Rényi formulation of entanglement criteria for continuous variables

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Entanglement criteria for an n-partite quantum system with continuous variables are formulated in terms of Rényi entropies. Rényi entropies are widely used as a good information measure due to many nice properties. Derived entanglement criteria are based on several mathematical results such as the Hausdorff-Young inequality, Young's inequality for convolution and its converse. From the historical viewpoint, the formulations of these results with sharp constants were obtained comparatively recently. Using the position and momentum observables of subsystems, one can build two total-system measurements with the following property. For product states, the final density in each global measurement appears as a convolution of n local densities. Hence, restrictions in terms of two Rényi entropies with constrained entropic indices are formulated for n-separable states of an n-partite quantum system with continuous variables. Experimental results are typically sampled into bins between prescribed discrete points. For these aims, we give appropriate reformulations of the derived entanglement criteria.

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I. INTRODUCTION

Quantum entanglement is one of the fundamental properties of Nature at the microscopic level. This quantum-mechanical feature was concerned by founders in the Schrödinger "cat paradox" paper [1] and in the Einstein-Podolsky-Rosen paper [2]. In view of the role of entanglement in quantum theory, related questions deserve to be studied in detail (see, e.g., the review [3] and references therein). Due to progress in quantum information processing, both the detection and quantification of entanglement are very important. In the case of discrete variables, the positive partial transpose (PPT) criterion [4] and the reduction criterion [5] are very powerful. On the other hand, no universal criteria are known even for discrete variables. Say, the PPT criterion is necessary and sufficient for 2×2 and 2×3 systems, but ceases to be so in higher dimensions [6]. Separability conditions can be derived from various uncertainty relations [7–13]. These studies concerned finite-dimensional quantum systems. For systems with continuous variables, detection of entanglement is a more challenging task. Reasons for studying quantum information with continuous variables are originated in the fact that many quantum protocols can be efficiently implemented within current technologies of quantum optics [14,15].

Due to a practical importance, Gaussian states were well studied from the viewpoint of entanglement detection [16,17]. Properties of Gaussian entanglement with respect to information processing were discussed in [18–21]. Entanglement criteria of the second-order type deal with variations of certain observables. For systems with continuous variables, such criteria were proposed in [16,22–25]. The authors of [26] formulated an infinite hierarchy of conditions for positive partial transpose involving higher-order moments. The conditions of [26] were later amended in [27]. Such conditions provide a very powerful criterion, which is rather hard for implementation in experimental practice. The authors of [28,29] have formulated biseparability conditions in terms of differential entropies related to measurement statistics. This method differs from some previous studies, in which entropies of density operators were considered [30–33]. In particular, inequalities with Rényi entropies of density matrices can be treated as a condition for local realism [31]. The authors of [33] studied the relation between entanglement properties and conditional Rényi and Tsallis entropies for bipartite quantum systems in finite dimensions. Many covariance-matrix-based criteria and Shannon-entropy criteria can be unified within a general formalism proposed in [34].

When more than two parties are involved, the structure of entanglement is much richer in comparison with the bipartite case [35,36]. The authors of [36] developed a general framework for constructing multipartite entanglement tests. Separability eigenvalue equations of [36] allow one to witness partial and full entanglement in multipartite composed systems. This basic technique has been applied to examine multipartite entanglement of frequency-comb Gaussian states [37,38]. It was shown in [38] that there are two separable states which include all other forms of higher-order entanglement. This example illustrates significance of studying different categories of multipartite entanglement. In practice, we would often like to detect the entanglement of states that are partially or completely unknown. In this case, desired entanglement criteria should immediately be related to results of some measurements specially built for such purposes. Another approach is to construct the density operator via quantum tomography, but quantum tomography usually requires considerable effort. The separability problem with partial information was addressed in [34,39]. The formalism of [34] allows one to extend biseparability conditions to multimode case in both discrete- and continuous-variable systems.

The aim of this work is to formulate n-separability conditions for an n-partite system with continuous variables in terms of generalized entropies. We also study derived entanglement criteria from the viewpoint of sampling density functions into bins. The paper is organized as follows. In Sec. II, the required material is presented. Entropic functionals of the Rényi type are briefly discussed. Further, we recall Young's inequality and its converse, both formulations with sharp constants. In Sec. III, we formulate Rényi-entropy entanglement criteria for a multipartite quantum system with continuous variables. The global observables are constructed within a commonly accepted approach to deriving separability conditions for such systems. In Sec. IV, we examine the presented separability conditions from the viewpoint of their use in practice of quantum information processing. Appropriate reformulations are given for sampling measurement statistics into prescribed bins. In this case, separability conditions in terms of Tsallis entropies are also given. A utility of the derived criteria is illustrated with examples in Sec. V. It is shown that separability conditions in terms of generalized entropies sometimes lead to more robust detection of entanglement.

II. PRELIMINARIES

In this section, we recall the required material and describe the notation. Let $x \in \mathbb{R}$, and let $v(x)$ be probability density function of some continuous variable. Then, the differential Shannon entropy is defined as [40]

$$H_1(v) := -\int_{\mathbb{R}} v(x)\,\ln v(x)\,dx. \tag{1}$$

There are several fruitful generalizations of the standard Shannon entropy. For $0 < \alpha \neq 1$, the differential Rényi entropy is written as

$$H_\alpha(v) := \frac{1}{1-\alpha}\,\ln\left[\int_{\mathbb{R}} v(x)^\alpha\,dx\right]. \tag{2}$$

This entropy is a continuous analog of the α entropy introduced by Rényi in [41]. Entropies of discrete random variable will be used, when continuous variable is sampled into chosen bins. Let such bins be specified by the set of marks $\{\xi_i\}$. Hence, we have the intervals $\Delta\xi_i = \xi_{i+1} - \xi_i$ with the maximum $\Delta\xi = \max \Delta\xi_i$. We then introduce probabilities

$$q_i := \int_{\xi_i}^{\xi_{i+1}} v(x)\,dx. \tag{3}$$

To get a good exposition, the size of bins should be sufficiently small in comparison with a scale of considerable changes of $v(x)$. For the discrete distribution with probabilities (3), its Rényi α entropy is defined as [41]

$$H_\alpha(q) := \frac{1}{1-\alpha}\,\ln\left(\sum_i q_i^\alpha\right), \tag{4}$$

where $0 < \alpha \neq 1$. In the limit $\alpha \to 1$, we obtain the usual Shannon entropy

$$H_1(q) := -\sum_i q_i\,\ln q_i, \tag{5}$$

where $-0\ln 0 \equiv 0$ by definition. Many interesting properties of Rényi entropies with applications are discussed in [42].

