Traveling waves in the Euler-Heisenberg electrodynamics

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(Received 22 November 2016; published 14 April 2017)

We examine the possibility of traveling-wave solutions within the nonlinear Euler-Heisenberg electrodynamics. Since this theory resembles in its form the electrodynamics in matter, it is *a priori* not clear if there exist traveling-wave solutions with a new dispersion relation for $\omega(k)$ or if the Euler-Heisenberg theory stringently imposes $\omega = k$ for any arbitrary ansatz $\mathbf{E}(\xi)$ and $\mathbf{B}(\xi)$ with $\xi \equiv \mathbf{k}\cdot\mathbf{r} - \omega t$. We show that the latter scheme applies for the Euler-Heisenberg theory, but point out the possibility of new solutions with $\omega \neq k$ if we go beyond the Euler-Heisenberg theory, allowing strong fields. In case of the Euler-Heisenberg theory the quantum-mechanical effect of the traveling-wave solutions remains in $\hbar$ corrections to the energy density and the Poynting vector.

DOI: 10.1103/PhysRevA.95.043820

I. INTRODUCTION

In the presence of intense electromagnetic fields, quantum electrodynamics predicts that the vacuum behaves like a material medium. This happens since starting from the one-loop level light-light interaction becomes possible for an even number of photons. Due to this quantum effect, the linear Maxwell theory receives nonlinear corrections. If the electromagnetic field does not change too fast and the fields are below the so-called critical field $B_c = \frac{m_e^2}{e}$, then the lowest-order quantum corrections to classical electrodynamics are encoded in the Euler-Heisenberg Lagrangian [1–5]

$$\mathcal{L}_{\rm EH} = a[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2], \tag{1}$$

where

$$a = \frac{e^4}{360\pi^2 m_e^4}. \tag{2}$$

The breakdown of linearity is predicted to give rise to plenty of new effects which do not exist in classical electrodynamics in vacuum. At the optical level the polarization dependent refractive index of the vacuum in the presence of a magnetic or electric field is calculated in [6]. Calculations related to the change of the polarization of a wave due to the birefringence of the vacuum can be found in [6–9]. Other effects include vacuum dichroism [10], second-harmonic generation [11–14], parametric amplification [7,15], quantum vacuum reflection[16,17], slow light [18], photon acceleration in vacuum [19], pulse collapse [20,21], and more (see [22,23] for comprehensive reviews). Examples of waves that are solutions to the Euler-Heisenberg equations but not to the classical Maxwell equations are solitons [24,25] and shock waves [26,27]. Both these solutions are not traveling waves.

Worth mentioning are new developments concerning the equation of motion for a test body with either a charged massive particle giving rise to corrections in the Lorentz force [28], or massless photons who now "feel" the presence of an electromagnetic field and mimic, in a certain sense, the motion of a massless particle in general relativity [29–33]. Such a self-interaction of the electromagnetic quanta or the interaction of the photon with the field raises the question "what is the role of a plane wave within such a theory" or, more generally, what the role of traveling waves is. Comparing the nonlinear electrodynamics with general relativity, where plane waves as solutions exist only in the linearized version of the theory, it is *a priori* not clear what kind of traveling waves exist in the Euler-Heisenberg theory and what happens to the dispersion relation. It is evident that solutions for which the two gauge invariants $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$ are zero are also solutions of the Maxwell theory with $\omega = k$. More generally, keeping $\omega = k$, the Maxwell solution itself allows for nonzero values of the gauge invariants. The first question that we can put forward in such a context is whether these Maxwellian solutions are also solutions in the Euler-Heisenberg theory. We will show that the answer is affirmative if we impose a restriction. The second question of interest is if traveling-wave solutions exist in the Euler-Heisenberg theory which have no connection to the Maxwellian case, i.e., waves with a new dispersion relation, $\omega(k) \neq k$. We present a lengthy proof demonstrating that the only traveling-wave solutions in the Euler-Heisenberg theory are waves with $\omega(k) = k$, i.e., they are of Maxwellian type but with a restriction on the integration constants. Interestingly, this result is not due to some physical principle which would exclude all other solutions. From a purely mathematical point of view traveling waves exist with a new dispersion relation, but we have to reject them on physical grounds as in these solutions the strength of the fields exceeds the critical value allowed in the weak-field approximation. We touch upon the possibility that such a restriction can, in principle, be avoided by going beyond the Euler-Heisenberg theory. As far as the Euler-Heisenberg theory is concerned, the physical effect of traveling-wave solutions is a quantum-mechanical contribution to the energy density of the waves of the Poynting vector.

The paper is organized as follows. In Sec. II we review in full generality the Maxwellian case allowing for nonzero integration constants. In Sec. III we recall the salient features of the Euler-Heisenberg theory. In Sec. IV we present the algebraic equations of the Euler-Heisenberg theory with the traveling waves as an ansatz. Section V probes into the existence of traveling-wave solutions with $\omega = k$. In the Appendix we prove that this is the only viable case. In Sec. VI we discuss a mathematically viable but physically not acceptable solution with $\omega \neq k$. We present the case in order to argue in Sec. VII that a more general Lagrangian

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allowing strong fields would make a similar and analogous solution possible.

II. MAXWELL'S TRAVELING WAVES

The method of obtaining solutions in vacuum for the four Maxwell equations of classical electrodynamics is well known. It starts by taking the Maxwell equations, four linear first-order differential equations that involve the electric and magnetic fields, and combining them to form two waves equations, which are second-order differential equations, and then solving the wave equations. The answer is given by fields of the form

$$\mathbf{E} = \mathbf{E}(\xi), \tag{3}$$

$$\mathbf{B} = \mathbf{B}(\xi), \tag{4}$$

with

$$\xi \equiv \mathbf{k}\cdot\mathbf{r} - \omega t. \tag{5}$$

Waves with such a dependency on the space and time coordinates are called traveling waves.

In this paper we are interested in the traveling-wave solutions in the Euler-Heisenberg electrodynamics. In the Euler-Heisenberg case solving the wave equation is not the most useful approach to the problem. As a preparation for the next section and for the sake of comparison, we present a different way to solve the Maxwell equation in vacuum which does not make use of the wave equation. The same approach will be used later on to deal with the Euler-Heisenberg equations.

The magnetic Gauss, Faraday, electric Gauss, and Ampere-Maxwell laws for classical electrodynamics are

$$\nabla\cdot\mathbf{B} = 0, \tag{6}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{7}$$

$$\nabla\cdot\mathbf{E} = 0, \tag{8}$$

$$\nabla\times\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}. \tag{9}$$

Using a traveling-wave condition as an ansatz, we can write the Maxwell equation as

$$\mathbf{k}\cdot\frac{d\mathbf{B}}{d\xi} = 0, \tag{10}$$

$$\mathbf{k}\times\frac{d\mathbf{E}}{d\xi} = \omega\frac{d\mathbf{B}}{d\xi}, \tag{11}$$

$$\mathbf{k}\cdot\frac{d\mathbf{E}}{d\xi} = 0, \tag{12}$$

$$\mathbf{k}\times\frac{d\mathbf{B}}{d\xi} = \omega\frac{d\mathbf{E}}{d\xi}. \tag{13}$$

These equations can be directly integrated to give the following algebraic relations for the fields:

$$\mathbf{k}\cdot\mathbf{B} = C_B, \tag{14}$$

$$\mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega} + \mathbf{d}_B, \tag{15}$$

$$\mathbf{k}\cdot\mathbf{E} = C_E, \tag{16}$$

$$\mathbf{E} = -\frac{\mathbf{k}\times\mathbf{B}}{\omega} + \mathbf{d}_E. \tag{17}$$

where C_B, C_E, $\mathbf{d}_B$, and $\mathbf{d}_E$ are integration constants.

