Quantum coherence via skew information and its polygamy

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Quantifying coherence is a key task in both quantum-mechanical theory and practical applications. Here, a reliable quantum coherence measure is presented by utilizing the quantum skew information of the state of interest subject to a certain broken observable. This coherence measure is proven to fulfill all the criteria (especially the strong monotonicity) recently introduced in the resource theories of quantum coherence. The coherence measure has an analytic expression and an obvious operational meaning related to quantum metrology. In terms of this coherence measure, the distribution of the quantum coherence, i.e., how the quantum coherence is distributed among the multiple parties, is studied and a corresponding polygamy relation is proposed. As a further application, it is found that the coherence measure forms the natural upper bounds for quantum correlations prepared by incoherent operations. The experimental measurements of our coherence measure as well as the relative-entropy coherence and l_p-norm coherence are studied finally.

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I. INTRODUCTION

Quantum coherence stemmed from the state superposition principle is the most fundamental feature of quantum mechanics that distinguishes the quantum from the classical world. It is the root of all the other intriguing quantum features such as entanglement [1], quantum correlation [2,3], quantum nonlocality, and so on [4]. Coherence is also a vital physical resource with various applications in biology [5–10], thermodynamical systems [11–16], transport theory [17,18], and nanoscale physics [19,20]. Since the seminal work [21] defined the ingredients in the quantification of coherence such as "incoherent states," "incoherent operations," and criteria (null, monotonicity, and convexity) of a good coherence measure for resource theory, quantum coherence has attracted increasing interest in many aspects ranging from coherence measures [21–24], different understandings of coherence [25–29], and especially operational resource theory [30–33] and so on (see [34–40] and references therein).

However, coherence research is still quite limited. Coherence measure, first as a mathematical quantifier, has been only well understood based on the relative entropy and l_1 norm especially considering the strong monotonicity and the closed expression, while for experimental practice, only the relative-entropy coherence can be, in principle, exactly measured without the full quantum state tomography (QST) [41,42] (shown in Appendix C), even though the measurable bounds can be found for other coherence quantifiers such as the measure based on the l_1 norm (given in this paper) and the robustness of coherence (ROC) [24]. In fact, different quantifications of coherence can greatly enrich our understanding of coherence. For example, the relative-entropy coherence can be understood as the optimal rate for distilling a maximally coherent state from given states [30]. ROC is shown to quantify the advantage enabled by a quantum state in a phase discrimination task [24]. But the attempt based on quantum skew information (QSI) failed to quantify the coherence of a general state [23] (shown in Appendix A, also found in [35]), even though the Wigner-Yanase skew information [43–45] and the quantum Fisher information [46,47] are more accessible measures of relevance for quantum metrology as mentioned in [24]. So besides the expected operational meaning, how to revive the skew information for coherence measure is also of vital mathematical significance.

In addition, the relative-entropy coherence measure has been shown to be closely related to the entanglement [27], which has an important characteristic—the monogamy, that is, the entanglement in a multipartite system cannot be freely shared by several subsystems (see [48–50] and references therein). The simplest example is that once three qubits are maximally entangled, any two qubits among them cannot own any entanglement, or equivalently, two maximally entangled qubits are prohibited from entangling with the third qubit. Similarly, is the coherence freely shared among the multipartite system? Recently, the relative-entropy coherence with free reference basis was studied for multipartite systems in [36,40], in particular, [40] constructed the tradeoff relation (monogamy or polygamy) not only depending on the state but also accompanied by the basis-free coherence. How is the coherence distributed in terms of a different measure, especially completely by the basis-dependent measure (as the original purpose of coherence measure)? It is of immense importance to solve this question for understanding coherence both as a quantum-mechanical feature and as a useful physical resource.

In this paper, we employ quantum skew information to construct a quantum coherence measure which is valid for any quantum state. The most prominent advantage is that this coherence measure satisfies the strong monotonicity. Another advantage is that the coherence has an analytic (closed) expression which is similar to the relative-entropy coherence and l_1-norm coherence, but different from the nonanalytic ROC [24]. We employ this coherence measure to construct a clear polygamy relation that dominates the coherence distribution among multipartite systems. As a further application, we consider the tradeoff relation between quantum coherence and quantum discord and find the natural upper bounds of quantum discord. Furthermore, our coherence measure inherits the property of QSI, so a close relation with the

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quantum metrology is founded. Finally the measurement for the experimental practice is considered for various coherence measures.

II. COHERENCE VIA QSI

To begin with, we would like to first introduce the strict definition of coherence [21]. Given a reference basis $\{|i\rangle\}$, a state $\hat{\delta}$ is incoherent if $\hat{\delta}=\sum_i \delta_i|i\rangle\langle i|$. The states with other forms are coherent. The incoherent state set is denoted by $\mathcal{I}$. The incoherent operations are defined by the incoherent completely positive and trace preserving mapping (ICPTP), i.e., the Kraus operator $\sum_n K_n^\dagger K_n=\mathbb{I}$, if $K_n\sigma_I K_n^\dagger\in\mathcal{I}$ for $\forall\sigma_I\in\mathcal{I}$. Thus a good coherence measure $C(\rho)$ of the state ρ should (a) (null) be zero for incoherent states; (b1) (strong monotonicity) not increase under selective ICPTP $\$_I(\rho)=\sum_n K_n\rho K_n^\dagger$, i.e., $C(\rho)\geqslant\sum_n p_n C(\rho_n)$ with $p_n=\mathrm{Tr}K_n\rho K_n^\dagger$ and $\rho_n=K_n\rho K_n^\dagger/p_n$; (b2) (monotonicity) not increase under ICPTP, i.e., $C(\rho)\geqslant C[\$_I(\rho)]$; and (c) (convexity) not increase under classical mixing, i.e., $\sum_n q_n C(\varrho_n)\geqslant C(\varrho)$ with $\varrho=\sum_n q_n\varrho_n$, $\sum_n q_n=1$, $q_n>0$.

It is obvious that in such a framework the definition of coherence strongly depends on the basis. This can be easily understood because the bases could not be arbitrarily changed in some particular scenarios. For example, in an experiment the standard control-NOT (CNOT) gate of two qubits takes the right effect only within some fixed bases. Thus the CNOT gate can transform the coherent joint state $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle$ to the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, but do nothing on the incoherent joint state $|0\rangle|0\rangle$ [27]. This provides an explicit meaning for the basis dependence of the coherence.

Since the states without off-diagonal entries in the basis are incoherent, the usual and intuitive way to quantify the coherence is to measure the distance between the given state and its closest incoherent state according to different (pseudo-) distance norms, as done in almost all the above-mentioned coherence measures. In fact, whether the density matrix is diagonal or not in a basis can be directly revealed by the commutation relation between the density matrix of interest and the given (nondegenerate) observable which equivalently (unambiguously) determines a group of basis. In the following, we establish our coherence measure just by quantifying to what degree the density matrix does not commute with some given (broken) observable.

Theorem 1. The quantum coherence of ρ in the computational basis $\{|k\rangle\}$ can be quantified by

$$C(\rho)=\sum_{k=0}^{N_D-1} I(\rho,|k\rangle\langle k|), \tag{1}$$

where $I(\rho,|k\rangle\langle k|)=-\frac{1}{2}\mathrm{Tr}\{[\sqrt{\rho},|k\rangle\langle k|]\}^2$ represents the skew information subject to the projector $|k\rangle\langle k|$ (N_D-1 is usually omitted if no confusion occurs). $C(\rho)$ is a strongly monotonic coherence measure.

Before the proof of Theorem 1, we first introduce two very useful lemmas.

Lemma 1. Define the function $f(\rho,\sigma)=\mathrm{Tr}\sqrt{\rho}\sqrt{\sigma}$ for two arbitrary density matrices ρ and σ, and the coherence $C(\rho)$ can be expressed as

$$C(\rho)=1-\sum_k\langle k|\sqrt{\rho}|k\rangle^2 \tag{2}$$

$$=1-\big[\max_{\hat{\delta}\in\mathcal{I}} f(\rho,\hat{\delta})\big]^2. \tag{3}$$

In particular, $\hat{\delta}=\hat{\delta}^o=\sum_k\frac{\langle k|\sqrt{\rho}|k\rangle^2}{\sum_{k'}\langle k'|\sqrt{\rho}|k'\rangle^2}|k\rangle\langle k|$ is the optimal incoherent state that achieves the maximal value.

Proof. At first, one can easily find that Eq. (2) is valid by expanding $I(\rho,|k\rangle\langle k|)$ in Eq. (1). So the details are omitted here.

