, D, and B_s mesons
$\{M^{(2)},M_L^{(1)}\}$	F_6	Never violates
$\{\bar{M}^{(2)},M_L^{(1)}\}$	F_6	Never violates
$\{M_1^{(2)},M_L^{(1)}\}$	F_7	Never violates
$\{M_2^{(2)},M_L^{(1)}\}$	F_8	Never violates

$z = 1/\tau_{B_s} \geqslant 17$, which agrees with a naïve estimate obtained from the simplified inequality (41). One can also obtain inequality (40) by choosing events $\mathcal{S}_1(t_0)$ and $\mathcal{S}_2(t)$ in the form $\mathcal{S}_1(t_0) = \{M_1^{(2)},\bar{M}^{(1)}\}$, $\mathcal{S}_2(t) = \{M_1^{(2)},\bar{M}^{(1)}\}$. The event $\mathcal{S}_3(t_0)$ remains the same.

The explicit form of the functions $F_N(x,r,\zeta,\lambda)$ is shown in Appendix D. The inequality (42) in systems of neutral B_s mesons is violated not only while choosing events to which the function F_1, corresponds, but also for sets of events, to which correspond the functions F_2 and F_5. The correspondence between sets of events and the functions is given in Table II. The behavior of the function F_5 for B_s mesons is shown in Fig. 1 right. The dependence of functions F_1 and F_5 on z is almost identical due to the small magnitude of CP-violation effects. Unfortunately a test of the violation of the inequality (42) for $z \geqslant 17$ requires large statistics due to the exponential character of B_s-meson decays.

Now consider systems of neutral kaons. According to Table II, for K mesons the inequality (42) is violated when choosing sets of events which lead to the functions F_3, F_4, and F_5. As one can see from Fig. 2, a significant violation of inequality (42) holds for $z \sim 1$, which makes the systems of neutral kaons good candidates for an experimental test of the hypothesis of realism.

In the case of entangled states of $D^0\bar{D}^0$ mesons, the following functions lead to violation of inequality (42): F_3, F_4, and F_5, which were already considered for entangled kaons. This happens due to the fact that for neutral K and D mesons, the real part of the relation $\frac{q}{p}$ is close to $+1$. From Fig. 3, for D mesons even with $z \sim 40$, the functions F_3 and F_5 remain almost linear. Note that the corresponding functions for the K mesons demonstrate exponential growth already for $z \geqslant 1$ (see Fig. 2). The difference in the behavior of the functions $F_{3,4,5}(x,r,\zeta,\lambda)$ for K and D mesons is stipulated by the value of the relation $|\Delta\Gamma|/\Gamma$, which sets the scale of the magnitude of the functions. For K mesons, $(\frac{|\Delta\Gamma|}{\Gamma})_K \approx 2$, while for D mesons that parameter is lower by almost two orders of magnitude, equal to $(\frac{|\Delta\Gamma|}{\Gamma})_D \approx 10^{-2}$. For B_s mesons, $(\frac{|\Delta\Gamma|}{\Gamma})_{B_s} \approx 0.13$, and the value of $z \sim 15$ when the functions $F_{1,2,5}(x,r,\zeta,\lambda)$ become exponential. Hence they are an intermediate state between the values of z for K and D mesons.

The analysis described above shows that from the experimental point of view, violation of inequality (42) is more suitable to observe in systems of entangled K and D mesons. Due to oscillations, for B_s mesons inequality (42) is violated for $z \geqslant 17$ and its observation requires quite large statistics.

V. CONCLUSION

Using the notion of conditional probability in the framework of Kolmogorov's axiomatics and Bayes' theorem, we obtained the static equality (13) and the time-dependent