Multiplying Eqs. (15) and (17) by $\mathbf{k}\cdot$, we see these constants are not independent, but instead obey the relations

$$C_B = \mathbf{k}\cdot\mathbf{d}_B, \tag{18}$$

$$C_E = \mathbf{k}\cdot\mathbf{d}_E. \tag{19}$$

To find further relations among the quantities involved, we now insert Eq. (17) into Eq. (15):

$$\mathbf{B} = \frac{\mathbf{k}\times}{\omega}\left(-\frac{\mathbf{k}\times\mathbf{B}}{\omega} + \mathbf{d}_E\right) + \mathbf{d}_B, \tag{20}$$

and after some rearranging of the terms we obtain

$$\mathbf{B}\left(1 - \frac{k^2}{\omega^2}\right) = -\frac{C_B}{\omega^2}\mathbf{k} + \mathbf{d}_B + \frac{\mathbf{k}\times\mathbf{d_E}}{\omega}. \tag{21}$$

Similarly, we can insert Eq. (15) into Eq. (17) to obtain for the electric field

$$\mathbf{E}\left(1 - \frac{k^2}{\omega^2}\right) = -\frac{C_E}{\omega^2}\mathbf{k} + \mathbf{d}_B - \frac{\mathbf{k}\times\mathbf{d_B}}{\omega}. \tag{22}$$

A similar algebraic equation will emerge in the Euler-Heisenberg theory when we make the traveling-wave ansatz.

The right-hand sides of Eqs. (21) and (22) are constants. Therefore the only way these equations do not lead to trivial constant solutions is to have the well-known dispersion relation for the classical traveling wave $k = \omega$. In this way Eqs. (21) and (22) become algebraic equations that relate the constants which appear in the problem, namely,

$$\mathbf{d}_B = \frac{C_B}{\omega^2}\mathbf{k} - \frac{\mathbf{k}\times\mathbf{d_E}}{\omega}, \tag{23}$$

$$\mathbf{d}_E = \frac{C_E}{\omega^2}\mathbf{k} + \frac{\mathbf{k}\times\mathbf{d_B}}{\omega}. \tag{24}$$

Note that if $\mathbf{d}_B = \mathbf{d}_E = 0$, Eqs. (15) and (17) reduce to

$$\mathbf{B} = \mathbf{k}\times\mathbf{E}, \tag{25}$$

$$\mathbf{E} = -\mathbf{k}\times\mathbf{B}, \tag{26}$$

which is the well-known result that $\mathbf{k}$ and the undulatory parts of $\mathbf{E}$ and $\mathbf{B}$ form a right-handed triplet of orthogonal vectors. This fact together with the dispersion relations are the main results for the classical waves.

Finally, we want to find expressions for the quantities $\mathbf{E}\cdot\mathbf{B}$ and $B^2 - E^2$, which are of great importance for the generalizations of classical electrodynamics. The first one can be obtained by direct computation. Multiplying Eq. (13) by $\mathbf{E}$ or Eq. (15) by $\mathbf{B}$ we get

$$\mathbf{E}\cdot\mathbf{B} = \mathbf{E}\cdot\mathbf{d_B} = \mathbf{d_E}\cdot\mathbf{B}. \tag{27}$$

For $B^2 - E^2$ we can start by squaring Eq. (15):

$$B^2 = \left(\frac{\mathbf{k}\times\mathbf{E}}{\omega} + \mathbf{d}_B\right)^2$$
$$= E^2 - \frac{C_E}{\omega^2} + d_B^2 - 2\mathbf{E}\cdot(\widehat{\mathbf{k}}\times\mathbf{d_B})$$
$$= E^2 + \frac{C_E}{\omega^2} + d_B^2 - \mathbf{E}\cdot\mathbf{d_E}, \tag{28}$$

or we can square Eq. (17) to have

$$E^2 = B^2 - \frac{C_B^2}{\omega^2} + d_E + 2\mathbf{B}\cdot(\widehat{\mathbf{k}}\times\mathbf{d_E})$$
$$= B^2 + \frac{C_B^2}{\omega^2} + d_E^2 - \mathbf{B}\cdot\mathbf{d_B}. \tag{29}$$

With this at hand we can write $B^2 - E^2$ in a few different ways:

$$B^2 - E^2 = \frac{C_E}{\omega^2} - d_B^2 + 2\mathbf{E}\cdot(\widehat{\mathbf{k}}\times\mathbf{d_B})$$
$$= -\frac{C_E}{\omega^2} - d_B^2 + \mathbf{E}\cdot\mathbf{d_E}$$
$$= -\frac{C_B^2}{\omega^2} + d_E + 2\mathbf{B}\cdot(\widehat{\mathbf{k}}\times\mathbf{d_E})$$
$$= \frac{C_B^2}{\omega^2} + d_E^2 - \mathbf{B}\cdot\mathbf{d_B}. \tag{30}$$

As we will encounter a similar situation in the Euler-Heisenberg case, a comment on the integration constants $\mathbf{d_E}$ and $\mathbf{d_B}$ is in order. First, we mention that due to the superposition principle in the linear Maxwell equations we can interpret these constants as part of constant fields which then enter the full solutions. The fact that, e.g., $\mathbf{d_E}$ is part of a constant field can be seen by writing $\mathbf{B} = \mathbf{B}_0(\xi) + \mathbf{d_B}'$ and $\mathbf{E} = \mathbf{E}_0(\xi) + \mathbf{d_E}'$. Using Faraday's law we obtain $\mathbf{B} = \mathbf{k}\times\mathbf{E}_0 + \mathbf{d_B} + \mathbf{k}\times\mathbf{d_E}'$ where $\mathbf{d_B} + \mathbf{k}\times\mathbf{d_E}'$ is the constant magnetic field (a similar consideration can be done for the electric field). Therefore, even if $\mathbf{k}\times\mathbf{d_E}'$ is zero, we are left with a constant magnetic contribution. Thus we can interpret the integration constants as parts of constant fields in which the electromagnetic wave propagates. Secondly, we recall that the photon represented by $\mathbf{A} = \epsilon e^{ikx}$ with $\mathbf{k}\cdot\epsilon = 0$ has two degrees of freedom with respect to $\mathbf{k}$ (two independent polarization vectors ϵ). Classically this is in correspondence with the number of parameters required to specify a plane wave in classical electrodynamics. Keeping the constant fields increases the number of parameters required to specify the classical field since every constant arbitrary vector has three free directions. This, however, does not imply that the degrees of freedom for the photon have changed as a photon which moves in a classical electromagnetic field (and every constant electromagnetic field can be considered as classical; see page 15 of [34]) still has only two polarization modes [7].

There might exist yet another interpretation regarding the integration constants which introduce additional degrees of freedom if we drop our previous interpretation of a wave in constant fields. One such degree of freedom could be accounted for by the breaking of the conformal symmetry at quantum level [35]. A detailed examination of this possibility will be attempted elsewhere.

III. EULER-HEISENBERG ELECTRODYNAMICS

As in the classical electrodynamics, the Euler-Heisenberg theory consists of four equations that determine the evolution of the electric and the magnetic fields. The magnetic Gauss and Faraday laws remain the same as in the classical case, namely,

$$\nabla\cdot\mathbf{B} = 0, \tag{31}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}. \tag{32}$$

These equations serve to define the electromagnetic potentials and are independent of any Lagrangian. The second set of equations, ones that replace the classical electric Gauss and the Ampere-Maxwell laws, are derived after a variation of the Lagrangian [34]. They can be written, in the absence of electric charges and currents, as

$$\nabla\cdot\mathbf{D} = 0, \tag{33}$$

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}, \tag{34}$$

where the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$ are given by

$$\mathbf{D} = \mathbf{E} + 4\pi\frac{\partial\mathcal{L}_{\text{EH}}}{\partial\mathbf{E}}$$
$$= \mathbf{E} + \eta[2\mathbf{E}(E^2 - B^2) + 7\mathbf{B}(\mathbf{E}\cdot\mathbf{B})], \tag{35}$$
$$\mathbf{H} = \mathbf{B} - 4\pi\frac{\partial\mathcal{L}_{\text{EH}}}{\partial\mathbf{B}}$$
$$= \mathbf{B} + \eta[2\mathbf{B}(E^2 - B^2) - 7\mathbf{E}(\mathbf{E}\cdot\mathbf{B})], \tag{36}$$

with

$$\eta = \frac{e^4}{45\pi m_e^4}. \tag{37}$$

As is customary in classical electrodynamics, the four first-order differential equations can be combined to create two second-order wave equations [25]. In this paper we will not use the wave equations; we will focus on the first-order Eqs. (31)–(34).