Next, let us prove Eq. (3). Within the computational basis $\{|k\rangle\}$, the incoherent state $\hat{\delta}$ can be explicitly written as

$$\hat{\delta}=\sum_{k=0}^{N_D-1}\hat{\delta}_{kk}|k\rangle\langle k|. \tag{4}$$

Thus we have

$$f(\rho,\hat{\delta})=\sum_{k=0}^{N_D-1}\langle k|\sqrt{\rho}|k\rangle\sqrt{\hat{\delta}_{kk}}$$

$$=Q\sum_{k=0}^{N_D-1}\frac{\langle k|\sqrt{\rho}|k\rangle}{Q}\sqrt{\hat{\delta}_{kk}} \tag{5}$$

with $Q=\sqrt{\sum_{k=0}^{N_D-1}\langle k|\sqrt{\rho}|k\rangle^2}$. According to the Cauchy-Schwarz inequality, we have

$$\left(\sum_{k=0}^{N_D-1}\frac{\langle k|\sqrt{\rho}|k\rangle}{Q}\sqrt{\hat{\delta}_{kk}}\right)^2 \leqslant\left(\sum_{k=0}^{N_D-1}\frac{\langle k|\sqrt{\rho}|k\rangle^2}{Q^2}\right)\left(\sum_{k=0}^{N_D-1}\hat{\delta}_{kk}\right)=1 \tag{6}$$

with the inequality saturated for

$$\sqrt{\hat{\delta}_{kk}}=\frac{\langle k|\sqrt{\rho}|k\rangle}{Q}. \tag{7}$$

Substituting Eq. (6) into Eq. (5), one will find

$$f(\rho,\hat{\delta})\leqslant Q$$

or

$$\big[\max_{\hat{\delta}\in\mathcal{I}} f(\rho,\hat{\delta})\big]^2=Q^2=\sum_{k=0}^{N_D-1}\langle k|\sqrt{\rho}|k\rangle^2. \tag{8}$$

Comparing Eqs. (2) and (8), one can immediately find that Eq. (3) is satisfied.

In addition, since Eq. (7) saturates Eq. (6), one can find that the optimal incoherent state can be directly obtained by substituting Eq. (7) into Eq. (4), which completes the proof. ■

Lemma 2. Let $\$=\{M_n\}$ denote any quantum channel given in the Kraus representation with $\sum_{n=0}M_n^\dagger M_n=\mathbb{I}$; then, for any two density matrices ρ and σ,

$$f(\rho,\sigma)\leqslant\sum_n\sqrt{p_n q_n}f(\rho_n,\sigma_n), \tag{9}$$

with $p_n=\mathrm{Tr}M_n\rho M_n^\dagger$, $q_n=\mathrm{Tr}M_n\sigma M_n^\dagger$, and $\rho_n=M_n\rho M_n^\dagger/p_n,\sigma_n=M_n\sigma M_n^\dagger/q_n$.

Proof. At first, one can note that the function $f(\rho,\sigma) = \mathrm{Tr}\sqrt{\rho}\sqrt{\sigma}$ is closely related to the QSI and has many useful properties [51].

(I) $f(\rho\otimes\tau,\sigma\otimes\tau) = \mathrm{Tr}\sqrt{\rho}\sqrt{\sigma} = f(\rho,\sigma)$ for any density matrix τ.

(II) $f(U\rho U^\dagger,U\sigma U^\dagger) = f(\rho,\sigma)$ for any unitary operation.

(III) (joint concavity) $f(\rho,\sigma) \leqslant f(\$[\rho],\$[\sigma])$ for any quantum channel $\$$.

With the above properties, we can begin our proof as follows. Any quantum channel $\$$ can always be implemented by first utilizing a proper unitary evolution on the composite system composed of the system of interest and an auxiliary system and then performing a proper projective measurement on the auxiliary system, i.e.,

$$M_n\rho M_n^\dagger \otimes |n\rangle_a\langle n| = \|n\rangle_a\langle n\|U(\rho\otimes|0\rangle_a\langle 0|)U^\dagger\|n\rangle_a\langle n\|, \tag{10}$$

where $\|n\rangle_a = \mathbb{I}\otimes|n\rangle_a$ denotes the orthonormal basis in the auxiliary space (labeled by a), and U is a unitary operation on the composite system determined by $\$$. Explicitly, we have $M_n = \langle n\|_a U\|0\rangle_a$.

According to the properties I and II, we have

$$f(\rho,\sigma) = f(U(\rho\otimes\tau_a)U^\dagger, U(\sigma\otimes\tau_a)U^\dagger). \tag{11}$$

Let $\tau_a = |0\rangle_a\langle 0|$ and $\$' = \{\|n\rangle_a\langle n\|\}$, then the property III and Eq. (10) imply

$$\begin{aligned} f(\rho,\sigma) &\leqslant f(\$'[U(\rho\otimes\tau_a)U^\dagger],\$'[U(\sigma\otimes\tau_a)U^\dagger]) \\ &= f\left(\sum_n M_n\rho M_n^\dagger\otimes|n\rangle_a\langle n|, \sum_{n'} M_{n'}\sigma M_{n'}^\dagger\otimes|n'\rangle_a\langle n'|\right) \\ &= \sum_n f(M_n\rho M_n^\dagger, M_n\sigma M_n^\dagger) = \sum_n \sqrt{p_n q_n} f(\rho_n,\sigma_n), \end{aligned} \tag{12}$$

with $p_n = \mathrm{Tr}M_n\rho M_n^\dagger$, $q_n = \mathrm{Tr}M_n\sigma M_n^\dagger$, and $\rho_n = M_n\rho M_n^\dagger/p_n, \sigma_n = M_n\sigma M_n^\dagger/q_n$. Here we use the orthonormalization of $\{|n_a\rangle\}$ to derive Eq. (12) which closes the proof. ■

With Lemmas 1 and 2, now we can prove the theorem 1 as follows.

Proof of Theorem 1. To prove Theorem 1, we need to show the coherence measure $C(\rho)$ satisfies all the required criteria (a), (b1), (b2), and (c).

It is clear that quantum skew information $I(\rho,A)$ has many good properties such as vanishing iff $[\rho,A]=0$, convexity on the classical mixing of the states, and so on [43–45]. $C(\rho)$ inherits all the properties, so $C(\rho)=0$ is the sufficient and necessary condition for incoherent states and $C(\rho)$ is convex under the mixing of states. That is, the criteria (a) and (c) are automatically satisfied. In addition, one can note that since the coherence measure is convex, the monotonicity on selective ICPTP (strong monotonicity) will automatically imply the monotonicity on ICPTP. So the remaining task of the proof is to prove that $C(\rho)$ satisfies (b1)—*the strong monotonicity.*

To do so, let us consider a density matrix ρ with its coherence $C(\rho)$ defined by Eq. (3). Meanwhile, we let $\hat{\delta}^o$ denote the optimal incoherent state achieving the maximal value in Eq. (3). Define *the incoherent selective quantum operations* $\$_I$ given by the Kraus operators as M_n. Suppose $\$_I$ is performed on the state ρ, then the postmeasurement ensemble can be given by $\{p_n,\rho_n\}$ with $p_n = \mathrm{Tr}M_n\rho M_n^\dagger$ and $\rho_n = M_n\rho M_n^\dagger/p_n$. Therefore, the average coherence can be given by

$$\sum_n p_n C(\rho_n) = 1 - \sum_n p_n\left[\max_{\hat{\delta}_n\in\mathcal{I}} f\left(\frac{M_n\rho M_n^\dagger}{p_n},\hat{\delta}_n\right)\right]^2. \tag{13}$$

Since the incoherent operation cannot prepare the coherence from an incoherent state, for the optimal incoherent state $\hat{\delta}^o$, we have $\hat{\delta}_n^o = \frac{M_n\hat{\delta}^o M_n^\dagger}{q_n} \in \mathcal{I}$ with $q_n = \mathrm{Tr}M_n\hat{\delta}^o M_n^\dagger$ for any incoherent operation M_n. Thus for such a particular $\hat{\delta}_n^o$, it is natural that

$$f(\rho_n,\hat{\delta}_n^o) \leqslant \max_{\hat{\delta}_n\in\mathcal{I}} f(\rho_n,\hat{\delta}_n). \tag{14}$$

Thus Eq. (13) can be rewritten as

$$\sum_n p_n C(\rho_n) \leqslant 1 - \sum_n p_n f^2(\rho_n,\hat{\delta}_n^o). \tag{15}$$

For the probability distribution $\{q_n\}$, the Cauchy-Schwarz inequality implies

$$\sum_n p_n f^2(\rho_n,\hat{\delta}_n^o) \geqslant \left[\sum_n \sqrt{p_n q_n} f(\rho,\hat{\delta}^o)\right]^2. \tag{16}$$

Based on Eq. (9) given by Lemma 2, we have

$$\sum_n p_n C(\rho_n) \leqslant 1 - f^2(\rho,\hat{\delta}^o) = C(\rho), \tag{17}$$

which is the strong monotonicity. The convexity of $C(\rho)$ directly shows $C(\rho) \geqslant C(\sum_{n=1} p_n\rho_n) = C(\$_I[\rho])$, that is, the monotonicity. ■

III. CONNECTION WITH *K* COHERENCE FOR QUBITS

The K coherence of a density matrix ρ subject to a given observable K is defined by [23]

$$C_K(\rho) = -\tfrac{1}{2}\mathrm{Tr}\{[\sqrt{\rho},K]\}^2. \tag{18}$$

Needless to say, whether the K coherence is strongly monotonic or not, it is obvious that $C_K(\rho)$ depends on both the eigenvalue and the eigenvectors (basis) of K. So once the observable K has a degenerate subspace, the coherence of the state ρ in the corresponding subspace will not be revealed. However, our coherence measure $C(\rho)$ depends on the broken instead of the original observable, so it is independent of the eigenvalues of the observable. In other words, it is not affected by the degeneracy of the observable and so is unambiguously defined for a certain basis. This is the obvious difference between the K coherence and ours. However, next we will show that the K coherence is only valid for the qubit system because it is equivalent to our measure $C(\rho)$ for qubits.