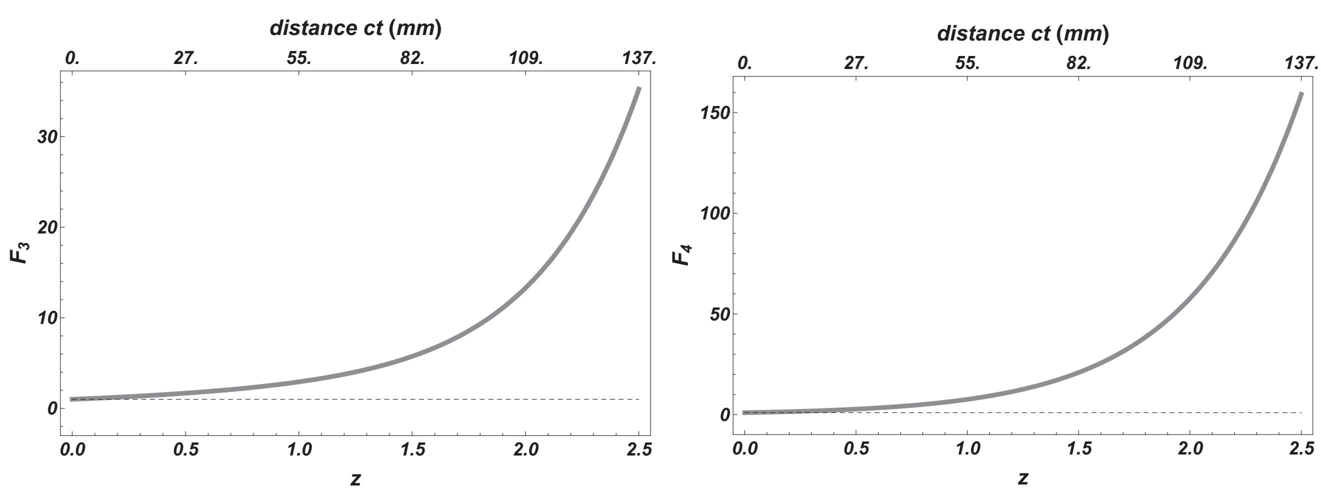

FIG. 2. Functions $F_{3,4}(x,r,\zeta,\lambda)$ for neutral K mesons. The top axis corresponds to ct (in mm), the bottom axis to the time in units of lifetime $z = (\Gamma_S + \Gamma_L)t/2 = \Gamma t = t/\tau_K$, where t is calculated in the K-meson rest frame. The plots are produced for $r = 0.997$ and $\zeta = -0.18^\circ$.

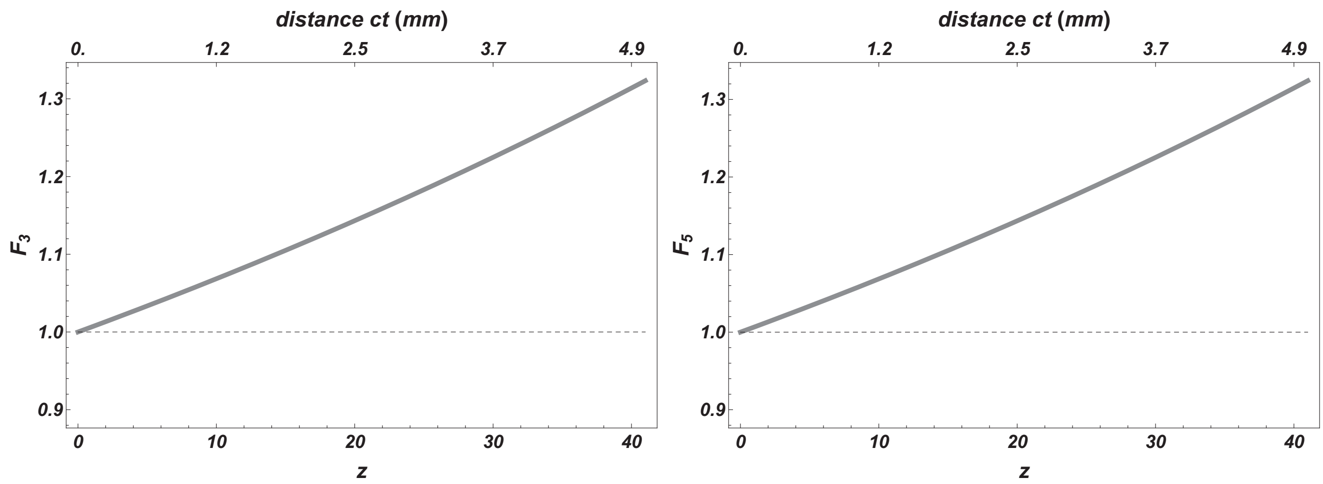

FIG. 3. Functions $F_{3,5}(x,r,\zeta,\lambda)$ for neutral D mesons. The top axis corresponds to ct (in mm), the bottom axis to the time in units of lifetime $z = (\Gamma_H + \Gamma_L)t/2 = \Gamma t = t/\tau_D$, where t is calculated in D-meson rest frame. The plots are produced for $r = 1.1$ and $\zeta = -10^\circ$.

inequality (15) which allow experimental demonstration of the unsoundness of the hypothesis of realism for quantum systems.