The symmetric gauge invariant energy-momentum tensor of this theory [36,37] is

$$T_{\mu\nu} = H^{\mu\nu}F_\nu^\alpha - \mathcal{L}g_{\mu\nu}, \tag{38}$$

where the dielectric tensor $H^{\mu\nu}$ is given by

$$H^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}, \tag{39}$$

and can be obtained in a simple way from $F^{\mu\nu}$ by the replacement $E_i \to D_i$ and $B_i \to H_i$.

We follow [38] and write the energy and momentum components of the energy-momentum tensor as

$$T^{00} = A\left(\frac{E^2 + B^2}{8\pi}\right) + \frac{\tau}{4}, \tag{40}$$

$$T^{0i} = A\frac{(\mathbf{E}\times\mathbf{B})_i}{4\pi}, \tag{41}$$

where, for the weak-field Euler-Heisenberg Lagrangian, the dielectric function A and the trace τ are

$$A \equiv 1 + 2\eta(E^2 - B^2), \tag{42}$$

$$\tau \equiv a[(E^2 - B^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2]. \tag{43}$$

IV. TRAVELING WAVES IN EULER-HEISENBERG THEORY

Our procedure is again a straightforward one, i.e., trying the ansatz $\mathbf{E} = \mathbf{E}(\xi)$ and $\mathbf{B} = \mathbf{B}(\xi)$ into the differential Euler-Heisenberg equations. Since the classical dispersion relation is not *a priori* guaranteed to be obeyed, we look for what conditions $\mathbf{k}$ and ω must satisfy. We can integrate the Euler-Heisenberg equations in the same way as we did for the Maxwell equations in Sec. I. We obtain

$$\mathbf{k}\cdot\mathbf{B} = C_B, \tag{44}$$

$$\mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega} + \mathbf{d_B}, \tag{45}$$

$$\mathbf{k}\cdot\mathbf{D} = C_D, \tag{46}$$

$$\mathbf{D} = -\frac{\mathbf{k}\times\mathbf{H}}{\omega} + \mathbf{d_D}, \tag{47}$$

where C_B, C_D, $\mathbf{d_D}$, and $\mathbf{d_B}$ are constants related by taking the scalar product of Eqs. (45) and (47) with $\mathbf{k}$:

$$C_B = \mathbf{k}\cdot\mathbf{d}_B, \tag{48}$$

$$C_D = \mathbf{k}\cdot\mathbf{d}_D. \tag{49}$$

We look for the Euler-Heisenberg equivalent of Eq. (22). Let us start by noticing that the auxiliary fields can be written as

$$\mathbf{D} = A\mathbf{E} + 7\eta(\mathbf{E}\cdot\mathbf{d_B})\mathbf{B}, \tag{50}$$

$$\mathbf{H} = A\mathbf{B} - 7\eta(\mathbf{E}\cdot\mathbf{d_B})\mathbf{E}, \tag{51}$$

where A is the dielectric function defined in Eq. (42). With Eqs. (50) and (51) Eq. (47) can be written as

$$A\mathbf{E} + 7\eta(\mathbf{E}\cdot\mathbf{d_B})\mathbf{d_B} = -A\frac{\mathbf{k}}{\omega}\times\mathbf{B} + \mathbf{d_D}, \tag{52}$$

where we have used Eq. (45) to transform the terms $7\eta(\mathbf{E}\cdot\mathbf{d_B})\mathbf{B}$ and $7\eta(\mathbf{E}\cdot\mathbf{d_B})\frac{\mathbf{k}}{\omega}\times\mathbf{E}$ into $7\eta(\mathbf{E}\cdot\mathbf{d_B})\mathbf{d_B}$. Replacing $\mathbf{B}$ using Eq. (45) we arrive at an algebraic equation in which only the electric field appears:

$$A\left(1 - \frac{k^2}{\omega^2}\right)\mathbf{E} = \mathbf{d_D} - A\left\{\frac{(\mathbf{k}\cdot\mathbf{E})}{\omega^2}\mathbf{k} + \frac{\mathbf{k}\times\mathbf{d_B}}{\omega}\right\} - 7\eta\mathbf{d_B}(\mathbf{E}\cdot\mathbf{d_B}). \tag{53}$$

The dielectric function can also be put solely in terms of $\mathbf{E}$ as

$$A = 1 + 2\eta\left[E^2\left(1 - \frac{k^2}{\omega^2}\right) + \frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2} + \frac{2\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})}{\omega} - d_B^2\right]. \tag{54}$$

Let us note that Eq. (53) reduces to Eq. (22) in the limit $\eta \to 0$, as it should be.

V. MAXWELLIAN CASE ($k = \omega$) IN EULER-HEISENBERG THEORY

It is well known that some solutions of the Maxwell equations are also solutions of the Euler-Heisenberg equations [6]. The simplest examples are waves with $E^2 - B^2 = \mathbf{E}\cdot\mathbf{B} = 0$, where the Euler-Heisenberg equations trivially reduce to the classical Maxwell ones (physically this corresponds to the fact that in QED a single free photon can propagate undisturbed [41]). We shall now see that this fact can be obtained directly from Eq. (53). Looking for Maxwellian solutions we put $k = \omega$ into Eq. (53) to obtain

$$0 = \mathbf{d_D} - A(\widehat{\mathbf{k}}\cdot\mathbf{E})\widehat{\mathbf{k}} - A\widehat{\mathbf{k}}\times\mathbf{d_B} - 7\eta\mathbf{d_B}(\mathbf{E}\cdot\mathbf{d_B}). \tag{55}$$

Let us first assume that $\mathbf{d_B}$ is not parallel to $\widehat{\mathbf{k}}$, then we can take the scalar product of Eq. (55) with $\widehat{\mathbf{k}}$, $\mathbf{d_B}$, and $\widehat{\mathbf{k}}\times\mathbf{d_B}$ (which we take as basis) to obtain the following three equations:

$$0 = \mathbf{d_D}\cdot\widehat{\mathbf{k}} - A(\widehat{\mathbf{k}}\cdot\mathbf{E}) - 7\eta(\widehat{\mathbf{k}}\cdot\mathbf{d_B})(\mathbf{E}\cdot\mathbf{d_B}), \tag{56}$$

$$0 = \mathbf{d_D}\cdot\mathbf{d_B} - A(\widehat{\mathbf{k}}\cdot\mathbf{E})(\widehat{\mathbf{k}}\cdot\mathbf{d_B}) - 7\eta d_B^2(\mathbf{E}\cdot\mathbf{d_B}), \tag{57}$$

$$0 = \mathbf{d_D}\cdot\widehat{\mathbf{k}}\times\mathbf{d_B} - A\left[d_B^2 - (\widehat{\mathbf{k}}\cdot\mathbf{d_B})^2\right]. \tag{58}$$

From Eq. (58) it follows that $A = \text{const}$. Meanwhile, Eqs. (56) and (57) have $\widehat{\mathbf{k}}\cdot\mathbf{E}$ and $\mathbf{E}\cdot\mathbf{d_B}$ as unknowns. Since Eqs. (56) and (57) are algebraically independent (due to our choice $\widehat{\mathbf{k}}\times\mathbf{d_B} \neq 0$), we can solve $\widehat{\mathbf{k}}\cdot\mathbf{E}$ and $\mathbf{E}\cdot\mathbf{d_B}$ in terms of constants. Finally, from Eq. (54) $\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})$ is also a constant. We have a case where there is no undulatory solution at all.