For a *qubit* state ρ and an observable K with the eigendecomposition $K = \sum_{k=0}^1 a_k|k\rangle\langle k|$ where a_k is the eigenvalue and $\{|k\rangle\}$ denotes the set of eigenvectors, our coherence measure $C(\rho)$ subject to the basis $\{|k\rangle\}$ is given by

$$C(\rho) = -\frac{1}{2}\sum_{k=0}^{1}\mathrm{Tr}\{[\sqrt{\rho},|k\rangle\langle k|]\}^2 \tag{19}$$

and the K coherence is given as the same form as Eq. (18). Any two-dimensional observable can be decomposed as $K = \frac{1}{2}\mathrm{Tr}K\cdot\mathbb{I} + \tilde{K}$ with $\tilde{K} = \lambda(|0\rangle\langle 0| - |1\rangle\langle 1|)$ where $|0\rangle$ and $|1\rangle$, respectively, denote the common eigenvectors of K and $\tilde{K}$, λ represents the positive eigenvalue of $\tilde{K}$, and $a_{0/1}$ can be rewritten by $\frac{TrK}{2} \pm \lambda$. Therefore, Eq. (18) can also be rewritten based on $\tilde{K}$ as

$$\begin{aligned} C_K(\rho) &= -\frac{1}{2}\mathrm{Tr}\left\{\frac{1}{2}\mathrm{Tr}K[\sqrt{\rho},\mathbb{I}] + [\sqrt{\rho},\tilde{K}]\right\}^2 \\ &= -\frac{\lambda^2}{2}\mathrm{Tr}\{[\sqrt{\rho},|0\rangle\langle 0| - |1\rangle\langle 1|]\}^2 \\ &= -\frac{\lambda^2}{2}\bigg(\frac{1}{2}\mathrm{Tr}\{[\sqrt{\rho},\mathbb{I} - 2|1\rangle\langle 1|]\}^2 \\ &\quad + \frac{1}{2}\mathrm{Tr}\{[\sqrt{\rho},2|0\rangle\langle 0| - \mathbb{I}]\}^2\bigg) \\ &= 2\lambda^2 C(\rho), \end{aligned} \tag{20}$$

which exhibits the equivalence between the two coherence measures for qubit systems if neglecting a constant $2\lambda^2$. Thus K coherence is valid for qubit systems (satisfying the strong monotonicity), since our coherence measure $C(\rho)$ is strongly monotonic.

IV. CONNECTION WITH QUANTUM METROLOGY

In the following, we will demonstrate how our coherence measure can be related to some quantum metrology scheme. This also provides an operational meaning for our coherence measure $C(\rho)$.

The scheme is described as follows. Suppose we have an n-dimensional state ρ and then let the state undergo a unitary operation $U_{\varphi_k} = e^{-i\varphi_k|k\rangle\langle k|}$ which will endow an unknown phase φ_k to the state ρ as $\rho_k = U_{\varphi_k}\rho U_{\varphi_k}^{\dagger}$. We aim to estimate φ_k in ρ_k by $N >> 1$ runs of detection on ρ_k. The question is what the measurement precision is.

In the above scheme, the measurement precision of φ_k is characterized by the uncertainty of the estimated phase φ_k^{est} defined by

$$\delta\varphi_k = \left\langle\left(\frac{\varphi_k^{\mathrm{est}}}{\left|\partial\langle\varphi_k^{\mathrm{est}}\rangle/\partial\varphi_k\right|} - \varphi_k\right)^2\right\rangle^{1/2}, \tag{21}$$

which, for an unbiased estimator, is just the standard deviation [52–54]. Based on the quantum parameter estimation [52–54], $\delta\varphi_k$ is limited by the quantum Cramér-Rao bound as

$$(\delta\varphi_k)^2 \geqslant \frac{1}{NF_{Qk}}, \tag{22}$$

where $F_{Qk} = \mathrm{Tr}\{\rho_\varphi L_\varphi^2\}$ is the quantum Fisher information with L_φ being the symmetric logarithmic derivative defined by $2\partial_\varphi\rho_\varphi = L_\varphi\rho_\varphi + \rho_\varphi L_\varphi$ [52]. It was shown in [52–54] that this bound can always be reached asymptotically by maximum likelihood estimation and a projective measurement in the eigenbasis of the "symmetric logarithmic derivative operator." Thus one can let $(\delta\varphi_k^o)^2$ denote the optimal variance which achieves the Cramér-Rao bound, i.e., $(\delta\varphi_k^o)^2 = \frac{1}{NF_{Qk}}$. Reference [55] showed that the Fisher information F_{Qk} is well bounded by the skew information as

$$I(\rho,|k\rangle\langle k|) \leqslant \frac{F_{Qk}}{4} \leqslant 2I(\rho,|k\rangle\langle k|), \tag{23}$$

which directly leads to

$$4NI(\rho,|k\rangle\langle k|) \leqslant \frac{1}{(\delta\varphi_k^o)^2} \leqslant 8NI(\rho,|k\rangle\langle k|). \tag{24}$$

If we repeat this scheme N times, respectively, corresponding to the different $|k\rangle\langle k|$, we can sum Eq. (24) over k as

$$4NC(\rho) \leqslant \sum_k \frac{1}{(\delta\varphi_k^o)^2} \leqslant 8NC(\rho), \tag{25}$$

where we have used $C(\rho) = \sum_k I(\rho,|k\rangle\langle k|)$. If we define $\frac{1}{(\Delta_\varphi^o)^2} = \sum_k \frac{1}{(\delta\varphi_k^o)^2}$, Eq. (25) can be rewritten as

$$\frac{1}{8NC(\rho)} \leqslant (\Delta_\varphi^o)^2 \leqslant \frac{1}{4NC(\rho)}, \tag{26}$$

which shows that quantum coherence $C(\rho)$ contributes to the upper and lower bounds of the "average variance" $(\Delta_\varphi^o)^2$ that characterizes the contributions of all the inverse optimal variances of the estimated phases.

In fact, one can recognize that the practical variance $\delta\varphi_k$ usually deviates from the optimal one $\delta\varphi_k^o$ because the experimental measurement strategy cannot be as ideal as we expect theoretically, so that $\delta\varphi_k \geqslant \delta\varphi_k^o$. Thus, one can replace $\delta\varphi_k^o$ in Eqs. (24) and (25) by $\delta\varphi_k$ and obtain the other two relations as

$$\frac{1}{(\delta\varphi_k)^2} \leqslant 8NI(\rho,|k\rangle\langle k|) \tag{27}$$

and

$$\sum_k \frac{1}{(\delta\varphi_k)^2} \leqslant 8NC(\rho). \tag{28}$$

Equations (27) and (28) mean that no matter what kind of measurement strategy is employed, with the fixed N the measurement cannot be infinitely precise. The variance φ_k is well restricted by the skew information $I(\rho,|k\rangle\langle k|)$ (of course by the corresponding Fisher information), while the sum of $\frac{1}{\varphi_k^2}$ (or the corresponding $\frac{1}{(\Delta_\varphi)^2}$) is just constrained by our coherence $C(\rho)$.

V. DISTRIBUTION OF COHERENCE

In this section, we will consider how the coherence is distributed among a multipartite system. This essentially requires us to extend the coherence to the multipartite system and establish the tradeoff relation between the coherence among different subsystems and even the relation with other quantum features. Such a question was considered by [40], but the tradeoff relation as mentioned at the beginning includes both the basis-free coherence measure and the basis-dependent coherence measure; especially, this relation depends on the state (monogamous for some states and polygamous for other states). This indeed benefits our recognition of coherence, but strictly speaking should be the property of the state instead of the coherence. So how to establish a tradeoff relation describing a certain property (monogamy or polygamy) with the unified measure is very important no matter if it serves as a physical feature or a physical resource. In order to keep the consistent reference basis (similar to the monogamy of entanglement via the same entanglement quantifier [48,49]), we will restrict ourselves to the computational basis with which our coherence can be directly used. Therefore, the polygamy relation of bipartite pure states can be given as follows.

Theorem 2. For a bipartite pure state $|\Psi\rangle_{AB}$, let $\rho_{A/B}$ denote the reduced density matrix for A or B, then

$$1 - C(|\Psi\rangle_{AB}) \leqslant [1 - C(\rho_A)][1 - C(\rho_B)], \tag{29}$$

which is saturated by product states.