The structure of time-dependent inequality (15) gives a principal possibility to avoid the contextuality loophole in contemporary experiments that test Bell's, Wigner's, and the Leggett-Garg inequalities.

The possibility to experimentally test the violation of formulas (13) and (15) is studied with two examples: the behavior of correlated spins in a constant and homogeneous magnetic field; and the behavior of pairs of correlated pseudoscalar mesons. Some factors that can prevent such tests at the LHC experiments, Belle II, and ϕ factories are considered.

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APPENDIX A: CORRELATED SPINS IN AN EXTERNAL MAGNETIC FIELD: MAIN FORMULAS

At the initial time $t_0 = 0$, a pseudoscalar particle at rest decays into a positron (index "1") and an electron (index "2"). If such a decay is described by Hamiltonians (3), then for $t_0 = 0$ the e^+e^- pair is in Bell state $|\Psi^-\rangle$ with zero full spin (12).

Choose spatial direction $\vec{n} = (\sin\theta_n \cos\varphi_n, \sin\theta_n \sin\varphi_n, \cos\theta_n)$. At time $t_0 = 0$, the state vectors of the positron and the electron, related to the spin projections $\pm 1/2$ onto axis $\vec{n}$, are

$$\left|\frac{1}{2}, n_+^{(i)}\right\rangle = \begin{pmatrix} \cos(\theta_n/2)e^{-i\varphi_n/2} \\ \sin(\theta_n/2)e^{i\varphi_n/2} \end{pmatrix} \quad \text{and} \quad \left|\frac{1}{2}, n_-^{(i)}\right\rangle = \begin{pmatrix} -\sin(\theta_n/2)e^{-i\varphi_n/2} \\ \cos(\theta_n/2)e^{i\varphi_n/2} \end{pmatrix}, \tag{A1}$$

where $i = \{1,2\}$. Then

$$|\Psi^-(t_0)\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} \cos(\theta_n/2)e^{-i\varphi_n/2} \\ \sin(\theta_n/2)e^{i\varphi_n/2} \end{pmatrix}^{(2)} \begin{pmatrix} -\sin(\theta_n/2)e^{-i\varphi_n/2} \\ \cos(\theta_n/2)e^{i\varphi_n/2} \end{pmatrix}^{(1)} - \begin{pmatrix} -\sin(\theta_n/2)e^{-i\varphi_n/2} \\ \cos(\theta_n/2)e^{i\varphi_n/2} \end{pmatrix}^{(2)} \begin{pmatrix} \cos(\theta_n/2)e^{-i\varphi_n/2} \\ \sin(\theta_n/2)e^{i\varphi_n/2} \end{pmatrix}^{(1)}\right]. \tag{A2}$$

To illustrate the violation of relations (13) and (16) in NRQM, it is enough to measure fermionic spin projections onto two or three nonparallel directions lying in plane (x,z) [we use a standard rectangular coordinate system (x,y,z)]. We map the unit vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ to these directions. In this case $\varphi_a = \varphi_b = \varphi_c = 0$.

Now put this spin singlet e^+e^- state into a constant, homogeneous magnetic field with strength $\vec{\mathcal{H}}$ directed along the y axis. Require the electron and positron to propagate strictly along y. This requirement avoids unnecessary complications related to the rotation of charged particles in the magnetic field.

The spins of the electron and positron will begin to precess around the y axis. Given initial condition (A1), the state vectors of the electron which describe its spin projections onto $\vec{n}$ at an arbitrary moment of time may be written as

$$\left|\psi_{n_+}^{(2)}(t)\right\rangle = \begin{pmatrix} \cos{}^{\theta_n}/2 \cos(\omega t) e^{-i\varphi_n/2} - \sin{}^{\theta_n}/2 \sin(\omega t) e^{i\varphi_n/2} \\ \cos{}^{\theta_n}/2 \sin(\omega t) e^{-i\varphi_n/2} + \sin{}^{\theta_n}/2 \cos(\omega t) e^{i\varphi_n/2} \end{pmatrix}^{(2)} \quad \text{and}$$

$$\left|\psi_{n_-}^{(2)}(t)\right\rangle = \begin{pmatrix} -\sin{}^{\theta_n}/2 \cos(\omega t) e^{-i\varphi_n/2} - \cos{}^{\theta_n}/2 \sin(\omega t) e^{i\varphi_n/2} \\ -\sin{}^{\theta_n}/2 \sin(\omega t) e^{-i\varphi_n/2} + \cos{}^{\theta_n}/2 \cos(\omega t) e^{i\varphi_n/2} \end{pmatrix}^{(2)}. \tag{A3}$$