If, on the other hand, $\mathbf{k}$ and $\mathbf{d_B}$ are parallel then Eq. (55) reduces to

$$0 = \mathbf{d_D} - \left(A - 7\eta d_B^2\right)(\widehat{\mathbf{k}}\cdot\mathbf{E})\widehat{\mathbf{k}}. \tag{59}$$

Equation (59) tells us that $\mathbf{d_D}$ has to be parallel to $\widehat{\mathbf{k}}$. Furthermore, using Eq. (54) we can write for A

$$A = 1 + 2\eta\left[(\widehat{\mathbf{k}}\cdot\mathbf{E})^2 - d_B^2\right]. \tag{60}$$

Then Eq. (59) together with Eq. (60) implies that $\widehat{\mathbf{k}}\cdot\mathbf{E}$ and A are constants. This still leaves us with enough freedom for the components of $\mathbf{E}$ orthogonal to $\widehat{\mathbf{k}}$. Since $\widehat{\mathbf{k}}\cdot\mathbf{E}$ and A are constants, it can be checked that the Euler-Heisenberg equations reduce to the Maxwell equations. For example, the following set,

$$\mathbf{E} = \mathbf{E}_0(\xi) + d_E\widehat{\mathbf{k}}, \tag{61}$$

$$\mathbf{B} = \mathbf{B}_0(\xi) + d_B\widehat{\mathbf{k}}, \tag{62}$$

with $\widehat{\mathbf{k}}\cdot\mathbf{E}_0 = \widehat{\mathbf{k}}\cdot\mathbf{B}_0 = 0$ and $\mathbf{B}_0 = \widehat{\mathbf{k}}\times\mathbf{E}_0$, is a solution of both the Maxwell and Euler-Heisenberg equations. Notice, however, a subtle difference. Whereas $\mathbf{d_B}$ was an arbitrary constant, in the Euler-Heisenberg theory its direction is fixed by $\mathbf{d_B} \propto \hat{\mathbf{k}}$.

At the end of Sec. II we have commented on the interpretation of integration constants in the Maxwell case. In the Euler-Heisenberg theory constant fields are also solutions of the corresponding equations. What we do not have here is a general superposition principle due to the nonlinearities of the equations. Interpreting the constants in Eqs. (61) and (62) as constant fields, we could say that these equations represent a restricted superposition principle where a traveling wave and constant field can be added together to form a new solution if and only if the direction of the constant field is parallel to **k**. An analogous situation exists for two or more waves, in the sense that they can be added together to form a new solution to the Euler-Heisenberg equations only if they travel in the same direction [41]. The physical interpretation given to this last effect is that the photons which travel in the same direction do not scatter from each other. We can then interpret Eqs. (61) and (62) as a photon propagating undisturbed through a constant electromagnetic field if and only if the photon's motion is parallel to the direction of the background field.

Although waves of Eqs. (61) and (62) are also present in the classical theory, their energy and momentum content are different in the Euler-Heisenberg theory. For example, using Eq. (41) we can write their momentum components as

$$T^{0i} = \left[1 + 2\eta\left(d_E^2 - d_B^2\right)\right]\frac{(\mathbf{E}\times\mathbf{B})_i}{4\pi}. \tag{63}$$

We can see from Eq. (63) that the photon-photon interaction codified in the Euler-Heisenberg Lagrangian implies that the wave's momentum density is slightly bigger when compared to the classical Poynting vector $T^{0i}_{\text{Maxwell}} = \frac{(\mathbf{E}\times\mathbf{B})_i}{4\pi}$, if d_E^2 is bigger than d_B^2 and vice versa.

The energy density is also changed from the classical $T^{00}_{\text{Maxwell}} = \frac{E^2+B^2}{8\pi}$ to

$$T^{00} = \left[1 + 2\eta\left(d_E^2 - d_B^2\right)\right]\left(\frac{E^2+B^2}{8\pi}\right) + \frac{a}{4}\left[\left(d_E^2 - d_B^2\right)^2 + 7(d_E d_B)^2\right]. \tag{64}$$

The new terms in the energy density and the Poynting vector proportional to η and a are quantum mechanical in origin. They are small unless the fields become very strong, but that takes us outside the weak-field limit of the Euler-Heisenberg Lagrangian.

In the Appendix we examine all cases with $\omega \neq k$ and $A \neq 0$ and show that they lead to trivial constant field solutions. The proof makes use of the fact that we can use the integration constant vectors and **k** (or some other combinations involving cross products) as basis and decompose the electric and magnetic fields in terms of projections in this basis.

VI. OFF LIGHT CONE WAVES ($A = 0$)

There is a formal way to invalidate the proof presented in the Appendix (this proof demonstrates that no traveling-wave solutions with $\omega \neq k$ exist in the Euler-Heisenberg theory). Indeed it suffices to set the dielectric function A to zero. However, it is important to bring to attention that $A = 0$ is physically not viable. Indeed, such an equation would result in strong fields violating the restriction on the theory. On the other hand, if the weak-field restriction is the only obstacle to obtain physically valid solutions, it makes sense to generalize the $A = 0$ condition to more general Lagrangians where the weak-field restriction is not implemented. This seems, in principle, possible as the Euler-Heisenberg Lagrangian (1) is a weak-field version of a more general one. As shown below, $A = 0$ goes hand in hand with $\omega \neq k$, i.e., we have traveling-wave solutions off the light cone.

For these reasons it is illustrative to consider here the $A = 0$ case as in the more general Lagrangian the steps would be similar. Taking $A = 0$ in the algebraic Eq. (53) gives us the conditions

$$1 + 2\eta(E^2 - B^2) = 0, \tag{65}$$

$$\mathbf{E}\cdot\mathbf{B} = \mathbf{E}\cdot\mathbf{d_B} = \beta = \text{const.} \tag{66}$$

We will call "off light cone waves" the waves that obey conditions (65) and (66).

It is easy to check that conditions (65) and (66) give us a solution to the full set of Euler-Heisenberg equations. Using Eqs. (65) and (66) the auxiliary fields become

$$\mathbf{D} = 7\eta\beta\mathbf{B}, \tag{67}$$

$$\mathbf{H} = -7\eta\beta\mathbf{E}, \tag{68}$$

and we have the strange case where the vector **D** is associated with the magnetic field while the vector **H** is associated with the electric field, the opposite of what one would usually expect in electrodynamics (see, however, [39]).

With the vectors (67) and (68), the modified electric Gauss law (33) and the Ampere-Maxwell law (34) become the classical magnetic Gauss and Faraday laws:

$$7\eta\beta\nabla\cdot\mathbf{B} = 0, \tag{69}$$

$$7\eta\beta\nabla\times\mathbf{E} = -7\eta\beta\frac{\partial\mathbf{B}}{\partial t}. \tag{70}$$

Notice that choosing $\beta = 0$ we end up with $\mathbf{D} = \mathbf{H} = 0$. Provided $A = 0$, this configuration is mathematically a solution of the Euler-Heisenberg equations.

Finally, the condition (65) gives us an intensity dependent dispersion relation. Indeed, using Eq. (54) we can write

$$0 = 1 + 2\eta\left[E^2\left(1 - \frac{k^2}{\omega^2}\right) + \frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2} + \frac{2\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})}{\omega} - d_B^2\right]. \tag{71}$$

As an example, consider the fields

$$\mathbf{E} = E_0[\cos(\xi)\widehat{\mathbf{x}} + \sin(\xi)\widehat{\mathbf{y}}], \tag{72}$$

$$\mathbf{B} = \frac{kE_0}{\omega}[-\sin(\xi)\widehat{\mathbf{x}} + \cos(\xi)]\widehat{\mathbf{y}}, \tag{73}$$

with $\mathbf{k} = \widehat{\mathbf{z}}$. The fields form an off light cone wave solution to the Euler-Heisenberg equations as long as Eq. (71) is true. Since for this example $d_B^2 = \mathbf{k}\cdot\mathbf{E} = 0$, we can calculate a dispersion relation of the form

$$\frac{k^2}{\omega^2} = 1 + \frac{1}{2\eta E_0^2}. \tag{74}$$

Though unusual, the relevant energy-momentum components would simply read

$$T^{00} = \frac{\tau}{4}, \tag{75}$$

$$T^{0i} = 0. \tag{76}$$

However, as previously stated, the off light cone waves are not well-defined physical solutions. The vanishing of the dielectric function (65) implies fields stronger than allowed by the weak-field approximation of the Euler-Heisenberg Lagrangian, i.e.,

$$\frac{B^2}{\eta} > 1, \tag{77}$$

whereas physically acceptable fields should range below the critical limit $B_c = \frac{m_e^2}{e}$.