Proof. The pure state $|\Psi\rangle_{AB}$ has the Schmidt decomposition as $|\Psi\rangle_{AB} = \sum_i \lambda_i |\mu_i\rangle|\nu_i\rangle$ from which we can rewrite $|\Psi\rangle_{AB} = \sum_i \lambda_i U_A \otimes U_B |\mu_i\rangle|\nu_i\rangle$ with λ_i the Schmidt coefficients, so the reduced density matrices can be, respectively, given by $\rho_A = \sum_i \lambda_i^2 U_A|\mu_i\rangle\langle\mu_i|U_A^\dagger$ and $\rho_B = \sum_i \lambda_i^2 U_B|\nu_i\rangle\langle\nu_i|U_B^\dagger$. Thus one can always calculate the coherence for $|\Psi\rangle_{AB}$ and its reduced matrices ρ_A and ρ_B (within the basis $|k\rangle$ and $|k'\rangle$ instead of the Schmidt basis $|\mu_i\rangle$ and $|\nu_i\rangle$) as

$$1 - C(|\Psi\rangle_{AB}) = \sum_{kk'} \left| \sum_i \lambda_i \langle k|U_A|\mu_i\rangle\langle k'|U_B|\nu_i\rangle \right|^4, \tag{30}$$

$$1 - C(\rho_A) = \sum_k \left[\sum_i \lambda_i |\langle k|U_A|\mu_i\rangle|^2 \right]^2, \tag{31}$$

$$1 - C(\rho_B) = \sum_{k'} \left[\sum_i \lambda_i |\langle k'|U_B|\nu_i\rangle|^2 \right]^2. \tag{32}$$

From these three equations, we can find that, for each k and k',

$$\begin{aligned}&\left(\sum_i \lambda_i |\langle k|U_A|\mu_i\rangle|^2\right)\cdot\left(\sum_i \lambda_i |\langle k'|U_B|\nu_i\rangle|^2\right)\\ &\geq \left(\sum_i \lambda_i |\langle k|U_A|\mu_i\rangle|\cdot|\langle k'|U_B|\nu_i\rangle|\right)^2\\ &\geqslant \left|\sum_i \lambda_i \langle k|U_A|\mu_i\rangle\langle k'|U_B|\nu_i\rangle\right|^2.\end{aligned} \tag{33}$$

Therefore, squaring both sides of Eq. (33) and summing over k and k', one will immediately arrive at Eq. (29). It is easy to show that the product states saturate the inequality. ■

From Theorem 2, it can be found that the coherence of a subsystem is not limited by the coherence of the composite system. A trivial case is that the incoherent composite quantum state means no coherence in its subsystems. However, the composite quantum state with the relatively large coherence does not restrict the coherence of the subsystems (which is different from the monogamy of entanglement). That is, the subsystems could also have relatively large coherence. A typical example is the maximally coherent state, e.g., $|\Psi\rangle_{AB} = \frac{1}{3}\sum_{i,j=0}^2 |ij\rangle$. One can find that $C(|\Psi\rangle_{AB}) = \frac{8}{9}$ but $C(\rho_A) = C(\rho_B) = \frac{2}{3}$, which is the maximal coherence in three-dimensional space corresponding to the reduced states $\rho_A = \rho_B = \frac{1}{3}\sum_{i,j=0}^2 |i\rangle\langle j|$. This example also implies that the subsystem with relatively large coherence does not restrict its ability to interact with another system and form a composite system with large coherence. These are the manifestations of the so-called *polygamy*. Theorem 2 can also be extended to mixed states and multipartite states as the following two corollaries.

Corollary 1. For bipartite mixed states ρ_{AB} with their reduced density matrices $\rho_{A/B}$, the coherences satisfy

$$[1 - C(\rho_A)][1 - C(\rho_B)] \geqslant \sum_{kk'} \langle kk'|\rho_{AB}|kk'\rangle^2 \tag{34}$$

$$= \mathrm{Tr}\rho_{AB}^2 - C_2(\rho_{AB}) \geqslant \lambda_{\min}[1 - C(\rho_{AB})] \tag{35}$$

with $|kk'\rangle$ being the fixed computational basis, $\lambda_{\min}$ denoting the minimal *nonzero* eigenvalue of ρ_{AB}, and $C_{l_k}(\rho)$ denoting the l_k-norm coherence. In addition, one can also have

$$[1 - C(\rho_A)]\left[r - \sum_{i=1}^r C(\rho_{Bi})\right] \geq 1 - C(\rho_{AB}), \tag{36}$$

$$\left[r - \sum_{i=1}^r C(\rho_{Ai})\right][1 - C(\rho_B)] \geq 1 - C(\rho_{AB}), \tag{37}$$

which can also lead to a symmetric form as

$$[1 - C(\rho_A)][1 - C(\rho_B)] \geqslant \frac{1}{c_s}[1 - C(\rho_{AB})]^2 \tag{38}$$

with $c_s = [r - \sum_i C(\rho_{Ai})][r - \sum_i C(\rho_{Bi})]$ where r is the rank of ρ_{AB} and ρ_{Ai}, and ρ_{Bi} denote the reduced density matrices of the ith eigenstate of ρ_{AB}.

Corollary 2. For an N-partite quantum state $\rho_{AB\cdots N}$, define the index set $S = \{A,B,C,\cdots,N\}$ corresponding to all the N subsystems. Let α represent a subset of S, i.e., $\alpha \subset S$, and let ρ_α denote the reduced density matrix by tracing over all subsystems corresponding to $\bar{\alpha}$, the complementary set of S. Thus for $\forall \alpha_i \subset S$ such that $\alpha_i \cap \alpha_j = \delta_{ij}\alpha_i$ and $\sum_{i=1} \alpha_i = S$, the coherences satisfy

$$\prod_i [1 - C(\rho_{\alpha_i})] \geqslant \lambda_M[1 - C(\rho_{AB\cdots N})], \tag{39}$$

$$\prod_i [1 - C(\rho_{\alpha_i})]^{n_i} \geqslant \frac{1}{c_{sT}}[1 - C(\rho_{AB\cdots N})]^2, \tag{40}$$

where n_i as well as λ_M and c_{sT} can be determined from Corollary 1 based on the concrete bipartite grouping of $\rho_{AB\cdots N}$.

The proofs of both Corollaries 1 and 2 are given in Appendix B, which also demonstrates how to determine n_i, λ_M, and c_{sT} . One can note that Eq. (35) can be understood as the general polygamy relation for both mixed and pure states since $\lambda_{\min} = 1$ for the pure state. In addition, no matter what $\lambda_M, c_{sT}, \lambda_{\min}$, and c_s are, they can always be some finite values. Therefore, similar to Theorem 2, *polygamy* is also clearly demonstrated by mixed states and multipartite states.

VI. BOUNDS ON QUANTUM DISCORD

Resource theory provides a platform to understand one quantum feature via another quantum feature. Quantum coherence can be understood by quantum discord [26]. That is, the coherence assisted by an incoherent auxiliary state can be converted by incoherent operations to the same amount of quantum discord. As an application of our coherence measure, here we revisit this question and find some similar bounds. As we know, quantum discord of a bipartite quantum state is initially defined by the discrepancy between quantum versions of two classically equivalent expressions for mutual information [2,3]. Even though the latter various measures of quantum discord have been presented [56], quantum discord with both good computability and good properties (e.g., contractivity) should count on local quantum uncertainty (LQU) based on quantum skew information [57]. We would like to emphasize that LQU was developed with the broken observable in [58]. In the following, we will restrict the quantum discord to the one given in [58].

The quantum discord in [58] is defined for a bipartite state ρ_{AB} as

$$D(\rho_{AB}) = \min_{\{|k\rangle_A\}} C_{\{|k\rangle_A\}}(\rho_{AB}), \tag{41}$$

where

$$C_{\{|k\rangle_A\}}(\rho_{AB}) = -\frac{1}{2}\sum_k \mathrm{Tr}[\sqrt{\rho_{AB}},|k\rangle_A\langle k|\otimes \mathbb{I}_B]^2 \tag{42}$$

and $\{|k\rangle_A\}$ denotes the fixed basis. We can understand $C_{\{|k\rangle_A\}}(\rho_{AB})$ as the coherence of the A subspace and thus $D(\rho_{AB})$ can be naturally considered as the minimal coherence of A subspace. Since $I(\rho_{AB},K\otimes \mathbb{I}_B) \geqslant I(\rho_A,K)$, one can immediately obtain

$$C_{\{|k\rangle_A\}}(\rho_{AB}) \geqslant D(\rho_{AB}) \geqslant C_{\{|\tilde{k}\rangle_A\}}(\rho_A) \tag{43}$$

with $\{|\tilde{k}\rangle_A\}$ denoting the optimal basis to achieve the quantum discord. This relation implies the quantum discord is upper bounded by its subspace coherence and lower bounded by the coherence of the subsystem subject to the optimal basis. To reveal all the quantum discords, the symmetric quantum discord can be similarly defined as

$$D_S(\rho_{AB}) = \min_{\{|k\rangle\}\{|k'\rangle\}} C_{\{|kk'\rangle\}}(\rho_{AB}) \tag{44}$$

with

$$C_{\{|kk'\rangle\}}(\rho_{AB}) = -\frac{1}{2}\sum_{kk'} \mathrm{Tr}[\sqrt{\rho_{AB}},|k\rangle_A\langle k|\otimes |k'\rangle_B\langle k'|]^2. \tag{45}$$

Analogously, $C_{\{|kk'\rangle\}}(\rho_{AB})$ is exactly the coherence of ρ_{AB} within the basis $\{|k\rangle|k'\rangle\}$ and quantum discord $D_S(\rho_{AB})$ is just the minimal coherence. With these concepts in mind, we can give the important results in the following rigorous way.

Theorem 3. Suppose an incoherent operation $\$_I$ that is performed on a bipartite product state $\sigma_A\otimes\sigma_B$ is a bipartite product state. The quantum discord of the postoperation state is bounded as

$$D_S(\$_I[\sigma_A\otimes\sigma_B]) \leqslant 1-[1-C(\sigma_A)][1-C(\sigma_B)]. \tag{46}$$

In particular, the upper bound is attained by $\$_I = \{U_I = \sum_{ij}|i,i\oplus j\rangle\langle i,j|\}$ and $\sigma_{B/A} = |k\rangle\langle k|$.