To pose required mathematics formally, we introduce convenient normlike functionals. For arbitrary $\alpha > 0$, one defines

$$\|f\|_\alpha := \left[\int_{\mathbb{R}} |f(x)|^\alpha\,dx\right]^{1/\alpha}. \tag{6}$$

Of course, we will further assume that such integrals exist. The right-hand side of (6) gives a legitimate norm only for $\alpha \geqslant 1$. The case $\alpha = \infty$ is allowed and leads to the essential supremum [43]. For the given discrete distribution and $\alpha > 0$, we also define

$$\|q\|_\alpha := \left(\sum_i q_i^\alpha\right)^{1/\alpha}, \tag{7}$$

including $\|q\|_\infty = \max q_i$. The α entropy (2) can be rewritten as

$$H_\alpha(v) = \frac{\alpha}{1-\alpha}\,\ln\|v\|_\alpha. \tag{8}$$

In a similar manner, we express (4) in terms of (7). Due to $q_i \leqslant 1$, for $\alpha > 1 > \beta$ we clearly have

$$\|q\|_\alpha \leqslant 1 \leqslant \|q\|_\beta. \tag{9}$$

Hence, Rényi entropies of discrete probability distributions are always positive including zero for deterministic distributions. This is not the case for differential entropies of the form (2). Despite of the normalization $\|v\|_1 = 1$, we cannot generally write a continuous counterpart of (9). The quantity (2) is not of definite sign and becomes negative for density functions with sufficiently large variations. Nevertheless, relations with such entropies may express nontrivial conditions. For instance, the differential Shannon entropy of phase with negative values was considered in [44]. Differential entropies are also an intermediate point in obtaining conditions for entropies with binning.

We will also use several mathematical results for functions of one scalar variable. First, we recall the Hausdorff-Young inequality with sharp constants. The question concerns relations between norms of a function and its Fourier transform. The sharp Hausdorff-Young inequality was found by Beckner [45] with using the previous result of Babenko [46]. We recall this result in a reformulation convenient for our aims. It deals with probability density functions and leads to entropic uncertainty relations for the position and momentum [47]. So, the Hausdorff-Young inequality with sharp constants leads to an improvement of the first entropic uncertainty relation of Hirschman [48]. Let two functions $\psi(x)$ and $\varphi(k)$ be connected by the Fourier transform, namely,

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(+ikx)\,\varphi(k)\,dk, \tag{10}$$

$$\varphi(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-ikx)\,\psi(x)\,dx. \tag{11}$$

They are treated as wave functions in the position and momentum spaces, respectively. Here, we accepted units in which $\hbar = 1$. The probability density functions are written as

$$v(x) = |\psi(x)|^2, \quad \tilde{v}(k) = |\varphi(k)|^2. \tag{12}$$

Introducing the Fourier transform by means of (11) is physically motivated. At the same time, the formula (11) slightly differs from the definition commonly used in the mathematical literature. So, we wrote the corresponding inequality of Beckner [45] in terms of the above wave functions and converted it into relations between density functions. We refrain from presenting the details here since this point has been addressed, e.g., in the works [49–51]. The result is posed as follows. Let positive indices α and β obey $1/\alpha + 1/\beta = 2$, and let $\alpha > 1 > \beta$. For any quantum state, normlike functionals of the position and momentum densities then obey

$$\|v\|_\alpha \leqslant \left(\frac{1}{\varkappa\pi}\right)^{(1-\beta)/\beta}\|\tilde{v}\|_\beta, \tag{13}$$

$$\|\tilde{v}\|_\alpha \leqslant \left(\frac{1}{\varkappa\pi}\right)^{(1-\beta)/\beta}\|v\|_\beta. \tag{14}$$

Here, the positive parameter $\varkappa$ is given by the formula

$$\varkappa^2 = \alpha^{1/(\alpha-1)}\beta^{1/(\beta-1)}. \tag{15}$$

It will be convenient to parametrize the indices α and β as $1/\alpha = 1-\tau$ and $1/\beta = 1+\tau$ with $\tau \in [0;1]$. By calculations, we then get

$$2\ln\varkappa(\tau) = \frac{1+\tau}{\tau}\ln(1+\tau) - \frac{1-\tau}{\tau}\ln(1-\tau). \tag{16}$$

The relations (13) and (14) hold, when wave functions are related via the Fourier transform with infinite limits. The corresponding observables obey the position-momentum commutation relation and have eigenvalues covering the real axis. Note that the sharp Hausdorff-Young inequality *per se* implies (13) and (14) only for pure states. However, they can immediately be extended to mixed states. The "twin" relations (13) and (14) lead to uncertainty relations in terms of Rényi's entropies as described in [49,51].

Dealing with convolutions, we should recall Young's inequality with sharp constants. It was found by Beckner [45] and by Brascamp and Lieb [52] with the use of different methods. We will closely follow the formulation of Brascamp and Lieb since they also gave the converse of Young's inequality with sharp constants. To each real index $a \geqslant 1$, we assign the conjugate index a' such that

$$\frac{1}{a} + \frac{1}{a'} = 1. \tag{17}$$

The Young inequality involves factors of the form $C(a)$ defined by

$$C(a)^2 = a^{1/a}\,(a')^{-1/a'}. \tag{18}$$

By $f * g$, we will mean the convolution of two functions of one scalar variable. Let the indices be such that $a_\ell, a \geqslant 1$ and their conjugate ones obey

$$\sum_{\ell=1}^{n}\frac{1}{a'_\ell} = \frac{1}{a'}. \tag{19}$$

For the convolution of n functions, one has

$$C(a)\,\|f_1 * \cdots * f_n\|_a \leqslant \prod_{\ell=1}^{n} C(a_\ell)\,\|f_\ell\|_{a_\ell}. \tag{20}$$

More results about the Young inequality as well as the Hausdorff-Young inequality can be found, e.g., in chapters 4 and 5 of the book by Lieb and Loss [43].

Inequalities converse to (20) generally involve indices, some of which are negative. Here, the definition should be reformulated. If indices b and b' are conjugate in the sense of (17), then [52]

$$C(b)^2 = |b|^{1/b}\,|b'|^{-1/b'}. \tag{21}$$

When $0 < b < 1$, the conjugate index b' is strictly negative. To emphasize this distinction, we prefer to mention (18) and (21) independently. Let the indices be such that $0 < b_\ell, b \leqslant 1$ and

$$\sum_{\ell=1}^{n}\frac{1}{b'_\ell} = \frac{1}{b'}. \tag{22}$$

For the convolution of n one-dimensional functions, we have [52]

$$\prod_{\ell=1}^{n} C(b_\ell)\,\|f_\ell\|_{b_\ell} \leqslant C(b)\,\|f_1 * \cdots * f_n\|_b. \tag{23}$$

This issue is connected with some previous results of Leindner and Prékopa (see, e.g., references in [52]). Calculating with factors of the form (18) and (21), we will often use the expression

$$2\ln C(b) = \frac{1}{b'}\ln\frac{1}{|b'|} - \frac{1}{b}\ln\frac{1}{|b|}. \tag{24}$$

In the following, both the inequalities (20) and (23) will be used in deriving separability conditions. One form of the Minkowski inequality will also be recalled when appropriate. Concerning this inequality, see corresponding sections of the book by Hardy *et al.* [53].