However, a more general Lagrangian, like the full Euler-Heisenberg case, can lift this restriction.

VII. MORE GENERAL LAGRANGIAN

The Euler-Heisenberg Lagrangian (1) is not the only proposed modification to the laws of classical electrodynamics. Indeed, we could consider the full version of the nonlinear electrodynamics arising from quantum corrections. To avoid the problem of pair production in such a case we could hypothetically consider an electric field below the pair production threshold and a strong magnetic field.

Let the correction to the Maxwell Lagrangian be given by the nonlinear Lagrangian:

$$\mathcal{L}_{\rm NL} = \mathcal{L}_{\rm NL}(\mathcal{F},\mathcal{G}^2), \tag{78}$$

where the electromagnetic invariants are given by

$$\mathcal{F} = \frac{B^2 - E^2}{2}, \tag{79}$$

$$\mathcal{G} = \mathbf{E}\cdot\mathbf{B}. \tag{80}$$

The pseudoscalar $\mathcal{G}$ always appears squared in the Lagrangian to preserve the parity invariance of the theory.

In a generic form, the auxiliary fields are

$$\begin{aligned}\mathbf{D} &= \mathbf{E} + 4\pi\frac{\partial\mathcal{L}_{\rm NL}}{\partial\mathbf{E}} \\ &= \mathbf{E} + 4\pi\frac{\partial\mathcal{L}_{\rm NL}}{\partial\mathcal{F}}\frac{\partial\mathcal{F}}{\partial\mathbf{E}} + 4\pi\frac{\partial\mathcal{L}_{\rm NL}}{\partial\mathcal{G}^2}\frac{\partial\mathcal{G}^2}{\partial\mathbf{E}} \\ &= \mathbf{E}\left(1 - 4\pi\frac{\partial\mathcal{L}_{\rm NL}}{\partial\mathcal{F}}\right) + 8\pi\frac{\partial\mathcal{L}_{\rm NL}}{\partial\mathcal{G}^2}\mathbf{B}(\mathbf{E}\cdot\mathbf{B}),\end{aligned} \tag{81}$$

$$\mathbf{H} = \mathbf{B}\left(1 - 4\pi\frac{\partial\mathcal{L}_{\rm NL}}{\partial\mathcal{F}}\right) - 8\pi\frac{\partial\mathcal{L}_{\rm NL}}{\partial\mathcal{G}^2}\mathbf{E}(\mathbf{E}\cdot\mathbf{B}). \tag{82}$$

We can again make the traveling-wave ansatz and look for solutions of the modified Maxwell equations (31)–(34).

Let us define $A \equiv 1 - 4\pi\frac{\partial\mathcal{L}_{\rm NL}}{\partial\mathcal{F}}$. Remembering that for traveling waves $\mathcal{G} = \mathbf{E}\cdot\mathbf{B} = \mathbf{E}\cdot\mathbf{d_B}$, we can see that the conditions $A = 0$ and $\mathbf{E}\cdot\mathbf{d_B} = 0$ guarantee vanishing auxiliary fields

$$\mathbf{D} = \mathbf{H} = 0, \tag{83}$$

and this is an immediate solution to the modified Maxwell equations. This generalizes the situation discussed in the last section without violating the weak-field restriction. Since the full Lagrangian is given in terms of an integral, it is difficult to derive analytical expressions. Moreover, we speculate that as in Sec. VI this solution would lead to physically realizable waves with a new dispersion relation. We leave the details to a future investigation.

We mention here that in [38] the dielectric function has been calculated to all orders for strong fields analytically up to an integral for $\mathbf{E} = 0$, $\mathbf{B} \neq 0$ and vice versa for $\mathbf{E} \neq 0$ and $\mathbf{B} = 0$. However, if in the Maxwell Lagrangian we also set, e.g., $\mathbf{E} = 0$ we would not obtain traveling-wave solutions and end up with static cases. A generalization of the results in [38] would be required.

ACKNOWLEDGMENTS

We thank the Faculty of Science at the Universidad de los Andes and the administrative department of Science, Technology, and Innovation of Colombia (Colciencias) for financial support.

APPENDIX

In this Appendix we investigate all cases of different choices of the integration constants and $\mathbf{k}$ assuming always $\omega \neq k$. We rely on the following equations derived in the main text:

$$\begin{aligned}A\left(1 - \frac{k^2}{\omega^2}\right)\mathbf{E} = \mathbf{d_D} - A\left\{\frac{(\mathbf{k}\cdot\mathbf{E})}{\omega^2}\mathbf{k} + \frac{\mathbf{k}\times\mathbf{d_B}}{\omega}\right\} \\ - 7\eta\mathbf{d_B}(\mathbf{E}\cdot\mathbf{d_B}),\end{aligned} \tag{A1}$$

$$A = 1+\eta\left[E^2\left(1 - \frac{k^2}{\omega^2}\right) + \frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2} + \frac{2\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})}{\omega} - d_B^2\right]. \tag{A2}$$

Case 1: $\mathbf{d_D}\cdot\mathbf{d_B} = \mathbf{k}\cdot\mathbf{d_B} = \mathbf{k}\cdot\mathbf{d_D} = 0$. We first analyze the case where $\mathbf{k}$, $\mathbf{d_B}$, and $\mathbf{d_D}$ form an orthogonal basis. Multiplying Eq. (A1) by $\mathbf{k}$, $\mathbf{d_B}$, and $\mathbf{d_D}$ we get, respectively,

$$A(\mathbf{k}\cdot\mathbf{E}) = 0, \tag{A3}$$

$$A\left(1 - \frac{k^2}{\omega^2}\right) = -7\eta d_B^2, \tag{A4}$$

$$A\left(1 - \frac{k^2}{\omega^2}\right)(\mathbf{E}\cdot\mathbf{d_D}) = d_D^2 - A\mathbf{d_D}\cdot\left(\frac{\mathbf{k}\times\mathbf{d_B}}{\omega}\right). \tag{A5}$$

We see from Eq. (A4) that A is given by a constant, hence we infer from Eq. (A3) that $\mathbf{k}\cdot\mathbf{E} = 0$ and from Eq. (A5) we get that $\mathbf{E}\cdot\mathbf{d_D}$ is given in terms of constants. As $\mathbf{k}$, $\mathbf{d_B}$, and $\mathbf{d_D}$ form an orthogonal basis, E^2 can be written as

$$E^2 = (\mathbf{E}\cdot\widehat{\mathbf{d_B}})^2 + (\mathbf{E}\cdot\widehat{\mathbf{d_D}})^2. \tag{A6}$$

Since $\mathbf{E}\cdot\mathbf{d_D}$ and A are constants, when we insert Eq. (A6) into Eq. (A5) we find that $\mathbf{E}\cdot\mathbf{d_B}$ is a constant. This case allows only trivial constants solutions.

Case 2: $\mathbf{k}\cdot\mathbf{d_B}=\mathbf{k}\cdot\mathbf{d_D}=0$ *and* $\mathbf{d_D}\cdot\mathbf{d_B}\neq 0$. Taking the scalar product of Eq. (A1) with $\mathbf{k}$, $\mathbf{d_B}$, $\mathbf{d_D}$, $\mathbf{E}$, and $\mathbf{k}\times\mathbf{d_B}$ we obtain, respectively,

$$A(\mathbf{k}\cdot\mathbf{E})=0, \tag{A7}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E}\cdot\mathbf{d_B}=\mathbf{d_D}\cdot\mathbf{d_B}-7\eta d_B^2, \tag{A8}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)(\mathbf{E}\cdot\mathbf{d_D})=d_D^2-A\mathbf{d_D}\cdot\left(\frac{\mathbf{k}\times\mathbf{d_B}}{\omega}\right)-7\eta\mathbf{d_D}\cdot\mathbf{d_B}(\mathbf{E}\cdot\mathbf{d_B}), \tag{A9}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})=\mathbf{d_D}\cdot(\mathbf{k}\times\mathbf{d_B})-A(\mathbf{k}\times\mathbf{d_B})^2. \tag{A10}$$