For the positron the analogous state vectors are

$$\left|\psi_{n_+}^{(1)}(t)\right\rangle = \begin{pmatrix} \cos{}^{\theta_n}/2 \cos(\omega t) e^{-i\varphi_n/2} + \sin{}^{\theta_n}/2 \sin(\omega t) e^{i\varphi_n/2} \\ -\cos{}^{\theta_n}/2 \sin(\omega t) e^{-i\varphi_n/2} + \sin{}^{\theta_n}/2 \cos(\omega t) e^{i\varphi_n/2} \end{pmatrix}^{(1)} \quad \text{and}$$

$$\left|\psi_{n_-}^{(1)}(t)\right\rangle = \begin{pmatrix} -\sin{}^{\theta_n}/2 \cos(\omega t) e^{-i\varphi_n/2} + \cos{}^{\theta_n}/2 \sin(\omega t) e^{i\varphi_n/2} \\ \sin{}^{\theta_n}/2 \sin(\omega t) e^{-i\varphi_n/2} + \cos{}^{\theta_n}/2 \cos(\omega t) e^{i\varphi_n/2} \end{pmatrix}^{(1)}, \tag{A4}$$

where $\omega = \frac{|e|\mathcal{H}}{2m_e c}$, the Larmor frequency of a fermion.

Using initial condition (A2) and the explicit form of the wave functions of the electron (A3) and positron (A4) in the magnetic field, we obtain for the spin-wave function of the e^+e^- pair for arbitrary time $t \geqslant t_0$

$$|\Psi^-(t)\rangle = \frac{1}{\sqrt{2}}\left(\left|\psi_{\mathbf{n}_+}^{(2)}(t)\right\rangle\left|\psi_{\mathbf{n}_-}^{(1)}(t)\right\rangle - \left|\psi_{\mathbf{n}_-}^{(2)}(t)\right\rangle\left|\psi_{\mathbf{n}_+}^{(1)}(t)\right\rangle\right). \tag{A5}$$

APPENDIX B: OSCILLATIONS OF NEUTRAL PSEUDOSCALAR MESONS: MAIN FORMULAS

The definitions used in this appendix are analogous to those from [45] and [46].

In contrast to spin states for which one can choose an infinite number of spatial directions for neutral pseudoscalar mesons $M = \{K, D, B_q\}$, where $q = \{d, s\}$, there are only three fixed "directions" with corresponding noncommuting "projectors."

As a first direction let us choose the flavor of the pseudoscalar meson. For D mesons, consider projections onto states $|D\rangle = |c\bar{u}\rangle$ and $|\bar{D}\rangle = |\bar{c}u\rangle$. Operators of charge $(\hat{C})$ and space $(\hat{P})$ conjugation act on the flavor states as follows:

$$\hat{C}\hat{P}|M\rangle = e^{i\alpha}|\bar{M}\rangle \quad \text{and} \quad \hat{C}\hat{P}|\bar{M}\rangle = e^{-i\alpha}|M\rangle,$$

where α is an arbitrary real phase. This phase should not appear in any experimentally testable relations. States $|M\rangle$ and $|\bar{M}\rangle$ are orthogonal to each other.

A second direction is specified by the states with definite values of CP parity:

$$\hat{C}\hat{P}|M_1\rangle = +|M_1\rangle, \quad \hat{C}\hat{P}|M_2\rangle = -|M_2\rangle,$$

which can be written using the states $|M\rangle$ and $|\bar{M}\rangle$ as

$$|M_1\rangle = \frac{1}{\sqrt{2}}(|M\rangle + e^{i\alpha}|\bar{M}\rangle), \quad |M_2\rangle = \frac{1}{\sqrt{2}}(|M\rangle - e^{i\alpha}|\bar{M}\rangle).$$

Note that $\langle M_1|M_2\rangle = 0$.