Proof. From Eq. (44), one can find that the discord is gotten by the minimization among all the potential basis, so it is natural that

$$D_S(\$_I[\sigma_A\otimes\sigma_B]) \leqslant C(\$_I[\sigma_A\otimes\sigma_B]). \tag{47}$$

Based on the monotonicity of the coherence, one will immediately arrive at

$$\begin{aligned} C(\$_I[\sigma_A\otimes\sigma_B]) &\leqslant C(\sigma_A\otimes\sigma_B)\\ &= 1-[1-C(\sigma_A)][1-C(\sigma_B)], \end{aligned} \tag{48}$$

which shows Eq. (46) is valid.

Next, we will show the upper bound is attainable as mentioned in the theorem. Let $\sigma_B = |\tilde{k}\rangle\langle\tilde{k}|$, so the initial state can be written as $\rho_0 = \rho_A\otimes|\tilde{k}\rangle\langle\tilde{k}|$. Suppose we employ the incoherence operation $\$_I = \{U_I = \sum_{ij}|i,i\oplus j\rangle\langle i,j|\}$. So the state after the operation is written by $\rho_f = U_I\rho_0 U_I^\dagger$. Considering the eigendecomposition of $\rho_A = \sum_i \lambda_i|\psi_i\rangle_A\langle\psi_i|$ with the eigenstate $|\psi_i\rangle = \sum_j a_j^i|j\rangle$ expanded by the basis $\{|j\rangle\}$, we can rewrite ρ_f as

$$\begin{aligned} \rho_f &= \sum_i \lambda_i U_I|\psi_i\rangle_A|\tilde{k}\rangle_B\langle\psi_i|_A\langle\tilde{k}|_B U_I^\dagger\\ &= \sum_i \lambda_i\left(\sum_j a_j^i|jj\oplus\tilde{k}\rangle\right)\left(\sum_j\langle jj\oplus\tilde{k}|a_j^{i*}\right). \end{aligned} \tag{49}$$

Based on our definition of quantum coherence, we can easily obtain the quantum coherence of ρ_A within the basis $\{|j\rangle\}$ as

$$\begin{aligned} C(\rho_A) &= 1-\sum_j\left(\sum_i\sqrt{\lambda_i}|\langle j|\psi_i\rangle|^2\right)^2\\ &= 1-\sum_j\left(\sum_i\sqrt{\lambda_i}|a_j^i|^2\right)^2. \end{aligned} \tag{50}$$

According to the definition of quantum correlation $D_S(\cdot)$, one can find that

$$\begin{aligned} 1-D_S(\rho_f) &= \max_{\{|kk'\rangle\}}\sum_{kk'}\left[\sum_i\sqrt{\lambda_i}\left(\sum_j a_j^i\langle kk'|jj\oplus\tilde{k}\rangle\right)\left(\sum_j\langle jj\oplus\tilde{k}|kk'\rangle a_j^{i*}\right)\right]^2\\ &= \max_{\{|kk'\rangle\}}\sum_{kk'}\left(\sum_i\sqrt{\lambda_i}\langle kk'|P_{\tilde{k}}\Lambda_i\otimes\mathbf{1}|\Phi\rangle\langle\Phi|P_{\tilde{k}}\Lambda_i^*\otimes\mathbf{1}|kk'\rangle\right)^2 \end{aligned}$$

$$= \max_{U,V} \sum_j \left(\sum_i \sqrt{\lambda_i} |\langle j|U^\dagger P_{\tilde{k}} \Lambda_i P_{\tilde{k}} V^* |j\rangle|^2 \right)^2$$

$$= \max_{U,V} \sum_j \left(\sum_i \sqrt{\lambda_i} \left| \sum_k [U^\dagger]_{jk} a_k^i [V^*]_{kj} \right|^2 \right)^2. \quad (51)$$

Here we first use the fact that $\sum_j a_j^i |jj \oplus \tilde{k}\rangle = (P_{\tilde{k}} \Lambda_i \otimes \mathbb{I})|\Phi\rangle$, where $|\Phi\rangle = \sum_j |jj\rangle$, $\Lambda_i = \mathrm{diag}(a_0, a_1, \cdots)$, and $P_{\tilde{k}} = \sum_j |\tilde{k} \oplus j\rangle\langle j|$. In addition, we also convert the optimization on the basis $\{|kk'\rangle\}$ to the unitary transformations by $|k\rangle = U|j\rangle$ and $|k'\rangle = V|j\rangle$. In the last line of Eq. (51), we omit $P_{\tilde{k}}$ because we force $P_{\tilde{k}}$ to be absorbed by the optimized unitary transformations U and V. By utilizing the Cauchy-Schwarz inequality to Eq. (51), one will find

$$D_S(\rho_f) \geq 1 - \max_U \sum_j \left(\sum_i \sqrt{\lambda_i} \sum_k |[U^\dagger]_{jk}|^2 |a_k^i|^2 \right)^2$$

$$\geq 1 - \max_U \sum_{jk} |[U^\dagger]_{jk}|^2 \left(\sum_i \sqrt{\lambda_i} |a_k^i|^2 \right)^2 \quad (52)$$

$$= 1 - \sum_j \left(\sum_i \sqrt{\lambda_i} |a_j^i|^2 \right)^2, \quad (53)$$

where inequality (52) comes from the convexity and the extreme value is achieved when we select the optimal basis $\{|kk'\rangle\} = \{|jj\rangle\}$. Comparing Eqs. (53) and (50), one can find

$$D_S(\rho_f) \geq C(\rho_A). \quad (54)$$

However, based on Eq. (46), we have $D_S(\rho_f) \leqslant C(\rho_A)$ for $\sigma_B = |\tilde{k}\rangle\langle\tilde{k}|$ and U_I. This means in this case $D_S(\rho_f) = C(\rho_A)$, which completes the proof. ■

In fact, if both σ_A and σ_B are coherent, one can find that the upper bound could not be attained generally for the fixed dimension of the state space. For example, $\sigma_A = \sigma_B = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$, a simple algebra can show $C(\sigma_A \otimes \sigma_B) = \frac{3}{4}$, but the maximal quantum discord in this fixed space is $D_S(\$_I[\sigma_A \otimes \sigma_B]) = \frac{1}{2}$ where $\$_I = [\mathbb{I}_2 \oplus i\sigma_y]$, and $\mathbb{I}_2$ and σ_y are, respectively, the two-dimensional identity matrix and Pauli matrix. However, if the state space is not fixed, the upper bound is obviously attainable, because one can always expand the state space as $\sigma_{A/B} \oplus 0$ as required, which, in some cases, is equivalent to attaching an auxiliary system as $\sigma_A \otimes \sigma_B \otimes |0\rangle_C\langle 0|$. In this sense, it is apparent that the coherence of $\sigma_A \otimes \sigma_B$ can be completely converted to the quantum discord between AB and C. One can perform a (incoherent) swapping operation on A and C and finally obtain the equal amount of quantum discord between A and BC (BC can be replaced by B with the equally expanded space). Finally we would like to emphasize that the similar Eq. (48) is also satisfied for multipartite states.

VII. DIRECTLY MEASURABLE COHERENCE

In this section, we will discuss the measurement of coherence in practical experiments. Like entanglement measure, the coherence measure *per se* is not an observable. In order to avoid so much cost (mainly in a high-dimensional system) for QST, the schemes for the direct measurement of entanglement and quantum discord have been presented in recent years by the simultaneous copies of the state [59–63] or by an auxiliary system [64], which provides a valuable reference for the coherence measure. For example, the relative-entropy coherence for an N_D -dimensional state ρ is given explicitly by

$$C_r(\rho) = \sum_i \lambda_i \log_2 \lambda_i - \sum_k \rho_{kk} \log_2 \rho_{kk} \quad (55)$$

with λ_i's denoting the eigenvalues of ρ and $\rho_{kk} = \langle k|\rho|k\rangle$ being the diagonal entries subject to the basis $\{|k\rangle\}$. Since λ_i's can be measured by the standard overlap measurement [64,65] and ρ_{kk} can be measured by the given projectors $\hat{P}_k = |k\rangle\langle k|$, $C_r(\rho)$ is experimentally measurable. The cost is $2(N_D - 1)$ measurements assisted by at most N_D copies of the state. The detailed measurement scheme is described for clarity in Appendix C.