Now we take a look at the projection. First, if $A\neq 0$ then from Eq. (A7) $\mathbf{k}\cdot\mathbf{E}=0$. Since $\mathbf{d_D}$ is orthogonal to $\mathbf{k}$, we can write

$$\mathbf{d_D}=a\mathbf{d_B}+b(\mathbf{k}\times\mathbf{d_B}), \tag{A11}$$

for some constant numbers a and b. Then,

$$\mathbf{E}\cdot\mathbf{d_D}=a\mathbf{E}\cdot\mathbf{d_B}+b\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B}). \tag{A12}$$

We can insert Eq. (A12) into Eq. (A9) to obtain

$$A\left(1-\frac{k^2}{\omega^2}\right)a\mathbf{E}\cdot\mathbf{d_B}+bA\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})=d_D^2-A\mathbf{d_D}\cdot\left(\frac{\mathbf{k}\times\mathbf{d_B}}{\omega}\right)-7\eta\mathbf{d_D}\cdot\mathbf{d_B}(\mathbf{E}\cdot\mathbf{d_B}). \tag{A13}$$

We can use now Eqs. (A8) and (A10) in Eq. (A13) to transform its left-hand side and obtain

$$a\left(\mathbf{d_D}\cdot\mathbf{d_B}-7\eta d_B^2\right)+b[\mathbf{d_D}\cdot(\mathbf{k}\times\mathbf{d_B})-A(\mathbf{k}\times\mathbf{d_B})^2]=d_D^2-A\mathbf{d_D}\cdot\left(\frac{\mathbf{k}\times\mathbf{d_B}}{\omega}\right)-7\eta\mathbf{d_D}\cdot\mathbf{d_B}(\mathbf{E}\cdot\mathbf{d_B}). \tag{A14}$$

Our next step consists in using Eq. (A8) to write Eq. (A4) only in terms of $\mathbf{d_B}\cdot\mathbf{E}$. The final equations read

$$(\mathbf{E}\cdot\mathbf{d_B})\left(\mathbf{d_D}\cdot\mathbf{d_B}-7\eta d_B^2\right)+b\left[(\mathbf{E}\cdot\mathbf{d_B})\mathbf{d_D}\cdot(\mathbf{k}\times\mathbf{d_B})-\frac{\left(\mathbf{d_D}\cdot\mathbf{d_B}-7\eta d_B^2\right)}{1-\frac{k^2}{\omega^2}}(\mathbf{k}\times\mathbf{d_B})^2\right]$$
$$=(\mathbf{E}\cdot\mathbf{d_B})d_D^2-\frac{\left(\mathbf{d_D}\cdot\mathbf{d_B}-7\eta d_B^2\right)}{1-\frac{k^2}{\omega^2}}\mathbf{d_D}\cdot\left(\frac{\mathbf{k}\times\mathbf{d_B}}{\omega}\right)-7\eta\mathbf{d_D}\cdot\mathbf{d_B}(\mathbf{E}\cdot\mathbf{d_B})^2. \tag{A15}$$

Equation (A15) is a polynomial equation with constant coefficients. Its solution gives $\mathbf{E}\cdot\mathbf{d_B}$ in terms of constants. The only way to avoid this conclusion is to have all the coefficients of each power in $\mathbf{E}\cdot\mathbf{d_B}$ to be zero individually. But it is impossible for the coefficient of the $(\mathbf{E}\cdot\mathbf{d_B})^2$ to be zero by the very same assumption we used at the beginning of this case.

Case 3: $\mathbf{d_D}\cdot\mathbf{d_B}=\mathbf{k}\cdot\mathbf{d_B}=0$ *and* $\mathbf{k}\cdot\mathbf{d_D}\neq 0$. Multiplying Eq. (A1) by $\mathbf{k}$, $\mathbf{d_B}$, and $\mathbf{d_D}$ we get, respectively,

$$A(\mathbf{E}\cdot\mathbf{k})=\mathbf{d_D}\cdot\mathbf{k}, \tag{A16}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)=-7\eta d_B^2, \tag{A17}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)(\mathbf{E}\cdot\mathbf{d_D})=d_D^2-A\left\{\frac{(\mathbf{E}\cdot\mathbf{k})}{\omega^2}\mathbf{k}\cdot\mathbf{d_D}-\frac{\mathbf{d_D}\cdot(\mathbf{k}\times\mathbf{d_B})}{\omega}\right\}. \tag{A18}$$

We immediately obtain from Eq. (A17) that A is a constant and we can use this fact in Eq. (A16) to find that $(\mathbf{E}\cdot\mathbf{k})$ is a constant. These two results together with Eq. (A18) tell us that $\mathbf{E}\cdot\mathbf{d_D}$ is a constant.

As $\mathbf{d_B}$ is orthogonal to $\mathbf{k}$ and $\mathbf{d_D}$ we can write

$$E^2=(\mathbf{E}\cdot\mathbf{d_B})^2+F((\mathbf{E}\cdot\mathbf{k}),(\mathbf{E}\cdot\mathbf{d_D})) \tag{A19}$$

where $F((\mathbf{E}\cdot\mathbf{k}),(\mathbf{E}\cdot\mathbf{d_D}))$ is just a constant. We now insert Eq. (A19) into Eq. (A2) to arrive at an expression for A:

$$A=1+\eta\left\{[(\mathbf{E}\cdot\mathbf{d_B})^2+F]\left(1-\frac{k^2}{\omega^2}\right)+\frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2}+\frac{2\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})}{\omega}-d_B^2\right\}. \tag{A20}$$

The expression $\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})^2$ is a constant since it can be written in terms of $(\mathbf{E}\cdot\mathbf{k})$ and $\mathbf{E}\cdot\mathbf{d_D}$. Therefore using Eq. (A20) we reach the conclusion that $\mathbf{E}\cdot\mathbf{d_B}$ is also a constant.

Case 4: $\mathbf{d_D}\cdot\mathbf{d_B}=\mathbf{k}\cdot\mathbf{d_D}=0$ *and* $\mathbf{k}\cdot\mathbf{d_B}\neq 0$. First note that $\mathbf{k}\times\mathbf{d_B}$ is proportional to $\mathbf{d_D}$. Hence we will write $\mathbf{k}\times\mathbf{d_B}=a\mathbf{d_D}$.

The scalar product of Eq. (A1) with $\mathbf{k}$, $\mathbf{d_B}$, and $\mathbf{d_D}$ gives, respectively,

$$A(\mathbf{E}\cdot\mathbf{k})=-7\eta(\mathbf{k}\cdot\mathbf{d_B})(\mathbf{E}\cdot\mathbf{d_B}), \tag{A21}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)(\mathbf{E}\cdot\mathbf{d_B})=-A\frac{(\mathbf{k}\cdot\mathbf{E})}{\omega^2}(\mathbf{k}\cdot\mathbf{d_B})-7\eta d_B^2(\mathbf{E}\cdot\mathbf{d_B}), \tag{A22}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)(\mathbf{E}\cdot\mathbf{d_D})=d_D^2-\frac{A}{\omega}ad_D^2, \tag{A23}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)E^2=\mathbf{E}\cdot\mathbf{d_D}-A\left\{\frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2}\mathbf{k}+\frac{a}{\omega}\mathbf{E}\cdot\mathbf{d_D}\right\}-7\eta(\mathbf{E}\cdot\mathbf{d_B})^2. \tag{A24}$$

Inserting Eq. (A21) into Eq. (A22) leads to

$$A\left(1-\frac{k^2}{\omega^2}\right)=\frac{7\eta}{\omega^2}(\mathbf{k}\cdot\mathbf{d_B})^2-7\eta d_B^2, \tag{A25}$$

and it follows that A is a constant. By virtue of Eq. (A23) this implies that $\mathbf{E}\cdot\mathbf{d_D}$ is a constant.

By using Eq. (A2) to write

$$\left(1-\frac{k^2}{\omega^2}\right)E^2=\frac{A-1}{\eta}-\frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2}-2\frac{a}{\omega}\mathbf{E}\cdot\mathbf{d_D}+d_B^2. \tag{A26}$$

and inserting Eq. (A26) into Eq. (A24),

$$A\left(\frac{A-1}{\eta}+d_B^2-\frac{a}{\omega}\mathbf{E}\cdot\mathbf{d_D}\right)=\mathbf{E}\cdot\mathbf{d_D}-7\eta(\mathbf{E}\cdot\mathbf{d_B})^2, \tag{A27}$$

we conclude that $\mathbf{E}\cdot\mathbf{d_B}$ is a constant.