A third direction corresponds to the states with definite lifetimes and masses. In terms of $|M\rangle$ and $|\bar{M}\rangle$, projections onto this direction may be written as

$$|M_L\rangle = p\left(|M\rangle + e^{i\alpha}\frac{q}{p}|\bar{M}\rangle\right) \quad \text{and}$$

$$|M_H\rangle = p\left(|M\rangle - e^{i\alpha}\frac{q}{p}|\bar{M}\rangle\right).$$

Using the normalization condition, we find the relation for complex coefficients p and q:

$$\langle M_L|M_L\rangle = \langle M_H|M_H\rangle = |p|^2 + |q|^2 = 1. \tag{B1}$$

It can be shown that $\langle M_L|M_H\rangle = |p|^2 - |q|^2 \neq 0$.

To automatically satisfy the normalization condition (B1) we introduce a new variable β:

$$|p| = \cos\beta, \quad |q| = \sin\beta, \quad \text{and} \quad \frac{q}{p} = \text{tg}\,\beta e^{i\zeta} \equiv r e^{i\zeta}.$$

From the definition it follows that $\beta \in [0, \pi/2]$.

Taking into account CPT invariance, the states $|M_L\rangle$ and $|M_H\rangle$ are eigenvectors of the Hamiltonian

$$\hat{H} = \begin{pmatrix} \mathcal{H} & H_{12}e^{-i\alpha} \\ H_{21}e^{i\alpha} & \mathcal{H} \end{pmatrix} = \begin{pmatrix} m - {}^i/2\Gamma & (m_{12} - {}^i/2\Gamma_{12})e^{-i\alpha} \\ (m_{12}^* - {}^i/2\Gamma_{12}^*)e^{i\alpha} & m - {}^i/2\Gamma \end{pmatrix},$$

with eigenvalues

$$E_L = m_L - {}^i/2\Gamma_L = \mathcal{H} - \sqrt{H_{12}H_{21}} = \mathcal{H} + {}^q/_p H_{12} \quad \text{and}$$

$$E_H = m_H - {}^i/2\Gamma_H = \mathcal{H} + \sqrt{H_{12}H_{21}} = \mathcal{H} - {}^q/_p H_{12}$$

accordingly. Finally we define parameters

$$\Delta M = M_H - M_L = -2\text{Re}\left(\frac{q}{p}H_{12}\right),$$

$$\Delta\Gamma = \Gamma_H - \Gamma_L = 4\text{Im}\left(\frac{q}{p}H_{12}\right).$$

Please note that the definition of $\Delta\Gamma$ here differs by a sign from the definition of $\Delta\Gamma$ in [75]. Experimental values of the parameters of CP violation are shown in Table I.

Decay of a neutral vector meson with quantum numbers $J^{PC} = 1^{--}$ into a pair of pseudoscalar mesons [experiments mostly deal with the decays $\phi(1020) \to K\bar{K}$, $\Upsilon(4S) \to B_d\bar{B}_d$, and $\Upsilon(5S) \to B_s\bar{B}_s$] produces a state of an $M\bar{M}$ pair at the time $t_0 = 0$ which is described by the Bell state vector

$$\begin{aligned} |\Psi^-(t_0)\rangle &= \frac{1}{\sqrt{2}}(|M^{(2)}\rangle|\bar{M}^{(1)}\rangle - |\bar{M}^{(2)}\rangle|M^{(1)}\rangle) \\ &= \frac{e^{-i\alpha}}{\sqrt{2}}\big(|M_2^{(2)}\rangle|M_1^{(1)}\rangle - |M_1^{(2)}\rangle|M_2^{(1)}\rangle\big) \\ &= \frac{1}{2\sqrt{2}pq}\big(|M_H^{(2)}\rangle|M_L^{(1)}\rangle - |M_L^{(2)}\rangle|M_H^{(1)}\rangle\big). \end{aligned} \quad \text{(B2)}$$

This state vector is fully analogous to the state vector (A2), entangled in the spin space [76–78].