III. ENTANGLEMENT CRITERIA FOR A MULTIPARTITE QUANTUM SYSTEM

In this section, we obtain n-separability conditions for an n-partite quantum system with continuous variables. Let subsystems of an n-partite system be labeled by $\ell = 1,\ldots,n$. The product $\mathcal{H}_{1:n} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is the total Hilbert space. Any quantum state of the total system is given by a density matrix $\boldsymbol{\rho}_{1:n}$ on $\mathcal{H}_{1:n}$. Density matrices are assumed to be normalized. We note that n-fold product states of the form $\boldsymbol{\rho}_1 \otimes \cdots \otimes \boldsymbol{\rho}_n$ have no correlations between subsystems. Recall that a bipartite mixed state $\boldsymbol{\rho}_{1:2}$ is called separable, when its density matrix can be written as a convex combination of product states [54,55]. For an n-partite system, we call $\boldsymbol{\rho}_{1:n}$ to be n separable, when it can be represented as a convex combination of product states of the form $\boldsymbol{\rho}_1 \otimes \cdots \otimes \boldsymbol{\rho}_n$. Such states are often called fully separable [35]. Without loss of generality, each separable state will be treated as a convex combination of only pure product states.

To formulate entanglement criteria, appropriate global observables will be built from local ones [16,24,25]. We first recall the formulation for a bipartite system. To each subsystem $\ell = 1,2$, one assigns the position and momentum variables x_ℓ and p_ℓ so that $[\mathsf{x}_\ell,\mathsf{p}_\ell] = i\mathbb{1}_\ell$, where $\mathbb{1}_\ell$ is the identity on $\mathcal{H}_\ell$. Using real θ_ℓ, we define the operators

$$\mathsf{r}_\ell := \cos\theta_\ell\,\mathsf{x}_\ell + \sin\theta_\ell\,\mathsf{p}_\ell, \tag{25}$$

$$\mathsf{s}_\ell := -\sin\theta_\ell\,\mathsf{x}_\ell + \cos\theta_\ell\,\mathsf{p}_\ell, \tag{26}$$

which also obey $[\mathsf{r}_\ell,\mathsf{s}_\ell] = i\mathbb{1}_\ell$. It is a linear canonical transformation in phase space, corresponding to a unitary transformation of the Hilbert space [56].

With the signs $\epsilon = \pm 1$ and $\bar\epsilon = \mp 1$, we further write

$$\mathsf{R}_\epsilon := \mathsf{r}_1 \otimes \mathbb{1}_2 + \epsilon\,\mathbb{1}_1 \otimes \mathsf{r}_2, \tag{27}$$

$$\mathsf{S}_{\bar\epsilon} := \mathsf{s}_1 \otimes \mathbb{1}_2 + \bar\epsilon\,\mathbb{1}_1 \otimes \mathsf{s}_2. \tag{28}$$

The observables R_ϵ and $\mathsf{S}_{\bar\epsilon}$ are commuting and jointly measurable. Let $|r_\ell\rangle$'s be eigenkets of r_ℓ normalized through Dirac's delta function. The observable (27) satisfies $\mathsf{R}_\epsilon|r_1,r_2\rangle = (r_1 + \epsilon\,r_2)|r_1,r_2\rangle$, where $|r_1,r_2\rangle = |r_1\rangle \otimes |r_2\rangle$. Let $\boldsymbol{\rho}_{1:2}$ be the state to be tested. For the observable (27), we get the probability density function after one integration of $\langle r_1,r_2|\boldsymbol{\rho}_{1:2}|r_1,r_2\rangle$. For any product state $\boldsymbol{\rho}_1\otimes\boldsymbol{\rho}_2$, we write $w_\ell(r_\ell)=\langle r_\ell|\boldsymbol{\rho}_\ell|r_\ell\rangle$ so that this integrand reads as $w_1(r_1)\,w_2(r_2)$. The probability density function of $r=r_1+\epsilon\, r_2$ then becomes $w_1*w_{\epsilon 2}$, where $w_{\epsilon 2}(r')=w_2(\epsilon r')$. Using this fact, the authors of [28] applied the entropy power inequality for the Shannon entropy. For the Rényi α entropy, a version of entropy power inequalities was given in [57] but only for $\alpha\geqslant 1$. The problem of extending the entropy power inequality to orders $0<\alpha<1$ remains open. To derive biseparability conditions in terms of generalized entropies, the authors of [29] used Young's inequality and its converse with sharp constants. Entropy power inequalities of the Rényi type with improved coefficients depend on number and dimensionality of involved random vectors [58]. As the dimensionality is explicitly used, such inequalities are relevant only for finite-dimensional variables.

Global observables for an n-partite system with continuous variables can be built in a similar way. To ℓth subsystem, we assign the operators $\widetilde{\mathsf{r}}_\ell$ and $\widetilde{\mathsf{s}}_\ell$, acting as r_ℓ and s_ℓ in $\mathcal{H}_\ell$ and as the identity in other subspaces. More precisely, we write

$$\widetilde{\mathsf{r}}_\ell=\mathbb{1}_1\otimes\cdots\otimes\mathbb{1}_{\ell-1}\otimes\mathsf{r}_\ell\otimes\mathbb{1}_{\ell+1}\otimes\cdots\otimes\mathbb{1}_n, \tag{29}$$

and similarly for $\widetilde{\mathsf{s}}_\ell$. Taking $\epsilon_\ell,\varepsilon_\ell\in\{+1,-1\}$, the two observables of interest are then defined as

$$\mathsf{R}_{\epsilon:\epsilon}:=\epsilon_1\widetilde{\mathsf{r}}_1+\cdots+\epsilon_n\widetilde{\mathsf{r}}_n, \tag{30}$$

$$\mathsf{S}_{\varepsilon:\varepsilon}:=\varepsilon_1\widetilde{\mathsf{s}}_1+\cdots+\varepsilon_n\widetilde{\mathsf{s}}_n. \tag{31}$$

The operators $\widetilde{\mathsf{r}}_\ell$ and $\widetilde{\mathsf{s}}_{\ell'}$ commute for $\ell\neq\ell'$, hence,

$$[\mathsf{R}_{\epsilon:\epsilon},\mathsf{S}_{\varepsilon:\varepsilon}]=i\mathbb{1}_{1:n}\sum_{\ell=1}^{n}\epsilon_\ell\,\varepsilon_\ell, \tag{32}$$

where $\mathbb{1}_{1:n}$ is the identity on $\mathcal{H}_{1:n}$. For even n, we can make (30) and (31) to be commuting. Say, we set up $\varepsilon_\ell=\epsilon_\ell$ for odd ℓ and $\varepsilon_\ell=-\epsilon_\ell$ for even ℓ.