In fact, the measurable evaluation of coherence (instead of the exact value as given above for the relative-entropy coherence) with less cost is also quite practical. We find that our $C(\rho)$ can also be effectively evaluated by the measurable upper and lower bounds. Based on the inequality $I(A,\rho) \geq -\frac{1}{4}\mathrm{Tr}\{[\rho,A]^2\}$ for any observable A and a density matrix ρ [23], we have

$$C(\rho) = \sum_k I(|k\rangle\langle k|,\rho)$$

$$\geq \frac{1}{2}\left(\mathrm{Tr}\rho^2 - \sum_k \langle k|\rho|k\rangle^2 \right) = \frac{1}{2} C_{l_2}(\rho) \quad (56)$$

with $\{|k\rangle\}$ defining the basis. Here

$$C_{l_2}(\rho) = \|\rho - \delta_I\|_2 = \sum_{i\neq j} |\rho_{ij}|^2$$

$$= \mathrm{Tr}\rho^2 - \sum_k \langle k|\rho|k\rangle^2 = \sum_k \left\{\lambda_k^2 - \langle k|\rho|k\rangle^2\right\}, \quad (57)$$

where $\|\cdot\|_2$ denotes the l_2 norm of a matrix, $\delta_I = \sum_k \rho_{kk}|k\rangle\langle k|$ is the closest incoherent state, and λ_k's are the eigenvalues of ρ. In addition, one can also find that $\langle k|\sqrt{\rho}|k\rangle \geqslant \langle k|\rho|k\rangle$ is satisfied for any $|k\rangle$. Thus one can have

$$C(\rho) = 1 - \sum_k \langle k|\sqrt{\rho}|k\rangle^2 \leqslant 1 - \sum_k \langle k|\rho|k\rangle^2. \quad (58)$$

Combining Eqs. (56) and (58), one will immediately obtain our second result:

$$\tfrac{1}{2}C_{l_2}(\rho) \leqslant C(\rho) \leqslant 1 - \mathrm{Tr}\rho^2 + C_{l_2}(\rho), \tag{59}$$

which provides both the upper and the lower bounds. Even though the coherence based on the l_2 norm is not a good measure, as one bound, it serves as a sufficient and necessary condition for the existence of quantum coherence. Since C_{l_2} is completely characterized by the eigenvalues λ_k and the diagonal entries $\langle k|\rho|k\rangle$ as seen from Eq. (57), one can find that both bounds are practically measurable similar to the above measurement scheme for the relative-entropy coherence. The cost is N_D measurements plus two copies of the state ρ.

In fact, l_1-norm coherence has also similar measurable bounds. As we know, for the N_D-dimensional density matrix ρ, we have

$$C_{l_1}(\rho) = \sum_{i\neq j}|\rho_{ij}| = \frac{1}{2}\sum_{i<j}|\rho_{ij}|. \tag{60}$$

Since $|\rho_{ij}| \leqslant 1$, we have $|\rho_{ij}|^2 \leqslant |\rho_{ij}|$, which leads to

$$C_{l_1}(\rho) \geqslant \frac{1}{2}\sum_{i<j}|\rho_{ij}|^2 = C_{l_2}(\rho). \tag{61}$$

Furthermore, the inequality $\left(\sum_{k=1}^{N_D} a_k\right)^2 \leqslant N_D \sum_{k=1}^{N_D} a_k^2$ for positive a_k directly implies that

$$C_{l_1}(\rho) \leqslant \sqrt{N_D(N_D-1)C_{l_2}(\rho)}. \tag{62}$$

Combining Eqs. (61) and (62) gives the bounds for $C_{l_1}(\rho)$ as

$$C_{l_2}(\rho) \leqslant C_{l_1}(\rho) \leqslant \sqrt{N_D(N_D-1)C_{l_2}(\rho)}. \tag{63}$$

Since $C_{l_2}(\rho)$ is measurable, the above bounds are naturally measurable. In addition, [24] also proposed a similar lower bound through the ROC and the improved lower bound rather than the exact coherence conditioned on the prior knowledge of the state of interest.

VIII. DISCUSSION AND CONCLUSIONS

Before concluding, we would like to first emphasize that the polygamy inequality shown in Theorem 2 has an elegant form for bipartite pure states, but the relation with the same form does not hold for a general bipartite mixed state of qubits, even though Eq. (14) provides a general polygamy relation. However, we would like to conjecture that it could hold for the bipartite mixed states with the dimension $N \geqslant 6$. The details can be seen from Appendix D.

In summary, we have presented a strongly monotonic coherence measure in terms of quantum skew information which characterizes the contribution of the commutation between the broken observable (basis) and the density matrix of interest. It is shown that the coherence measure has an operational meaning based on the quantum metrology. We also study the distribution of the coherence among a multipartite system by providing the polygamy inequalities and find that the coherence can serve as the natural upper bound on the quantum discord. Finally, we find that our coherence measure as well as the l_1 norm can induce the experimentally measurable bounds of coherence, but the relative-entropy coherence can be in principle exactly measured in experiment.

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APPENDIX A: AN EXAMPLE FOR *K* COHERENCE VIOLATING (STRONG) MONOTONICITY

Reference [23] defined the K coherence of a state subject to the observable K by the quantum skew information instead of the direct commutation. That is,

$$C_K(\rho) = -\tfrac{1}{2}\mathrm{Tr}[\sqrt{\rho},K]^2. \tag{A1}$$

However, the quantification of coherence given in Eq. (1) not only includes the contribution of the basis which the observable defines but also includes the contribution of the eigenvalues of the observable. In particular, once the observable is degenerate, the observable will not extract all the coherence of the state, even though it should be valid in its own right. Most important is that such a definition only serves as a good coherence measure in the qubit system, which will be shown in the following section. One can easily find that in the general case, this coherence measure satisfies neither criterion b1 nor b2 in the main text. So it is not a good coherence measure in general cases, which is also found in [35]. To see this, let us consider the state

$$\rho = \begin{pmatrix} 0.6309 & 0.0359 & 0.0858 \\ 0.0359 & 0.0441 & 0.1189 \\ 0.0858 & 0.1189 & 0.3250 \end{pmatrix} \tag{A2}$$

which undergoes the incoherent quantum channel $\$_I = \{M_n\}$ with $M_1 = \begin{pmatrix} 0 & 0.3 & 0 \\ 0 & 0 & 0.5 \\ 0.7 & 0 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 0 & 0.8660 \\ 0 & 0.9539 & 0 \\ 0.7141 & 0 & 0 \end{pmatrix}$ and $M_1^\dagger M_1 + M_2^\dagger M_2 = \mathbb{I}_3$. One can obtain the state $\rho_1 = M_1\rho M_1^\dagger/p_1$ with the probability $p_1 = \mathrm{Tr}M_1\rho M_1^\dagger$ and the state $\rho_2 = M_2\rho M_2^\dagger/p_2$ with the probability $p_2 = \mathrm{Tr}M_2\rho M_2^\dagger$. It is easy to find that the average coherence $\bar{C}_K = p_1C_K(\rho_1) + p_2C_K(\rho_2) = 1.2928$, and the coherence $C_K(\rho')$ of the final state $\rho' = p_1\rho_1 + p_2\rho_2$ is given by $C_K(\rho') = 0.3350$, while the coherence of the initial state $C_K(\rho) = 0.2277$ where the reference observable $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. It is apparent that criteria b1 and b2 are simultaneously violated.

APPENDIX B: PROOF OF THE POLYGAMY OF OUR COHERENCE

1. Proof of Corollary 1

From the proof of Theorem 2, one can find that

$$\langle k|\sqrt{\rho_A}|k\rangle\langle k'|\sqrt{\rho_B}|k'\rangle \geqslant \langle kk'|\Psi\rangle_{AB}\langle\Psi|kk'\rangle \tag{B1}$$

holds for pure $|\Psi\rangle_{AB}$. Considering a mixed state with a potential decomposition $\rho_{AB}=\sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$ and substituting every $|\psi_i\rangle_{AB}$ into Eq. (B1), one will arrive at

$$\sum_i p_i\langle k|\sqrt{\rho_{Ai}}|k\rangle\langle k'|\sqrt{\rho_{Bi}}|k'\rangle \geqslant \sum_i p_i\langle kk'|\psi_i\rangle_{AB}\langle\psi_i|kk'\rangle. \tag{B2}$$

Squaring both sides of Eq. (B2) and summing over all the kk', we have

$$\sum_{kk'}\left[\sum_i p_i\langle k|\sqrt{\rho_{Ai}}|k\rangle\langle k'|\sqrt{\rho_{Bi}}|k'\rangle\right]^2 \geq \sum_{kk'}\left[\sum_i p_i\langle kk'|\psi_i\rangle_{AB}\langle\psi_i|kk'\rangle\right]^2 \tag{B3}$$

with $\rho_{Ai/Bi}$ being the reduced matrix of $|\psi_i\rangle_{AB}\langle\psi_i|$ by tracing over A or B. Based on the Cauchy-Schwarz inequality, we have

$$\sum_{kk'}\sum_i p_i\langle k|\sqrt{\rho_{Ai}}|k\rangle^2\sum_i p_i\langle k'|\sqrt{\rho_{Bi}}|k'\rangle^2 \geq \sum_{kk'}\langle kk'|\rho_{AB}|kk'\rangle^2. \tag{B4}$$