The evolution of the state vectors $|M_L\rangle$ and $|M_H\rangle$ can be written as

$$\begin{aligned} |M_L(t)\rangle &= e^{-iE_L\Delta t}|M_L\rangle = e^{-im_L\Delta t - \Gamma_L\Delta t/2}|M_L\rangle, \\ |M_H(t)\rangle &= e^{-iE_H\Delta t}|M_H\rangle = e^{-im_H\Delta t - \Gamma_H\Delta t/2}|M_H\rangle, \end{aligned} \quad \text{(B3)}$$

where $\Delta t = t - t_0$. From the above one can find the evolution of the states $|M(t)\rangle$ and $|\bar{M}(t)\rangle$:

$$\begin{aligned} |M(t)\rangle &= g_+(\Delta t)|M\rangle - e^{i\alpha}\frac{q}{p}g_-(\Delta t)|\bar{M}\rangle \\ |\bar{M}(t)\rangle &= g_+(\Delta t)|\bar{M}\rangle - e^{-i\alpha}\frac{p}{q}g_-(\Delta t)|M\rangle \end{aligned}$$

and the time dependence of the state vectors $|M_1(t)\rangle$ and $|M_2(t)\rangle$:

$$\begin{aligned} |M_1(t)\rangle &= \frac{1}{\sqrt{2}}\bigg[\bigg(g_+(\Delta t) - \frac{p}{q}g_-(\Delta t)\bigg)|M\rangle \\ &\quad + e^{i\alpha}\bigg(g_+(\Delta t) - \frac{q}{p}g_-(\Delta t)\bigg)|\bar{M}\rangle\bigg], \\ |M_2(t)\rangle &= \frac{1}{\sqrt{2}}\bigg[\bigg(g_+(\Delta t) + \frac{p}{q}g_-(\Delta t)\bigg)|M\rangle \\ &\quad - e^{i\alpha}\bigg(g_+(\Delta t) + \frac{q}{p}g_-(\Delta t)\bigg)|\bar{M}\rangle\bigg], \end{aligned}$$

where $g_\pm(\tau) = \frac{1}{2}(e^{-iE_H\tau} \pm e^{-iE_L\tau})$. Function $g_\pm(\tau)$ satisfies the conditions

$$\begin{aligned} |g_\pm(\tau)|^2 &= \frac{e^{-\Gamma\tau}}{2}\bigg[\text{ch}\bigg(\frac{\Delta\Gamma\tau}{2}\bigg) \pm \cos(\Delta M\tau)\bigg] \\ g_+^*(\tau)g_-(\tau) &= -\frac{e^{-\Gamma\tau}}{2}\bigg[\text{sh}\bigg(\frac{\Delta\Gamma\tau}{2}\bigg) + i\sin(\Delta M\tau)\bigg], \end{aligned}$$

where $\Gamma = (\Gamma_H + \Gamma_L)/2$. Taking into account the initial condition (B2), for the state vector of the $M\bar{M}$ pair at an arbitrary time one can write

$$|\Psi^-(t)\rangle = e^{-i(m_H+m_L)\Delta t}e^{-\Gamma\Delta t}|\Psi^-(t_0)\rangle. \quad \text{(B4)}$$

For $t_0 = 0$ above, $\Delta t \equiv t$.

In systems of neutral pseudoscalar mesons, the magnitude of CP violation is small. If we neglect the CP violation which appears due to oscillations, then for K mesons, $(\frac{q}{p})_K = \frac{1-\epsilon}{1+\epsilon} \approx 1$; so $\cos\zeta_K = 1$. For B_q mesons the effective Hamiltonian of the oscillations is proportional to $(V_{tb}V_{tq}^*)^2$ [79]. Then

$$\bigg(\frac{q}{p}\bigg)_{B_q} = -\frac{H_{21}}{\sqrt{H_{12}H_{21}}} \approx -\bigg(\frac{V_{tb}^*V_{tq}}{|V_{tb}^*V_{tq}|}\bigg)^2 = -1,$$

hence $\cos\zeta_{B_q} = -1$. For D mesons, experimental data from BaBar [80] and Belle [81] are in agreement with the assumption that $\cos\zeta_D = 1$, so $\cos\zeta = \pm 1$ is a good approximation and the analysis of formulas (13) and (15) is much simplified.

APPENDIX C: OSCILLATIONS OF NEUTRAL PSEUDOSCALAR MESONS: TRANSITION PROBABILITIES

In this Appendix we collect the probabilities that are necessary for a test of the static equality (13) and time-dependent inequality (15) in systems of neutral pseudoscalar mesons.