Case 5: $d_D=0$ but $d_B\neq 0$. The scalar product of Eq. (A1) with $\mathbf{k}$ and $\mathbf{d_B}\cdot(\mathbf{k}\times\mathbf{d_B})$ results in the following equations:

$$0=A(\mathbf{k}\cdot\mathbf{E})+7\eta(\mathbf{k}\cdot\mathbf{d_B})(\mathbf{E}\cdot\mathbf{d_B}), \tag{A28}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)(\mathbf{E}\cdot\mathbf{d_B})=A\frac{(\mathbf{E}\cdot\mathbf{k})}{\omega^2}\mathbf{k}\cdot\mathbf{d_B}-7\eta d_B^2(\mathbf{E}\cdot\mathbf{d_B}), \tag{A29}$$

$$\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})=\frac{1}{\omega}(\mathbf{k}\times\mathbf{d_B})^2, \tag{A30}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)E^2=-A\left\{\frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2}-\frac{\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})}{\omega}\right\}-7\eta(\mathbf{E}\cdot\mathbf{d_B})^2. \tag{A31}$$

We can solve for $A(\mathbf{k}\cdot\mathbf{E})$ in Eq. (A28) and insert it into Eq. (A29) to obtain

$$A\left(1-\frac{k^2}{\omega^2}\right)=\frac{7\eta}{\omega^2}(\mathbf{k}\cdot\mathbf{d_B})\mathbf{k}\cdot\mathbf{d_B}-7\eta d_B^2. \tag{A32}$$

Again we arrive at the conclusion that A has to be a constant. Moreover, we can read directly from Eq. (A30) that $\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})$ is a constant. From Eq. (A2) we can write

$$\left(1-\frac{k^2}{\omega^2}\right)E^2=\frac{A-1}{\eta}-\frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2}-2\frac{1}{\omega}\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})+d_B^2, \tag{A33}$$

and inserting Eq. (A32) into Eq. (A31)

$$-7\eta(\mathbf{E}\cdot\mathbf{d_B})^2=A\left[\left(\frac{A-1}{\eta}\right)-\frac{1}{\omega}\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})+d_B^2\right]. \tag{A34}$$

Independent of the numerical value of the right-hand side, we easily see that $\mathbf{E}\cdot\mathbf{d_B}$ is a constant.

Case 6: $\mathbf{d_B}=d_B\mathbf{k}$ and $\mathbf{d_D}\neq 0$. In this case Eq. (A1) reduces to

$$A\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E}=\mathbf{d_D}-\left\{\frac{A}{\omega^2}-7\eta d_B^2\right\}(\mathbf{k}\cdot\mathbf{E})\mathbf{k}. \tag{A35}$$

We can choose $\mathbf{k}$, $\mathbf{k}\times\mathbf{d_D}$, and $\mathbf{k}\times(\mathbf{k}\times\mathbf{d_D})$ as a basis. To make the notation more concise, let us define $\mathbf{k}_\perp=\mathbf{k}\times(\mathbf{k}\times\mathbf{d_D})$. It is clear from Eq. (A35) that $\mathbf{E}$ does not have components in the $\mathbf{k}\times\mathbf{d_D}$ direction, and hence $\mathbf{E}$ can be written in the following form:

$$\mathbf{E}=(\widehat{\mathbf{k}}\cdot\mathbf{E})\widehat{\mathbf{k}}+(\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})\widehat{\mathbf{k}}_\perp. \tag{A36}$$

By the same token we have

$$\mathbf{d_D}=a\widehat{\mathbf{k}}+b\widehat{\mathbf{k}}_\perp \tag{A37}$$

for some numbers a and b.

Equation (A36) allows us to write

$$E^2=(\widehat{\mathbf{k}}\cdot\mathbf{E})^2+(\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})^2, \tag{A38}$$

and therefore

$$A=1+\eta\left\{[(\widehat{\mathbf{k}}\cdot\mathbf{E})^2+(\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})^2]\left(1-\frac{k^2}{\omega^2}\right)+\frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2}-d_B^2\right\}. \tag{A39}$$

The scalar product of Eq. (A35) with $\widehat{\mathbf{k}}$ and $\widehat{\mathbf{k}}_\perp$ leads to the following set of equations:

$$\left(1+\eta\left\{[(\widehat{\mathbf{k}}\cdot\mathbf{E})^2+(\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})^2]\left(1-\frac{k^2}{\omega^2}\right)+\frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2}-d_B^2\right\}\right)(\widehat{\mathbf{k}}\cdot\mathbf{E})=a-7\eta d_B^2(\widehat{\mathbf{k}}\cdot\mathbf{E})k^2, \tag{A40}$$

$$\left(1+\eta\left\{[(\widehat{\mathbf{k}}\cdot\mathbf{E})^2+(\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})^2]\left(1-\frac{k^2}{\omega^2}\right)+\frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2}-d_B^2\right\}\right)\left(1-\frac{k^2}{\omega^2}\right)(\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})=b. \tag{A41}$$

Equations (A40) and (A41) are algebraic independent polynomials for any (nonzero) value of the constants. This means that we cannot choose any relation among $k,d_B,a,$ and b to make Eq. (A40) proportional to Eq. (A41). By Bézout's theorem [40] the systems (A40) and (A41) have a finite number of solutions. These solutions will be functions of the coefficients of the polynomials, i.e., of constants. Therefore we have trivial constant solutions at hand.

On the other hand, if $\mathbf{d_B}$ is parallel to $\mathbf{k}$, then Eq. (A33) reduces further to

$$A\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E}=-\left\{\frac{A}{\omega^2}(\mathbf{k}\cdot\mathbf{E})-7\eta d_B^2(\mathbf{k}\cdot\mathbf{E})-d_D\right\}\mathbf{k}. \tag{A42}$$

There are two ways to solve Eq. (A42). The first is letting $k=\omega$, which leads to the condition $\mathbf{k}\cdot\mathbf{E}=$ const, which is identical to the classical Gauss law and also leads to a classical solution to the Maxwell equations. The other solution is to set $A = 0$, which also leads to $\mathbf{k}\cdot\mathbf{E} = \text{const}$, but we know from Sec. VI that this kind of wave is not a viable solution.

Case 7: $\mathbf{d_D} = d_D\mathbf{k}$ *and* $\mathbf{d_B} \neq 0$. For this case, Eq. (A1) reduces to

$$A\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E} = d_D\mathbf{k} - A\left\{\frac{(\mathbf{k}\cdot\mathbf{E})}{\omega^2}\mathbf{k} + \frac{\mathbf{k}\times\mathbf{d_B}}{\omega}\right\} - 7\eta\mathbf{d_B}(\mathbf{E}\cdot\mathbf{d_B}). \tag{A43}$$

By taking the dot product with $\mathbf{k}\times\mathbf{d_B}$ we get

$$\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B}) = C = \text{const}. \tag{A44}$$

Similar to the previous case, if $\mathbf{d_B}$ is not parallel to $\mathbf{k}$, then we can choose as a basis the vectors $\mathbf{k}$, $\mathbf{k}\times\mathbf{d_B}$, and, $\widehat{\mathbf{k}}_\perp$ where $\widehat{\mathbf{k}}_\perp = \mathbf{k}\times(\mathbf{k}\times\mathbf{d_B})$. In this way we can write $\mathbf{E} = (\mathbf{E}\cdot\widehat{\mathbf{k}})\widehat{\mathbf{k}} + (\mathbf{E}\cdot\widehat{\mathbf{k}}_\perp)\widehat{\mathbf{k}}_\perp$, and therefore

$$E^2 = (\widehat{\mathbf{k}}\cdot\mathbf{E})^2 + (\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})^2 + C, \tag{A45}$$

$$A = 1 + \eta\Big[((\widehat{\mathbf{k}}\cdot\mathbf{E})^2 + (\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})^2 + C) \tag{A46}$$

$$\times\left(\left(1-\frac{k^2}{\omega^2}\right) + \frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2} - d_B^2\right)\Big], \tag{A47}$$

$$\mathbf{E}\cdot\mathbf{d_B} = (\mathbf{E}\cdot\widehat{\mathbf{k}})\widehat{\mathbf{k}}\cdot\mathbf{d_B} + (\mathbf{E}\cdot\widehat{\mathbf{k}}_\perp)\widehat{\mathbf{k}}_\perp\cdot\mathbf{d_B}. \tag{A48}$$