Based on the joint concavity of the function $f(A,B)=TrX^\dagger A^t XB^{1-t}$ on both A and B (Lieb's theorem) [66], Eq. (B4) becomes

$$\sum_{kk'}\langle k|\sqrt{\rho_A}|k\rangle^2\langle k'|\sqrt{\rho_B}|k'\rangle^2 \geq \sum_{kk'}\langle kk'|\rho_{AB}|kk'\rangle^2 \tag{B5}$$

with $\rho_{A/B}$ denoting the reduced matrices of ρ_{AB}. So we have

$$[1-C(\rho_A)][1-C(\rho_B)] \geqslant \sum_{kk'}\langle kk'|\rho_{AB}|kk'\rangle^2 = \mathrm{Tr}\rho_{AB}^2 - C_{l_2}(\rho_{AB}), \tag{B6}$$

where $C_{l_2}(\rho_{AB})=\mathrm{Tr}\rho_{AB}^2-\sum_{kk'}\langle kk'|\rho_{AB}|kk'\rangle^2$ is the coherence measure based on the l_2 norm. One can easily find that Eq. (B6) will be reduced to Theorem 2 if ρ_{AB} is a pure state. In addition, in order to use the coherence to describe $\sum_{kk'}\langle kk'|\rho_{AB}|kk'\rangle^2$ or its lower bound, we now consider the eigendecomposition of ρ_{AB}, i.e., $\rho_{AB}=\sum_i\lambda_i|\psi_i\rangle_{AB}\langle\psi_i|$. Thus $\sum_{kk'}\langle kk'|\rho_{AB}|kk'\rangle^2$ can be rewritten as

$$\begin{aligned}\sum_{kk'}\langle kk'|\rho_{AB}|kk'\rangle^2 &= \sum_{kk'}\left(\sum_i\lambda_i\langle kk'|\psi_i\rangle_{AB}\langle\psi_i|kk'\rangle\right)^2\\ &\geqslant \sum_{kk'}\left(\sum_i\sqrt{\lambda_{\min}}\sqrt{\lambda_i}\langle kk'|\psi_i\rangle_{AB}\langle\psi_i|kk'\rangle\right)^2\\ &= \lambda_{\min}[1-C(\rho_{AB})],\end{aligned} \tag{B7}$$

where $\lambda_{\min}$ is the minimal nonzero eigenvalue of ρ_{AB}. This is the first conclusion in Corollary 1. It can be seen that Eq. (B7) will go back to Theorem 2 due to $\lambda_{\min}=1$ for the pure ρ_{AB}.

Considering the eigendecomposition of $\rho_{AB}=\sum_i\lambda_i|\psi_i\rangle_{AB}\langle\psi_i|$, one can obtain a series of equations akin to Eq. (B1). Multiplying $\sqrt{\lambda_i}$ on both sides of these equations and then summing over all i, we will have

$$\sum_i\sqrt{\lambda_i}\langle k|\sqrt{\rho_{Ai}}|k\rangle\langle k'|\sqrt{\rho_{Bi}}|k'\rangle \geq \sum_i\sqrt{\lambda_i}\langle kk'|\psi_i\rangle_{AB}\langle\psi_i|kk'\rangle. \tag{B8}$$

Squaring both sides of Eq. (B8) and summing over all the kk', we arrive at

$$\left(\sum_i\sqrt{\lambda_i}\langle k|\sqrt{\rho_{Ai}}|k\rangle\langle k'|\sqrt{\rho_{Bi}}|k'\rangle\right)^2 \geq \left(\sum_i\sqrt{\lambda_i}\langle kk'|\psi_i\rangle_{AB}\langle\psi_i|kk'\rangle\right)^2. \tag{B9}$$

According to the Cauchy-Schwarz inequality, Eq. (B9) becomes

$$\sum_k\langle k|\sqrt{\rho_A}|k\rangle^2\sum_{k'i}\langle k'|\sqrt{\rho_{Bi}}|k'\rangle^2 \geq \sum_{kk'}\langle kk'|\sqrt{\rho_{AB}}|kk'\rangle^2 \tag{B10}$$

and

$$\sum_{ki}\langle k|\sqrt{\rho_{Ai}}|k\rangle^2\sum_{k'}\langle k'|\sqrt{\rho_B}|k'\rangle^2 \geq \sum_{kk'}\langle kk'|\sqrt{\rho_{AB}}|kk'\rangle^2. \tag{B11}$$

A simple algebra can further show that Eq. (B10) leads to

$$[1-C(\rho_A)]\left[r-\sum_i C(\rho_{Bi})\right] \geq 1-C(\rho_{AB}) \tag{B12}$$

and Eq. (B11) leads to

$$\left[r-\sum_i C(\rho_{Ai})\right][1-C(\rho_B)] \geq 1-C(\rho_{AB}). \tag{B13}$$

Combining Eqs. (B12) and (B13), one will obtain a symmetric form

$$[1-C(\rho_A)][1-C(\rho_B)] \geqslant \frac{1}{c_s}[1-C(\rho_{AB})]^2, \tag{B14}$$

where r denotes the rank of ρ_{AB} and $c_s=[r-\sum_i C(\rho_{Ai})][r-\sum_i C(\rho_{Bi})]$ with $\sum_i C(\rho_{Ai/Bi})$ corresponding to the sum of the subsystematic (A/B) coherence of all the eigenstates. It is obvious that the inequality will be reduced to the case of pure states for pure ρ_{AB}. The proof of Corollary 1 is finished.

2. Proof of Corollary 2

Corollary 2 is the result of the direct application of Corollary 1, so it is sufficient to consider an example to demonstrate how to arrive at the expected inequalities and how to determine the coefficient λ_M and c_{sT}. Without loss of generality, let us consider a quadripartite quantum state ρ_{ABCD}. At first, we would like to consider ρ_{ABCD} as a bipartite state as $\rho_{(AB)(CD)}$ (or $\rho_{A(BCD)}$ and so on). Based on Corollary 1, we have

$$[1-C(\rho_{AB})][1-C(\rho_{CD})] \geqslant \lambda_{\min 1}[1-C(\rho_{ABCD})], \tag{B15}$$

where ρ_{AB} and ρ_{CD} are the reduced density matrices of $\rho_{(AB)(CD)}$ and $\lambda_{\min 1}$ is the minimal nonzero eigenvalue of

$\rho_{(AB)(CD)}$. One can also find similar results for ρ_{AB} and ρ_{CD}, that is,

$$[1-C(\rho_A)][1-C(\rho_B)] \geqslant \lambda_{\min 2}[1-C(\rho_{AB})], \quad \text{(B16)}$$

$$[1-C(\rho_C)][1-C(\rho_D)] \geqslant \lambda_{\min 3}[1-C(\rho_{CD})], \quad \text{(B17)}$$

where $\lambda_{\min 2}$ and $\lambda_{\min 3}$ are the minimal nonzero eigenvalues for ρ_{AB} and ρ_{CD}, respectively. Thus one can stop at Eq. (B15) where $\lambda_M = \lambda_{\min 1}$. One can combine Eqs. (B15) and (B16) and obtain

$$[1-C(\rho_A)][1-C(\rho_B)][1-C(\rho_{CD})] \geqslant \lambda_{\min 1}\lambda_{\min 2}[1-C(\rho_{ABCD})], \quad \text{(B18)}$$

where $\lambda_M = \lambda_{\min 1}\lambda_{\min 2}$. A similar conclusion can be reached if Eqs. (B15) and (B17) are combined. Of course, one can combine all three equations, and finally get to

$$\prod_{i=A,B,C,D}[1-C(\rho_i)] \geqslant \lambda_M[1-C(\rho_{ABCD})] \quad \text{(B19)}$$

with $\lambda_M = \lambda_{\min 1}\lambda_{\min 2}\lambda_{\min 3}$. This demonstrates how to obtain Eq. (24) in the main text.

Let us consider ρ_{ABCD} again and first look at it as a bipartite state, for example, $\rho_{A(BCD)}$. Based on Corollary 1, we have

$$[1-C(\rho_A)][1-C(\rho_{BCD})] \geqslant \frac{1}{c_{s1}}[1-C(\rho_{ABCD})]^2, \quad \text{(B20)}$$

where $c_{s1} = [r_1 - \sum_{i=1}^{r_1} C(\rho_{Ai})][r_1 - \sum_{i=1}^{r_1} C(\rho_{(BCD)i})]$ with ρ_{Ai} and $\rho_{(BCD)i}$ denoting the reduced density matrices of the ith eigenstate of ρ_{ABCD} and r_1 being the rank of ρ_{ABCD}. If one just wants to consider such a bipartite grouping, Eq. (B20) is the final description of polygamy with $c_{sT} = c_{s1}$ and $n_1 = n_2 = 1$. One can continue to consider ρ_{BCD} as a bipartite state $\rho_{(BC)D}$ and continue to use Corollary 1. Then we will obtain

$$[1-C(\rho_{BC})][1-C(\rho_D)] \geqslant \frac{1}{c_{s2}}[1-C(\rho_{BCD})]^2, \quad \text{(B21)}$$

where $c_{s2} = [r_2 - \sum_{i=1}^{r_2} C(\rho_{(BC)i})][r_2 - \sum_{i=1}^{r_2} C(\rho_{Di})]$ with $\rho_{(BC)i}$ and ρ_{Di} representing the reduced density matrices of the ith eigenstate of ρ_{BCD} and r_2 being the rank of ρ_{BCD}. Substituting Eq. (B21) into Eq. (B20), one will arrive at

$$[1-C(\rho_A)]\sqrt{[1-C(\rho_{BC})][1-C(\rho_D)]} \geqslant \frac{1}{c_{s1}c_{s2}}[1-C(\rho_{ABCD})]^2, \quad \text{(B22)}$$

with $c_{sT} = c_{s1}c_{s2}$. Thus we can see that $n_1 = 1, n_2 = n_3 = \frac{1}{2}$. Of course, one can continue to divide ρ_{BC} and obtain another inequality, which is omitted here.