In the framework of quantum theory using the normalization condition and the initial condition (B2), the following expressions for time-independent probabilities hold:

$$\begin{aligned} w\big(M_1^{(2)},\bar{M}^{(1)},t_0\big) &= \big|\langle M_1^{(2)}\big|\langle\bar{M}^{(1)}|\Psi^-(t_0)\rangle\big|^2 \\ &= \tfrac{1}{4} \equiv \tfrac{1}{4}(|p|^2 + |q|^2); \\ w\big(M_1^{(2)},M^{(1)},t_0\big) &= \big|\langle M_1^{(2)}\big|\langle M^{(1)}|\Psi^-(t_0)\rangle\big|^2 \\ &= \tfrac{1}{4} \equiv \tfrac{1}{4}(|p|^2 + |q|^2); \\ w\big(M_2^{(2)},\bar{M}^{(1)},t_0\big) &= \big|\langle M_2^{(2)}\big|\langle\bar{M}^{(1)}|\Psi^-(t_0)\rangle\big|^2 \\ &= \tfrac{1}{4} \equiv \tfrac{1}{4}(|p|^2 + |q|^2); \\ w\big(M_2^{(2)},M^{(1)},t_0\big) &= \big|\langle M_2^{(2)}\big|\langle M^{(1)}|\Psi^-(t_0)\rangle\big|^2 \\ &= \tfrac{1}{4} \equiv \tfrac{1}{4}(|p|^2 + |q|^2); \\ w\big(M_1^{(2)},M_H^{(1)},t_0\big) &= \big|\langle M_1^{(2)}\big|\langle M_H^{(1)}|\Psi^-(t_0)\rangle\big|^2 = \tfrac{1}{4}|p+q|^2; \\ w\big(M_2^{(2)},M_H^{(1)},t_0\big) &= \big|\langle M_2^{(2)}\big|\langle M_H^{(1)}|\Psi^-(t_0)\rangle\big|^2 = \tfrac{1}{4}|p-q|^2; \\ w\big(M_1^{(2)},M_L^{(1)},t_0\big) &= \big|\langle M_1^{(2)}\big|\langle M_L^{(1)}|\Psi^-(t_0)\rangle\big|^2 = \tfrac{1}{4}|p-q|^2; \\ w\big(M_2^{(2)},M_L^{(1)},t_0\big) &= \big|\langle M_2^{(2)}\big|\langle M_L^{(1)}|\Psi^-(t_0)\rangle\big|^2 = \tfrac{1}{4}|p+q|^2; \\ w\big(M_H^{(2)},\bar{M}^{(1)},t_0\big) &= \big|\langle M_H^{(2)}\big|\langle\bar{M}^{(1)}|\Psi^-(t_0)\rangle\big|^2 = \tfrac{1}{2}|p|^2; \\ w\big(M_H^{(2)},M^{(1)},t_0\big) &= \big|\langle M_H^{(2)}\big|\langle M^{(1)}|\Psi^-(t_0)\rangle\big|^2 = \tfrac{1}{2}|q|^2; \\ w\big(\bar{M}^{(2)},M_L^{(1)},t_0\big) &= \big|\langle\bar{M}^{(2)}|\langle M_L^{(1)}\big|\Psi^-(t_0)\rangle\big|^2 = \tfrac{1}{2}|p|^2; \\ w\big(M^{(2)},M_L^{(1)},t_0\big) &= \big|\langle M^{(2)}|\langle M_L^{(1)}\big|\Psi^-(t_0)\rangle\big|^2 = \tfrac{1}{2}|q|^2. \end{aligned} \quad \text{(C1)}$$

In order to test the time-dependent inequality (15) for correlated $M\bar{M}$ pairs, the following time-dependent probabilities are needed (for $t_0 = 0$):

$$\begin{aligned} w[M_1(0) \to M_1(t)] &= |\langle M_1(t)|M_1\rangle|^2 \\ &= \bigg|g_+(t) - \frac{1}{2}\bigg(\frac{q}{p} + \frac{p}{q}\bigg)g_-(t)\bigg|^2; \end{aligned}$$