The product state $|r_1,\ldots,r_n\rangle$ is an eigenstate of (30), corresponding to the eigenvalue $r=\epsilon_1r_1+\cdots+\epsilon_nr_n$. Like the case $n=2$, we define

$$V(r_1,\ldots,r_n)=\langle r_1,\ldots,r_n|\boldsymbol{\rho}_{1:n}|r_1,\ldots,r_n\rangle. \tag{33}$$

For brevity, we suppose $\epsilon_\ell=+1$ for all $\ell=1,\ldots,n$. To the observable (30), one assigns the probability density function

$$W(r)=\int\cdots\int_{r_1+\cdots+r_n=r}V(r_1,\ldots,r_n)\,dr_1\cdots dr_{n-1}. \tag{34}$$

For an n-fold product state $\boldsymbol{\rho}_1\otimes\cdots\otimes\boldsymbol{\rho}_n$, the expression (34) results in the convolution of n local densities,

$$W=w_1*\cdots*w_n. \tag{35}$$

For other choices of the signs ϵ_ℓ, we replace each w_ℓ with $w_{\epsilon\ell}$, where $w_{\epsilon\ell}(r')=w_\ell(\epsilon_\ell r')$. The same claims hold for probability density functions assigned to (31).

The above findings allow us to derive n-separability conditions in terms of Rényi entropies. We will follow the strategy already justified in the previous papers [50,51,59]. Here, the basic idea is to deal with inequalities between normlike functionals of the form (6). Desired relations in terms of suitable entropies are extracted only at the final step. It is important that our approach can naturally be combined with Young's inequality *per se*. It is therefore more direct than appealing to entropy power inequalities. Indeed, the entropy power inequality proved in [57] are mainly based on Young's inequality with sharp constants. The method of [57] was inspired by the earlier Lieb's proof of the entropy power inequality for the Shannon entropy [60]. The following statement takes place.

Proposition 1. Let positive indices a and b be defined by the formulas

$$\frac{1}{a}=1-t,\quad \frac{1}{b}=1+t, \tag{36}$$

where $t\in[0;1]$, and let

$$\ln\mathcal{K}(t)=\frac{1}{2}\left[\frac{1+t}{t}\,\ln(1+t)-\frac{1-t}{t}\,\ln(1-t)\right]. \tag{37}$$

Let W and U be density functions obtained, respectively, for the observables (30) and (31) with $n\geqslant 2$. If n-partite state $\boldsymbol{\rho}_{1:n}$ is n separable, then we have the inequality

$$H_a(W|\boldsymbol{\rho}_{1:n})+H_b(U|\boldsymbol{\rho}_{1:n})\geqslant\ln(n\mathcal{K}\pi), \tag{38}$$

and its "twin" with swapped W and U.

Proof. To simplify the notation, we will take $\epsilon_\ell=\varepsilon_\ell=+1$ for all $\ell=1,\ldots,n$. For other choices of the signs in (30) and (31), the desired results follow due to

$$\|w_{\epsilon\ell}\|_\alpha=\|w_\ell\|_\alpha,\quad \|u_{\varepsilon\ell}\|_\beta=\|u_\ell\|_\beta. \tag{39}$$

First, we will prove inequalities for arbitrary product state. Writing $\tau=t/n\in[0;1/n]$, we further use the parameters α and β such that

$$\frac{1}{\alpha}=1-\tau,\quad \frac{1}{\alpha'}=\tau,\quad \frac{1}{\beta}=1+\tau,\quad \frac{1}{\beta'}=-\tau. \tag{40}$$

In Young's inequality (20), we set $a_\ell=\alpha\geqslant 1$ for all $\ell=1,\ldots,n$, hence, the index restriction (19) gives

$$\frac{1}{a'}=\frac{n}{\alpha'}=n\tau,\quad \frac{1}{a}=1-n\tau, \tag{41}$$

consistently with (36) due to $t=n\tau$. Combining (20) with (35) then gives

$$\|W\|_a\leqslant C(\alpha)^n\,C(a)^{-1}\prod_{\ell=1}^{n}\|w_\ell\|_\alpha. \tag{42}$$

With each of the quantities $\|w_\ell\|_\alpha$, we use local uncertainty relations of the form (13). This step results in

$$\|W\|_a\leqslant\frac{C(\alpha)^n}{C(a)}\left(\frac{1}{\varkappa\pi}\right)^{n(1-\beta)/\beta}\prod_{\ell=1}^{n}\|u_\ell\|_\beta, \tag{43}$$

where $\varkappa(\tau)$ is defined by (16). To the convolution $U=u_1*\cdots*u_n$, we apply (23) with setting $0<b_\ell=\beta\leqslant 1$ for all $\ell=1,\ldots,n$. Consistently with (36), the index restriction (22) implies

$$\frac{1}{b'}=\frac{n}{\beta'}=-n\tau,\quad \frac{1}{b}=1+n\tau, \tag{44}$$

and the converse of Young's inequality reads as

$$\prod_{\ell=1}^{n}\|u_\ell\|_\beta\leqslant C(\beta)^{-n}\,C(b)\,\|U\|_b. \tag{45}$$

Combining (43) with (45) immediately gives

$$\|W\|_a \leqslant \frac{C(\alpha)^n\, C(b)}{C(a)\, C(\beta)^n}\left(\frac{1}{\varkappa\pi}\right)^t \|U\|_b. \tag{46}$$

It will be convenient to simplify factors that appeared in the right-hand side of (46).

Using expressions of the form (24), we further obtain

$$\begin{aligned}\frac{1}{t}\ \ln\frac{C(\beta)^n}{C(\alpha)^n} &= \frac{1}{2\tau}[-2\tau\ln\tau-(1+\tau)\ln(1+\tau)\\ &\quad+(1-\tau)\ln(1-\tau)]\\ &= -\ln\tau-\ln\varkappa,\end{aligned} \tag{47}$$

where $\ln\varkappa$ is written from (16). The expression for $\ln[C(b)/C(a)]$ is obtained from $\ln[C(\beta)/C(\alpha)]$ by replacing τ with t, hence,

$$\frac{1}{t}\ \ln\frac{C(a)}{C(b)} = \ln t+\ln\mathcal{K}. \tag{48}$$

Combining (47) and (48) with (37) finally gives

$$\frac{1}{t}\ \ln\frac{C(a)\, C(\beta)^n}{C(\alpha)^n\, C(b)}+\ln\varkappa\pi = \ln(n\mathcal{K}\pi). \tag{49}$$

Thus, we can finally rewrite (46) in the form

$$\|W\|_a \leqslant \left(\frac{1}{n\mathcal{K}\pi}\right)^t \|U\|_b. \tag{50}$$

By a parallel argument, we can obtain the "twin" of (50) with swapped W and U. The latter holds for each product state. Before completing the proof, we should extend our findings to separable states.