APPENDIX C: THE MEASURABLE RELATIVE-ENTROPY COHERENCE

Now we show that the relative-entropy coherence $C_r(\rho)$ can be directly measured in experiment.

$C_r(\rho)$ can be written as

$$C_r(\rho) = S(\rho^\star) - S(\rho) = \sum_j \lambda_j \log_2 \lambda_j - \sum_k \rho_{kk} \log_2 \rho_{kk} \quad \text{(C1)}$$

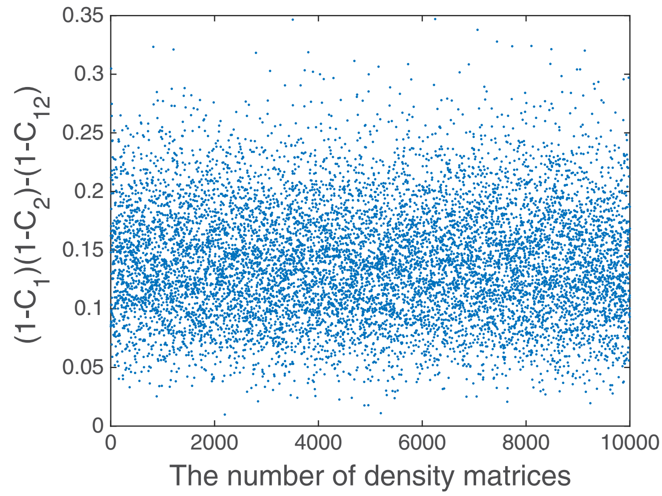

FIG. 1. All the density matrices ρ_{AB} are generated in $(2\otimes 3)$-dimensional Hilbert space.

where $\rho^\star$ denotes the state by deleting all off-diagonal entries of ρ, the λ_j's represent the eigenvalues of ρ, and $\rho_{kk} = \langle k|\rho|k\rangle$ are the diagonal entries of ρ within the reference basis $\{|k\rangle\}$. It is obvious that once the knowledge of λ_j and ρ_{kk} is extracted from an experiment, $C(\rho)$ is determined. This can be accomplished by the generalized standard overlap measurement [64,65] and simple projective measurements. To do so, we can define the generalized swapping operator V_n for natural number $n > 1$ as $V_n|\psi_1,\psi_2,\cdots,\psi_n\rangle = |\psi_n,\psi_1,\psi_2,\cdots,\psi_{n-1}\rangle$. So a controlled V_n gate can be constructed as $\mathbb{I}_2 \oplus V_n$ with a qubit as the control qubit. It is easy to find that $\mathrm{Tr}\rho^n = \mathrm{Tr}V_n\rho^{\otimes n}$. Now let us first prepare a probing qubit $|\varphi\rangle_p = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and n copies of measured state ρ. Then let the $n+1$ particles undergo the controlled V_n gate. Finally, let us measure σ_x on the probing qubit and obtain ± 1 with the probability $p_n^\pm = \frac{1\pm \mathrm{Tr}\rho^n}{2}$. Thus based on p_n^+ (or p_n^-) for $n = 2,3,\cdots,N_D$, with $\mathrm{Tr}\rho = 1$ all the λ_j's can be unambiguously determined and so can $\sqrt{\lambda_j}$'s. In

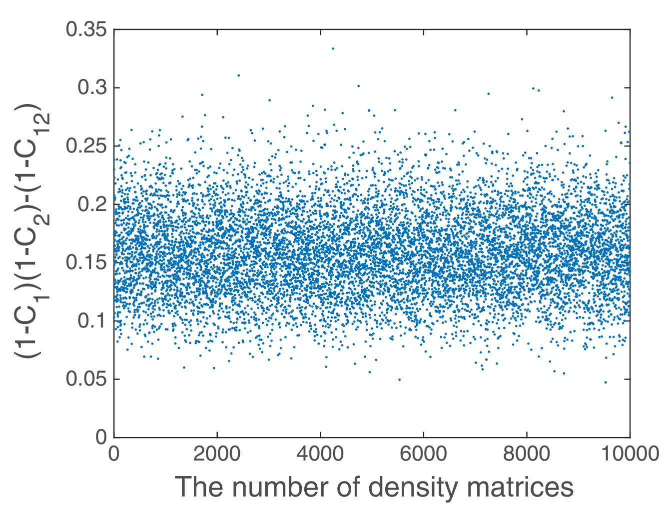

FIG. 2. All the density matrices ρ_{AB} are generated in $(3\otimes 3)$-dimensional Hilbert space.

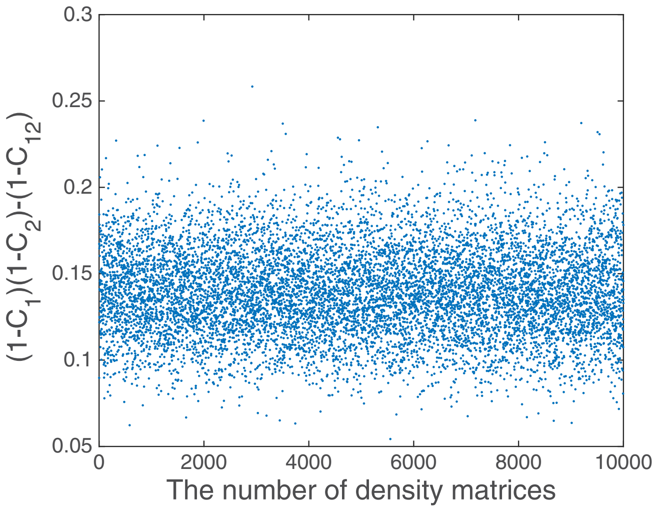

FIG. 3. All the density matrices ρ_{AB} are generated in $(3\otimes 4)$-dimensional Hilbert space.

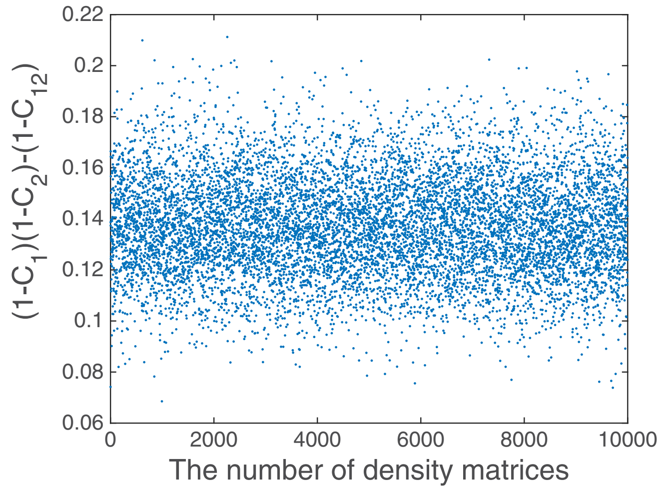

FIG. 4. All the density matrices ρ_{AB} are generated in $(4\otimes 4)$-dimensional Hilbert space.

addition, $\langle k|\rho|k\rangle$ can be measured directly by the projective measurement subject to the projectors $\hat{P}_k = |k\rangle\langle k|$. Therefore, $C(\rho)$ is obtained. Compared with $N_D^2 - 1$ observables in QST, the total cost is $N_D - 1$ controlled V_n gates plus $N_D - 1$ projective measurements assisted by at most N_D copies of the state.

APPENDIX D: THE CONJECTURE

The polygamy relation has an elegant form for the bipartite pure state, but one can easily find that such a relation does not hold for general mixed states. This can be seen as follows. Let us consider the qubit state $\rho_{AB} = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|$ with $|\psi_1\rangle = [-0.5612, -0.982, 0.8119, 0.1272]^T$, $|\psi_2\rangle = [0.8006, 0.1842, 0.5556, 0.1283]^T$, $\langle\psi_1|\psi_2\rangle = 0$, and $p = 0.0443$. A simple algebra can show that $C(\rho_1) = 0.2582$, $C(\rho_2) = 0.0909$, and $C(\rho) = 0.3242$ with $\rho_i = \mathrm{Tr}_{A/B}|\psi_i\rangle\langle\psi_i|$. Thus it is easy to check that $[1 - C(\rho_1)][1 - C(\rho_2)] = 0.7418 \times 0.9091 = 0.6744 < 0.6758 = 1 - C(\rho)$. However, through our numerical test, we conjecture that the same form of Theorem 2 for a $(N \geqslant 6)$-dimensional state could also be satisfied. In Figs. 1–4, we numerically test the inequality in high-dimensional systems, but we do not find the counterexample. In the figures, we use C_{12} to denote the bipartite state $C(\rho_{AB})$ and C_i to denote $C(\rho_i)$ with $\rho_i = \mathrm{Tr}_{A/B}\rho_{AB}$ representing the corresponding reduced density matrices. All the tested density matrices $\rho_{AB} = \frac{(A*A' + B*B')}{\mathrm{Tr}A*A' + B*B'}$ with $B = C + iD$ and A, C, D are randomly generated by MATLAB R2014b. One can find that in all the figures $(1 - C_1)(1 - C_2) - (1 - C_{12}) \geqslant 0$. Comparing the four figures, one can find that the minimal value of $(1 - C_1)(1 - C_2) - (1 - C_{12})$ in the figures is increased with the increasing of the dimension of the state. In this sense, we would like to conjecture that this relation should be satisfied in $(N \geqslant 6)$-dimensional systems.

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