$$
\begin{aligned}
w[M_2(0)\to M_1(t)] &= |\langle M_1(t)|M_2\rangle|^2 \\
&= \left|\frac{1}{2}\left(\frac{q}{p}-\frac{p}{q}\right)g_-(t)\right|^2; \\
w[M_2(0)\to M_2(t)] &= |\langle M_2(t)|M_2\rangle|^2 \\
&= \left|g_+(t)+\frac{1}{2}\left(\frac{q}{p}+\frac{p}{q}\right)g_-(t)\right|^2; \\
w[M_1(0)\to M_2(t)] &= |\langle M_2(t)|M_1\rangle|^2 \\
&= \left|\frac{1}{2}\left(\frac{q}{p}-\frac{p}{q}\right)g_-(t)\right|^2; \\
w[\bar{M}(0)\to \bar{M}(t)] &= |\langle \bar{M}(t)|\bar{M}\rangle|^2 = |g_+(t)|^2; \\
w[M(0)\to \bar{M}(t)] &= |\langle \bar{M}(t)|M\rangle|^2 = \left|\frac{p}{q}g_-(t)\right|^2; \\
w[M(0)\to M(t)] &= |\langle M(t)|M\rangle|^2 = |g_+(t)|^2; \\
w[\bar{M}(0)\to M(t)] &= |\langle M(t)|\bar{M}\rangle|^2 = \left|\frac{q}{p}g_-(t)\right|^2; \\
w\big(M_1^{(2)},\bar{M}^{(1)},t\big) &= \left|\langle M_1^{(2)}\right|\langle \bar{M}^{(1)}|\Psi^-(t)\rangle|^2 = \frac{1}{4}e^{-2\Gamma t}; \\
w\big(M_1^{(2)},M^{(1)},t\big) &= \left|\langle M_1^{(2)}\right|\langle M^{(1)}|\Psi^-(t)\rangle\big|^2 = \frac{1}{4}e^{-2\Gamma t}; \\
w\big(M_2^{(2)},\bar{M}^{(1)},t\big) &= \left|\langle M_2^{(2)}\right|\langle \bar{M}^{(1)}|\Psi^-(t)\rangle\big|^2 = \frac{1}{4}e^{-2\Gamma t}; \\
w\big(M_2^{(2)},M^{(1)},t\big) &= \left|\langle M_2^{(2)}\right|\langle M^{(1)}|\Psi^-(t)\rangle\big|^2 = \frac{1}{4}e^{-2\Gamma t}.
\end{aligned}
\tag{C2}
$$

APPENDIX D: OSCILLATIONS OF NEUTRAL PSEUDOSCALAR MESONS: FUNCTIONS F_N AND THEIR PROPERTIES

In this Appendix we show the explicit form of the functions F_N which enter time-dependent inequality (42). Also we provide in Table II correspondences between these functions and sets of events for neutral pseudoscalar mesons that violate (42),

$$
\begin{aligned}
F_1(x,r,\zeta,\lambda) &= \left|g_+(t)-\frac{1}{2}\left(\frac{q}{p}+\frac{p}{q}\right)g_-(t)\right|^2|g_+(t)|^2e^{2\Gamma t}; \\
F_2(x,r,\zeta,\lambda) &= \left|g_+(t)-\frac{1}{2}\left(\frac{q}{p}+\frac{p}{q}\right)g_-(t)\right|^2e^{-\Delta\Gamma t/2}e^{\Gamma t}; \\
F_3(x,r,\zeta,\lambda) &= \left|g_+(t)+\frac{1}{2}\left(\frac{q}{p}+\frac{p}{q}\right)g_-(t)\right|^2|g_+(t)|^2e^{2\Gamma t}; \\
F_4(x,r,\zeta,\lambda) &= \left|g_+(t)+\frac{1}{2}\left(\frac{q}{p}+\frac{p}{q}\right)g_-(t)\right|^2e^{-\Delta\Gamma t/2}e^{\Gamma t}; \\
F_5(x,r,\zeta,\lambda) &= |g_+(t)|^2e^{-\Delta\Gamma t/2}e^{\Gamma t}; \\
F_6(x,r,\zeta,\lambda) &= |g_+(t)|^2e^{+\Delta\Gamma t/2}e^{\Gamma t}; \\
F_7(x,r,\zeta,\lambda) &= \left|g_+(t)-\frac{1}{2}\left(\frac{q}{p}+\frac{p}{q}\right)g_-(t)\right|^2e^{+\Delta\Gamma t/2}e^{\Gamma t}; \\
F_8(x,r,\zeta,\lambda) &= \left|g_+(t)+\frac{1}{2}\left(\frac{q}{p}+\frac{p}{q}\right)g_-(t)\right|^2e^{+\Delta\Gamma t/2}e^{\Gamma t}.
\end{aligned}
$$

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