Each separable state can be represented as a convex combination of product states. Hence, we obtain

$$W(r) = \sum_{\lambda}\lambda\, W^{(\lambda)}(r), \tag{51}$$

and a similar expression for U. Of course, the weights are normalized here as $\sum_{\lambda}\lambda = 1$. Following [49,61], at this step we use the Minkowski inequality [53]. This inequality results in

$$\|W\|_a = \left\|\sum_{\lambda}\lambda\, W^{(\lambda)}\right\|_a \leqslant \sum_{\lambda}\lambda\, \|W^{(\lambda)}\|_a, \tag{52}$$

$$\sum_{\lambda}\lambda\|U^{(\lambda)}\|_b \leqslant \left\|\sum_{\lambda}\lambda\, U^{(\lambda)}\right\|_b = \|U\|_b, \tag{53}$$

where we recall $a>1>b>0$. For each λ, the quantities $\|W^{(\lambda)}\|_a$ and $\|U^{(\lambda)}\|_b$ satisfy (50). The latter remains therefore valid for the quantities $\|W\|_a$ and $\|U\|_b$ calculated in any separable state.

The final step is to convert (50) into entropic inequalities. We will first obtain entropic relations for $t>0$. The Shannon case $a=b=1$ is reached by taking the corresponding limit. The Rényi entropies are expressed via normlike functionals according to (8), hence,

$$H_a(W|\boldsymbol{\rho}_{1:n}) = -\frac{1}{t}\ \ln\|W\|_a, \tag{54}$$

$$H_b(U|\boldsymbol{\rho}_{1:n}) = \frac{1}{t}\ \ln\|U\|_b. \tag{55}$$

To reach (38), we take the logarithm of both the sides of (50) and use (54) and (55). The inequality with swapped W and U is obtained by a very parallel argument. ∎

We obtained n-separability conditions as a strictly positive lower bound on the sum of two Rényi entropies. With growth of n, the lower bound in the right-hand side of (38) increases as a logarithm. For $n=2$, our result is similar to the biseparability conditions derived in [29]. The latter generalizes the conditions in terms of differential Shannon entropies proved in [28]. Going from product states to separable ones, the authors of [28] used concavity of the Shannon entropy. In general, a reference to concavity is not relevant for Rényi entropies. It is for this reason that the Minkowski inequality was applied. In a similar manner, the Minkowski inequality was already used in deriving entropic uncertainty relations in [49,61]. Here, we again see a convenience of dealing with relations between normlike functionals.

With growth of $t\in[0;1]$, the parameter (37) decreases from $\mathcal{K}(0)=e$ up to $\mathcal{K}(1)=2$. In terms of differential Shannon entropies, we therefore have

$$H_1(W|\boldsymbol{\rho}_{1:n})+H_1(U|\boldsymbol{\rho}_{1:n}) \geqslant \ln(ne\pi). \tag{56}$$

For $n=2$, this inequality reduces to the main result of [28]. Thus, we have obtained an n-partite extension of Shannon-entropy entanglement criteria for continuous variables. Of course, concrete experimental setup is prescribed by the choice of the angles θ_ℓ in (25) and (26). For the input state of a n-partite system, we then measure commuting observables (30) and (31). Evaluating the densities W and U, we can check the condition (38) and its "twin" for various t. Their violation for any value of t will imply that the input is not n separable.

Using the entropy power inequality, the authors of [28] gave a combined inequality, which involves two global and four local densities. Such relations merely reflect the fact that the given density function is the convolution of two local densities. To use them in entanglement detection, we must *a priori* be sure that the input state is pure. This case will be exemplified in Sec. V. In general, however, the consideration of only pure states is too idealized.

IV. CRITERIA IN TERMS OF DISCRETIZED DISTRIBUTIONS

Previously, we have derived n-separability conditions for continuous variables in terms of Rényi entropies. Such relations are not applicable immediately in analysis of experimental data. Continuous-variable probability density functions are typically replaced with experimentally resolvable discrete probability distributions. In the following, we aim to reformulate our separability conditions due to the above reasons. In more detail, the problem of entanglement detection under coarse-grained measurements was examined in [62].

Entropic functions of the form (2) may generally take negative values. On the other hand, experiments typically result in discrete probability distributions obtained by sampling density functions of continuous variables. So, the density functions W and U will be used with a discretization into some bins. Let W be sampled with respect to the set of prescribed marks $\{\zeta_j\}$. Correspondingly, one puts the intervals $\Delta\zeta_j=\zeta_{j+1}-\zeta_j$ and $\Delta\zeta=\max\Delta\zeta_j$. The discrete distribution $p_{\Delta\zeta}$ is formed by the probabilities

$$p_j:=\int_{\zeta_j}^{\zeta_{j+1}}W(r)\,dr=\int_{\mathbb{R}}d_j^{(\zeta)}(r)\,W(r)\,dr. \tag{57}$$

Here, $d_j^{(\zeta)}(r)$ is a boxcar function equal to 1 for r between ζ_j and ζ_{j+1}. The probabilities (57) represent chances for the corresponding detection positions [62]. The quantity $\Delta\zeta$ characterizes the width of detectors in r space. Similarly, the distribution $q_{\Delta\xi}$ is gained by sampling U with respect to bins between marks ξ_i so that $\Delta\xi_i=\xi_{i+1}-\xi_i$ and $\Delta\xi=\max\Delta\xi_i$.

There are two ways to express results of measurements of the discussed type [62]. First, we can deal immediately with the discrete distributions $p_{\Delta\zeta}$ and $q_{\Delta\xi}$. Second, we can construct approximations to the original probability distributions $W(r)$ and $U(s)$, namely,

$$W_{\Delta\zeta}(r):=\sum_{j=-\infty}^{+\infty}\Delta\zeta_j^{-1}p_j\,d_j^{(\zeta)}(r), \tag{58}$$

$$U_{\Delta\xi}(s):=\sum_{i=-\infty}^{+\infty}\Delta\xi_i^{-1}q_i\,d_i^{(\xi)}(s). \tag{59}$$

The original probability densities are replaced with approximate continuous distribution in the histogram form. When the bins all tend to zero, these histograms reproduce the original distributions.

Assuming $a>1>b>0$, we can prove the inequalities [51,59]

$$\Delta\zeta_j^{1-a}\,p_j^a\leqslant\int_{\zeta_j}^{\zeta_{j+1}}W(r)^a\,dr, \tag{60}$$

$$\int_{\xi_i}^{\xi_{i+1}}U(s)^b\,ds\leqslant\Delta\xi_i^{1-b}\,q_i^b. \tag{61}$$

These formulas are based on theorem 192 of [53]. Another way refers to an integral analog of Jensen's inequality with weight functions of the form $d_j^{(\zeta)}(r)/\Delta\zeta_j$ [63]. It follows from (58) and (59) that

$$\|W_{\Delta\zeta}\|_a^a=\sum_{j=-\infty}^{+\infty}\Delta\zeta_j^{1-a}\,p_j^a, \tag{62}$$

$$\|U_{\Delta\xi}\|_b^b=\sum_{i=-\infty}^{+\infty}\Delta\xi_i^{1-b}\,q_i^b. \tag{63}$$

For $a>1>b$, we therefore obtain

$$\Delta\zeta^{1-a}\,\|p_{\Delta\zeta}\|_a^a\leqslant\|W_{\Delta\zeta}\|_a^a\leqslant\|W\|_a^a, \tag{64}$$

$$\|U\|_b^b\leqslant\|U_{\Delta\xi}\|_b^b\leqslant\Delta\xi^{1-b}\,\|q_{\Delta\xi}\|_b^b. \tag{65}$$

These inequalities follow from combining (60) with (62) and (61) with (63). Note also that Rényi's entropies of discrete distributions become unbounded, when the size of bins tends to zero. It can be observed from (64) and (65). In this limit, we will use entropic separability conditions of the form (38).