We can then write the equations for the projections in $\widehat{\mathbf{k}}$ and $\widehat{\mathbf{k}}_\perp$ to get

$$\left(1+\eta\left\{[(\widehat{\mathbf{k}}\cdot\mathbf{E})^2 + (\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})^2 + C^2]\left(1-\frac{k^2}{\omega^2}\right) + \frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2} - d_B^2\right\}\right)(\widehat{\mathbf{k}}\cdot\mathbf{E}) = d_D k - 7\eta(\widehat{\mathbf{k}}\cdot\mathbf{d_B})[(\mathbf{E}\cdot\widehat{\mathbf{k}})\widehat{\mathbf{k}}\cdot\mathbf{d_B} + (\mathbf{E}\cdot\widehat{\mathbf{k}}_\perp)\widehat{\mathbf{k}}_\perp\cdot\mathbf{d_B}], \tag{A49}$$

$$\left(1+\eta\left\{[(\widehat{\mathbf{k}}\cdot\mathbf{E})^2 + (\widehat{\mathbf{k}}_\perp\cdot\mathbf{E})^2 + C^2]\left(1-\frac{k^2}{\omega^2}\right) + \frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2} - d_B^2\right\}\right)\left(1-\frac{k^2}{\omega^2}\right)(\widehat{\mathbf{k}}_\perp\cdot\mathbf{E}) = 7\eta(\widehat{\mathbf{k}}_\perp\cdot\mathbf{d_B})[(\mathbf{E}\cdot\widehat{\mathbf{k}})\widehat{\mathbf{k}}\cdot\mathbf{d_B} + (\mathbf{E}\cdot\widehat{\mathbf{k}}_\perp)\widehat{\mathbf{k}}_\perp\cdot\mathbf{d_B}]. \tag{A50}$$

As in the previous case, Eqs. (A49) and (A50) are algebraically independent, and therefore only admit a finite number of constant solutions.

For $\mathbf{d_B}$ parallel to $\mathbf{k}$ we can write Eq. (A43) as

$$A\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E} = d_D\mathbf{k} - A\frac{(\mathbf{k}\cdot\mathbf{E})}{\omega^2}\mathbf{k} - 7\eta d_B^2(\mathbf{E}\cdot\mathbf{k})\mathbf{k}, \tag{A51}$$

but $\mathbf{E} = \frac{(\mathbf{k}\cdot\mathbf{E})}{k^2}\mathbf{k}$ and $A = 1 + \eta[E^2(1-\frac{k^2}{\omega^2}) + \frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2} - d_B^2] = 1 + \eta(\frac{(\mathbf{k}\cdot\mathbf{E})^2}{k^2} - d_B^2)$ and therefore we can write

$$\left\{1+\eta\left[\frac{(\mathbf{k}\cdot\mathbf{E})^2}{k^2} - d_B^2\right]\right\}\frac{(\mathbf{k}\cdot\mathbf{E})}{k} = d_D k - 7\eta d_B^2 k(\mathbf{E}\cdot\mathbf{k}), \tag{A52}$$

which is an algebraic equation for $(\mathbf{k}\cdot\mathbf{E})$ in terms of constant coefficients and therefore we again have a trivial constant solution for the fields.

Case 8: $\mathbf{k}$, $\mathbf{d_B}$, *and* $\mathbf{d_D}$ *are parallel.* This case is trivial. When $\mathbf{k}$, $\mathbf{d_B}$, and $\mathbf{d_D}$ are parallel and neither A nor $1-\frac{k^2}{\omega^2}$ vanish, then we can write Eq. (A1) as

$$A\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E} = d_D\mathbf{k} - A\frac{(\mathbf{k}\cdot\mathbf{E})}{\omega^2}\mathbf{k} - 7\eta d_B^2(\mathbf{E}\cdot\mathbf{k})\mathbf{k}. \tag{A53}$$

But $\mathbf{E} = \frac{(\mathbf{k}\cdot\mathbf{E})}{k^2}\mathbf{k}$ and $A = 1 + \eta[E^2(1-\frac{k^2}{\omega^2}) + \frac{(\mathbf{k}\cdot\mathbf{E})^2}{\omega^2} - d_B^2] = 1 + \eta(\frac{(\mathbf{k}\cdot\mathbf{E})^2}{k^2} - d_B^2)$ and therefore we can write

$$\left\{1+\eta\left[\frac{(\mathbf{k}\cdot\mathbf{E})^2}{k^2} - d_B^2\right]\right\}\frac{(\mathbf{k}\cdot\mathbf{E})}{k} = d_D k - 7\eta d_B^2 k(\mathbf{E}\cdot\mathbf{k}), \tag{A54}$$

which is an algebraic equation for $(\mathbf{k}\cdot\mathbf{E})$ in terms of constant coefficients and therefore we again have a trivial constant solution for the fields.

Case 9: None of $\mathbf{k}$, $\mathbf{d_B}$, *and* $\mathbf{d_D}$ *are parallel or orthogonal to any of the others.* Taking the scalar product of Eq. (A1) with $\mathbf{k}$, $\mathbf{d_B}$, $\mathbf{k}\times\mathbf{d_B}$, and $\mathbf{E}$ we get, respectively,

$$A(\mathbf{E}\cdot\mathbf{k}) = \mathbf{d_D}\cdot\mathbf{k} - 7\eta(\mathbf{k}\cdot\mathbf{d_B})(\mathbf{E}\cdot\mathbf{d_B}), \tag{A55}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)(\mathbf{E}\cdot\mathbf{d_B}) = \mathbf{d_D}\cdot\mathbf{d_B} + A\frac{(\mathbf{E}\cdot\mathbf{k})}{\omega^2}\mathbf{k}\cdot\mathbf{d_B} - 7\eta d_B^2(\mathbf{E}\cdot\mathbf{d_B}), \tag{A56}$$

$$A\left(1-\frac{k^2}{\omega^2}\right)\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B}) = \mathbf{d_D}\cdot\mathbf{k}\times\mathbf{d_B} + \frac{A}{\omega}(\mathbf{k}\times\mathbf{d_B})^2. \tag{A57}$$

As $\mathbf{k}$, $\mathbf{d_B}$, and $\mathbf{k}\times\mathbf{d_B}$ are not parallel they form a basis and we can write any other vector, like $\mathbf{E}$ and $\mathbf{d_D}$, as a linear combination of them. This means that E^2 (and therefore A) can be written in terms of $\mathbf{E}\cdot\mathbf{k}$, $\mathbf{E}\cdot\mathbf{d_B}$, and $\mathbf{E}\cdot(\mathbf{k}\times\mathbf{d_B})$. Moreover, E^2 (and therefore A) will contain a term $[\mathbf{E}\cdot(\widehat{\mathbf{k}\times\mathbf{d_B}})]^2$, and therefore Eq. (A57) will have a term $[\mathbf{E}\cdot(\widehat{\mathbf{k}\times\mathbf{d_B}})]^3$. This cubic term cannot be eliminated by any choice of the constants, and therefore Eq. (A57) cannot be reduced to Eq. (A55) or Eq. (A56). Using the same argument, Eq. (A56) will have a cubic term of the form $(\mathbf{E}\cdot\mathbf{d_B})^3$ that cannot be eliminated and therefore Eq. (A56) cannot be reduced to Eq. (A55). We have then a system of three algebraically independent equations for the three unknowns. We can use Bézout's theorem to say that the system allows only for a finite number of solutions that will be given in terms of constants. Therefore, this case also leads to a trivial constant solution.

This completes our proof that all $\omega \neq k$ cases lead to trivial constant solutions assuming $A \neq 0$.

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