Combining (50) with (64) and (65), the following conclusions take place. For n-separable states of an n-partite system, the result (50) and its "twin" remain valid for the histogram functions (58) and (59). Further, we get the inequality

$$\|p_{\Delta\zeta}\|_a\leqslant\left(\frac{\Delta\zeta\,\Delta\xi}{n\mathcal{K}\pi}\right)^t\|q_{\Delta\xi}\|_b, \tag{66}$$

and its "twin" with swapped $p_{\Delta\zeta}$ and $q_{\Delta\xi}$. Here, the indices a and b are again defined by (36). Entropic separability conditions with binning are derived from (66) similarly to the way by which the result (38) follows from (50).

Proposition 2. Let positive indices a and b be defined for $t\in[0;1]$ by (36), and let $\mathcal{K}(t)$ be defined by (37). Let $p_{\Delta\zeta}$ and $q_{\Delta\xi}$ be distributions obtained, respectively, by sampling the density functions in measurements of (30) and (31) with $n\geqslant 2$. Let $W_{\Delta\zeta}$ and $U_{\Delta\xi}$ be the corresponding histogram functions. If n-partite state $\boldsymbol{\rho}_{1:n}$ is n separable, then we have the inequalities

$$H_a(W_{\Delta\zeta}|\boldsymbol{\rho}_{1:n})+H_b(U_{\Delta\xi}|\boldsymbol{\rho}_{1:n})\geqslant\ln(n\mathcal{K}\pi), \tag{67}$$

$$H_a(p_{\Delta\zeta}|\boldsymbol{\rho}_{1:n})+H_b(q_{\Delta\xi}|\boldsymbol{\rho}_{1:n})\geqslant\ln\left(\frac{n\mathcal{K}\pi}{\Delta\zeta\,\Delta\xi}\right), \tag{68}$$

and their "twins" with swapped histogram functions and probability distributions.

The above entropic n-separability conditions are formulated using distributions with discretization. The condition (68) extends Rényi-entropy biseparability conditions derived in [29]. As experimentally resolvable distributions are typically discrete, relations with such distributions are more appropriate in practice. In the case $t=0$, the formula (67) reads as

$$H_1(W_{\Delta\zeta}|\boldsymbol{\rho}_{1:n})+H_1(U_{\Delta\xi}|\boldsymbol{\rho}_{1:n})\geqslant\ln(ne\pi). \tag{69}$$

We derived a one-parameter family of n-separability conditions for an n-partite continuous-variable system. A utility of entropic expressions with freely variable parameters was noted in [64] with respect to uncertainty relations. A family of relations is more informative in the sense that it generally provides stronger restrictions on involved probabilities. In entanglement detection, a dependence on entropic parameter can be used in slightly another manner. Varying the control parameter t, we try to observe the violation of separability conditions. When the violation has happened, we do detect entanglement of the state to be tested. Of course, the violation of separability conditions is sufficient but not necessary. There exist entangled states that will escape the entanglement detection by particular criteria. Nevertheless, a range of detectability will generally increase with adding more separability conditions.

Note that the inequality (66) also leads to separability conditions in terms of Tsallis entropies. Such conditions can be obtained due to the minimization task of [65]. We present only the final result since the derivation *per se* was in detail considered in [51,59]. For $0<\alpha\neq 1$, the Tsallis α entropy of distribution $\{q_i\}$ is defined as [66]

$$T_\alpha(q) := \frac{1}{1-\alpha}\left(\sum_i q_i^\alpha - 1\right) = -\sum_i q_i^\alpha \ln_\alpha(q_i). \tag{70}$$

For brevity, we use here the α logarithm $\ln_\alpha(x) = (x^{1-\alpha} - 1)/(1-\alpha)$. Let positive indices a and b be defined for $t \in [0;1]$ by (36), and let $\mathcal{K}(t)$ be defined by (37). If n-partite state $\boldsymbol{\rho}_{1:n}$ is n separable, then we have the inequality

$$T_a(p_{\Delta\zeta}|\boldsymbol{\rho}_{1:n}) + T_b(q_{\Delta\xi}|\boldsymbol{\rho}_{1:n}) \geqslant \ln_a\left(\frac{n\mathcal{K}\pi}{\Delta\zeta\,\Delta\xi}\right), \tag{71}$$

and its "twin" with swapped $p_{\Delta\zeta}$ and $q_{\Delta\xi}$. Tsallis entropies of continuously changed variables were also considered in the literature. However, the minimization problem of [65] is not applicable for differential entropies.

Using entropies of discrete probability distributions, we can take into account possible inefficiencies of the detectors used. In practice, measurement devices inevitably suffer from losses. Hence, some discussion of cases with nonzero probability of the no-click event is of interest. The following simple model will be considered. Let the parameter $\eta \in [0;1]$ characterize a detector efficiency. To the given value η and probability distribution $\{q_i\}$, we assign a "distorted" distribution such that

$$q_i^{(\eta)} = \eta q_i\ , \quad q_\varnothing^{(\eta)} = 1-\eta\ . \tag{72}$$

Here, the probability $q_\varnothing^{(\eta)}$ corresponds to the no-click event. We further assume that in both the measurements the inefficiency-free distributions are altered according to (72). So, an efficiency of detection is taken to be equal for all bins. The above formulation is inspired by the first model of detection inefficiencies used by the authors of [67] in the context of cycle scenarios of the Bell type. Since the separability conditions (68) and (71) involve different entropic parameters, we restrict a consideration to the Shannon entropies. It is easy to check that

$$H_1(q^{(\eta)}) = \eta H_1(q) + h_1(\eta), \tag{73}$$

where $h_1(\eta) = -\eta\ln\eta - (1-\eta)\ln(1-\eta)$ is the binary Shannon entropy. If n-partite state $\boldsymbol{\rho}_{1:n}$ is n separable, then

$$H_1\big(p_{\Delta\zeta}^{(\eta)}|\boldsymbol{\rho}_{1:n}\big) + H_1\big(q_{\Delta\xi}^{(\eta)}|\boldsymbol{\rho}_{1:n}\big) \geqslant \eta\ln\left(\frac{ne\pi}{\Delta\zeta\,\Delta\xi}\right) + 2h_1(\eta). \tag{74}$$

Thus, detector inefficiencies will produce additional uncertainties in the entropies of actually measured data. With decreasing $\eta > \frac{1}{2}$, the first term in the right-hand side of (74) reduces, whereas the second term increases. When η does not approach 1 sufficiently closely, this feature will prevent a robust detection of entanglement.

V. EXAMPLES OF APPLICATION OF THE DERIVED CRITERIA

Finally, we shall illustrate a relevance of the presented entanglement criteria with entropic parameters. The aim is not a study of numerous types of continuous-variable states in full detail. We rather wish to exemplify principal features of the new n-separability conditions. Nevertheless, considered states may be of interest in practice.

Our first example concerns n-partite states that are similar to dephased cat states. By $|z\rangle$, we mean the coherent state corresponding to complex number z. For $0 \leqslant c \leqslant 1$, we define

$$\begin{aligned}\varrho_{1:n}(z) = \mathcal{N}(z)\big\{&|z^{\otimes n}\rangle\langle z^{\otimes n}| + |(-z)^{\otimes n}\rangle\langle(-z)^{\otimes n}| \\ &- (1-c)[|z^{\otimes n}\rangle\langle(-z)^{\otimes n}| + |(-z)^{\otimes n}\rangle\langle z^{\otimes n}|]\big\},\end{aligned} \tag{75}$$

where $|z^{\otimes n}\rangle$ denotes the n-fold product state and $\mathcal{N}(z)$ is the normalization factor. For $n=2$, the formula (75) gives a bipartite dephased cat state. Such states were used in order to test entanglement criteria in terms of Shannon entropies [28] and Rényi entropies [29]. Applications of cat states in quantum information processing with continuous variables are reviewed in [68].

To relate with the results of [28,29], we substitute $\theta_\ell = 0$ in (25) and (26). As was already mentioned, for even n the observables (30) and (31) can be made commuting. For $n = 2m \geqslant 2$, we take the observables

$$\mathsf{R}_{\epsilon:\epsilon} = \sum_{\ell=1}^{n}(-1)^{\ell-1}\widetilde{\mathsf{r}}_\ell, \tag{76}$$

$$\mathsf{S}_{\varepsilon:\varepsilon} = \sum_{\ell=1}^{n}\widetilde{\mathsf{s}}_\ell. \tag{77}$$

When $n=2$, the formulas (76) and (77), respectively, lead to (27) and (28) with $\epsilon = -1$ and $\bar{\epsilon} = +1$. Namely, these commuting observables were used in [28,29]. Further, the family of states (75) will be considered for even n and positive real z. To study the violation of separability conditions, we introduce the characteristic quantity

$$Q_a(z) := \ln(n\mathcal{K}\pi) - H_a(U|\varrho_{1:n}) - H_b(W|\varrho_{1:n}). \tag{78}$$

Here, the indices a and b are linked by (36) and $\mathcal{K}(t)$ is defined by (37). Strictly positive values of $Q_a(z)$ will show that the tested state is entangled.

The characteristic quantity $Q_a(z)$ is drawn in Fig. 1 for $n=4$, $c=\frac{1}{2}$, and five values of a. Only positive values are shown here. We also restrict a consideration to values $z \in [0;4]$ since an asymptotic behavior already appears on the right sides of curves. For a very large range of z, the separability conditions in terms of Rényi entropies allow to detect entanglement. We also see that the undetectable region is almost the same for all curves. Although the border of detectable values becomes leftmost for the standard case $a=1$, this difference is quite small and hardly significant in practice. Indeed, all real devices are inevitably exposed to noise. In opposite, the size of violation essentially depends on entropic indices. With growth of a, the size of violation for sufficiently large z is increased more than three times. Thus, separability conditions in terms of generalized entropies lead to more robust detection of entanglement.

The obtained entanglement criteria have also been tested for even $n > 4$. A behavior of the curves is very similar to what we saw in Fig. 1. With growing a in the range considered, the size of violation for sufficiently large z is increased. Also, the curves drawn for different a have almost the same undetectable region. So, we again see a significance of entanglement criteria

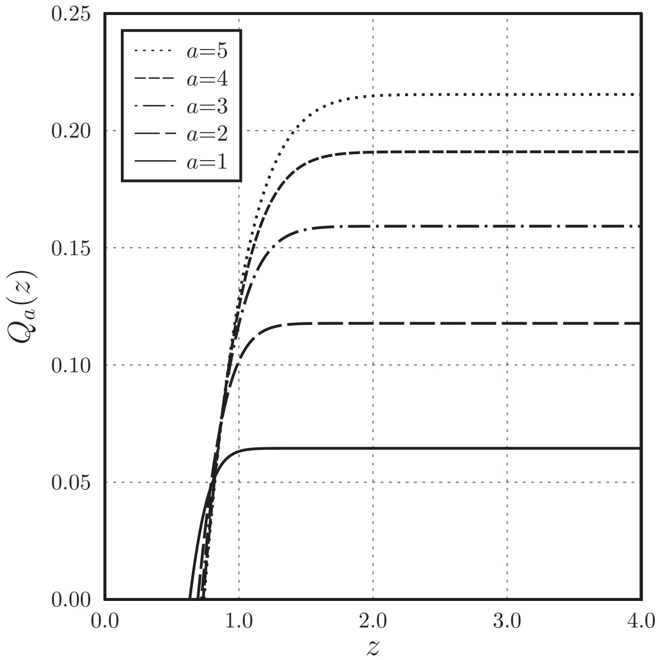

FIG. 1. $Q_a(z)$ as a function of positive real z for $n = 4$, $c = \frac{1}{2}$, and $a = 1,2,3,4,5$.

in terms of Rényi entropies. On the other hand, the border of detectable values goes to the left with growth of n. Increasing n, the undetectable region of entanglement criteria becomes more and more narrow. For example, we present $Q_a(z)$ in Fig. 2 for $n = 10$, $c = \frac{1}{2}$, and five values of a. Comparing Figs. 1 and 2, one sees that the curves with growth of n try to approach a form like Heaviside's step function.

The curves were presented for $c = \frac{1}{2}$, but for other values $c \neq 1$ we have seen a similar picture. Using generalized entropies generally increases the size of violation. On the other hand, growing c implies a decrease of the factor $1 - c$. Here, the curves reveal the following tendencies. When other parameters are fixed, positive values of $Q_a(z)$ are visibly reduced in size. Furthermore, the undetectable region on the z axis is widened due to increasing c. The character of dependence on c allows one to explain some natural relation between detectability of entanglement for different values of n. Taking the partial trace, we will obtain states of the same type (75) but with lesser n. When other parameters are fixed, the size of violation increases and the undetectable region narrows with growth of n. Another point is that the term $1 - c$ will be decreased due to tracing out some particles. Chances to detect entanglement of states of the considered type cannot be improved by applying the above entropic criteria to partial traces.

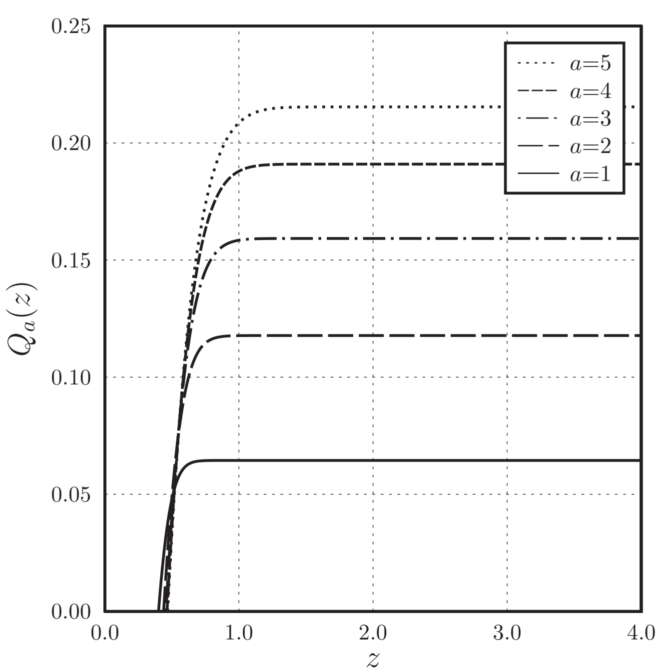

FIG. 2. $Q_a(z)$ as a function of positive real z for $n = 10$, $c = \frac{1}{2}$, and $a = 1,2,3,4,5$.

The size of violation of separability conditions is significant due to the following reasons. In practice, the original density functions U and W can additionally be masked in experiments due to a finiteness of resolution and external noise. These features can only increase the amount of uncertainty. Instead of the theoretical violation (78), measurements result in another quantity $\widetilde{Q}_a(z)$ such that $\widetilde{Q}_a(z) \leqslant Q_a(z)$. Since actually observed violation is reduced, our possibilities to detect entanglement essentially depend on the size of violation of separability conditions. To reach robust detection of entanglement, one will try to maximize the characteristic quantity with respect to entropic indices. So, the presented separability conditions are of interest in practice of quantum information processing. In this regard, the new criteria provided an extension of basic results of [29] to the case of multipartite systems.

Let us consider briefly applications of the derived entropic criteria to pure states. In this case, we can use the inequalities (42) and (45) separately. Separable pure states are written in the form

$$|\Phi_{1:n}\rangle = \bigotimes_{\ell=1}^{n} |\phi_\ell\rangle. \tag{79}$$

For such states, the condition (42) can be reformulated as

$$H_a(W|\,\Phi_{1:n}) \geqslant \frac{1}{t}\,\ln\frac{C(a)}{C(\alpha)^n} + \frac{1}{n}\sum_{\ell=1}^{n} H_\alpha(u_\ell|\phi_\ell), \tag{80}$$

where the indices $a \geqslant 1$ and $\alpha \geqslant 1$ are such that their conjugate ones obey (41). Assuming $n = 2m \geqslant 2$, we shall test the criterion (80) with states of the form

$$|\Psi_{1:n}(z)\rangle = \sqrt{\Omega(z)}\{|z^{\otimes m}\rangle \otimes |(-z)^{\otimes m}\rangle - |(-z)^{\otimes m}\rangle \otimes |z^{\otimes m}\rangle\}, \tag{81}$$

where $z \neq 0$ and $\Omega(z)$ is the normalization factor. We also rewrite the sum (76) with the same sign for all summands. Numerical calculations showed that we cannot reach a valuable increase of violation of (80) by varying entropic parameters. This situation is opposite to the curves drawn in Figs. 1 and 2. When the separability condition (80) is applied to states of the form (81), a picture somehow depends on the number of subsystems. Due to these reasons, we further take $t = \tau = 0$ and $a = \alpha = 1$. On the other hand, curves for different values of n will be compared. Similarly to (78), we

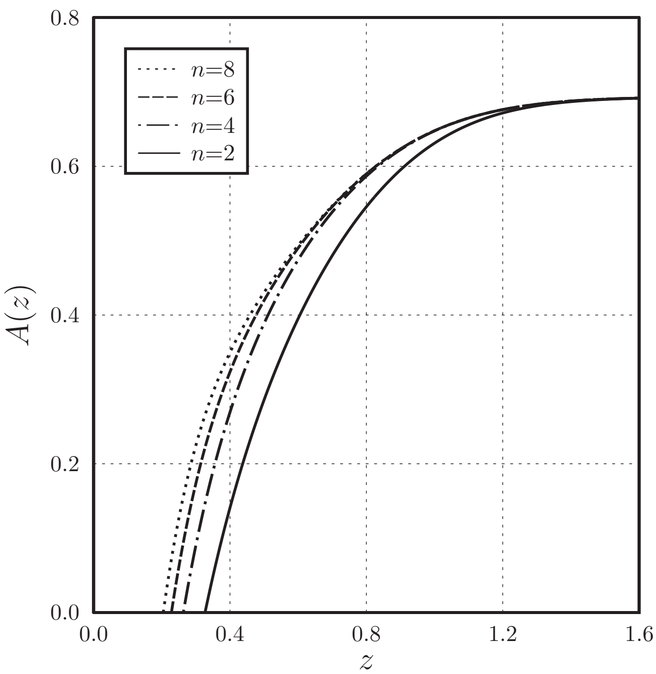

FIG. 3. $A(z)$ as a function of positive real z for $n = 2,4,6,8$.

put the characteristic quantity

$$A(z) := \frac{\ln n}{2} + \frac{1}{n}\sum_{\ell=1}^{n} H_\alpha(w_\ell|\boldsymbol{\rho}_\ell) - H_1(W|\Psi_{1:n}), \quad (82)$$

where $\boldsymbol{\rho}_\ell$ is the corresponding partial trace. In Fig. 3, we present positive values of $A(z)$ for $n = 2,4,6,8$ and positive real z. For sufficiently large z, all the curves approach the limiting value $\ln 2 \approx 0.693$. So, we restrict a consideration to values $z \in [0; 1.6]$. Like the above examples, the border of detectable values goes to the left with growth of n. At the same time, a distinction between the curves is not so essential as in Figs. 1 and 2. Except for a small region, we have seen a violation of the n-separability condition (80) up to arbitrary positive z.

Finally, we shall discuss possible applications of the presented n-separability conditions in multipartite entanglement detection. Our entanglement criteria are expressed in terms of experimentally measured quantities with the use of sufficiently simple and universal setup. In principle, states of an n-partite system may be n' separable with $1 < n' \leqslant n$. The equality $n' = n$ implies that the given state is fully separable. Our separability conditions allow one to test full separability immediately. Their violation is sufficient for the conclusion that the tested state is not fully separable. They may also be used in a more complicated manner, when corresponding partial traces of the input $\boldsymbol{\rho}_{1:n}$ will be tested with respect to full separability. This complexity is natural since the separability problem increases substantially in the context of multipartite systems [35–38].

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