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# Scaling of entanglement during the quantum phase transition for Ising spin systems on triangular and Sierpiński fractal lattices

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Adopting concurrence as entanglement measure, we study entanglement and quantum phase transition of the Ising spin systems on the triangular lattice and Sierpiński fractal lattices by using the quantum renormalization-group method. It is found that the ground-state entanglement between two spins (or spin blocks) depends on the following factors: the size of system, the magnetic field, the exchange coupling, and the structure of lattice. As the size of the system becomes large, (a) the range of the magnetic field, in which the entanglement exists, contracts gradually and focuses on the critical point; and (b) the first derivative of entanglement shows singular behavior, and its maximum or minimum is approaching to the critical point gradually. The scaling behaviors of entanglement on the different lattices are similar but the scaling relations are diverse. For the triangular lattice, the space dimensionality determines the scaling relationship between the critical exponent of the entanglement and the critical exponent correlation length. However, for fractal lattices, it is the fractal dimensionality but not the space one to determine this relationship.

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## I. INTRODUCTION

As an important nonlocal quantum correlation phenomenon, quantum entanglement has been used to realize quantum computation and quantum communication [1–3]. Meanwhile, it also receives much attention in the field of theoretical physics because it can depict the properties of quantum mechanics beyond the classical physics [4–7]. Entanglement exhibits fragility to the decoherence induced by the environment, and therefore it is important to discover the robust entanglement in real systems at low or finite temperature. At very low temperature even close to absolute zero temperature, one can manipulate quantum system to produce the entangled states by changing the physical parameters such as an external magnetic field. Meanwhile this controlling procedure may lead to the occurrence of quantum phase transition (QPT) [8]. It is natural to investigate the relation between the entanglement and QPT in various systems, which has been attracted much attention in the field of condensed-matter physics [9–11].

Solid spin systems cannot only capture the properties of real systems such as magnetic materials or ultracold atoms in optical lattices but also can exhibit some interesting phase-transition and nonclassical correlation characteristics. The entanglement properties in spin systems are of much interest in different fields because the related studies can provide support in designing quantum information processing tasks [12] and can also reveal correlation behaviors around the quantum critical points (QCPs) for QPT in many-body systems, for example, the superconduction and quantum Hall systems [13,14]. Recent investigation has pointed out that the ground-state entanglements in spin chains can describe the QPT successfully. It is found that the discontinuous extremum behavior of entanglement is relevant to the first-order QPT, while the nonanalytic and scaling behavior of their derivative is related to second-order QPT [10,15–18].

Recently, there are some works which have been devoted to studying the pairwise or block entanglement on two-dimensional (2D) or low-symmetry spin systems [19–28]. Fractals can generally characterize physical self-similar structures in noninteger dimensions which break the translation symmetry, and fractal lattices are effective to model the random magnetic properties of materials or magnetic domain [29,30], which have also been used to construct the networks for quantum computation and communication [31,32]. It motivates us to propose the following questions: Does the entanglement exist between two blocks in triangular or fractal lattices? How does the entanglement change as the sizes of these systems become large? Are there the finite-size scaling behaviors of entanglement in fractal lattices? How does the entanglement vary during the QPT and how does the quantum fluctuation influence the critical entanglement? Dose the space dimension or fractal dimension affect the critical behavior of entanglement on fractal lattice? However, it is very difficult to calculate directly exact results of the entanglement in many-body complicated lattices, especially the fractal lattices.

The entanglement behaviors of spin chains at ground states have been studied by using renormalization group (RG) methods such as the density matrix RG [10,15], Kadanoff's block RG [16–18,28], and decimation RG [33,34]. These works enlighten us on studying the entanglement on complicated lattices. In this paper, we use the quantum RG method to respectively study the entanglement of spin blocks on the two-dimensional (2D) triangular and Sierpiński fractal lattices by taking the concurrence as measure method. The critical behaviors of entanglement, especially the scaling behaviors, are found.

The remainder of this paper is organized as follows: In the next section the spin model and quantum RG method is introduced briefly. In Secs. III and IV, the critical entanglement of spin system on the triangular lattice and fractal

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lattice are studied, respectively. The summary is given in Sec. V.

## II. MODEL AND METHOD

Consider the transverse-field Ising model on the triangular or fractal lattices. Its Hamiltonian reads

$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x, \tag{1}$$

where $\sigma_i^\alpha (\alpha = x,z)$ are the spin-1/2 Pauli operators at the site $i$, and the ferromagnetic exchange coupling $J > 0$ as well as the transverse field $h \geqslant 0$. The first sum $\sum_{\langle i,j \rangle}$ is over all the nearest neighbors and the second sum is over all sites on the systems.

These systems cannot be exactly calculated. The RG method can be used to solve the difficulty of directly calculating exact results for the complex many-body lattices. The quantum RG method (based on the Kadanoff's block-site transformation) has been applied to study the quantum critical phenomena in a one-dimensional (1D) spin chain, which can keep the most important degrees of freedom in the low-energy spectrum while eliminating the rest through an iterative process. Recently some quantum RG schemes for translation-symmetry lattices and even fractal lattices have been developed, which can estimate the accurate correlation length exponent [35,36]. In this paper, we will utilize this quantum RG method to investigate the ground-state entanglement of the spin system near the critical point on triangular or fractal lattices and to explore the scaling behavior of entanglement. In order to preserve the symmetry of the system and the closed form of Hamiltonian, the choice of the structure of block and its corresponding Hamiltonian should be reasonable (see the scheme proposed in Ref. [36] for triangular lattice as shown in Fig. 1(a) in this paper). After defining the transverse field strength normalized to the exchange interaction as

$$g = h/J \tag{2}$$

and implementing the geometric mean of all renormalized coupling strengths [36], the renormalized transverse field strength for triangular lattice can be obtained, i.e.,

$$g_t' = \frac{g^3}{2^{\frac{1}{3}}(1+g^2)^{\frac{1}{6}}(\sqrt{1+g^2}+1)^{\frac{2}{3}}}. \tag{3}$$

Adopting the similar RG scheme in Ref. [36], we also investigate the cases of Sierpiński fractal lattices with the fractal dimension $d_f = \frac{\log(d+1)}{\log 2}$ (where $d = 2$ or 3 is the spatial dimension). According to the corresponding block-site transformation of Sierpiński triangle ($d_f = 1.585$) and Sierpiński pyramid($d_f = 2$) as shown in Fig. 1(b), the renormalized transverse field strength can also be calculated as

$$g_s' = g^{\frac{3d+1}{d+1}}(1+g^{-2})^{-\frac{d(d-1)}{2(d+1)}}. \tag{4}$$

According to the RG equation, the critical point $g_c$ can be obtained by solving $g' = g$, i.e., the nontrivial fixed point. This also permits us to calculate the correlation length critical exponent $\nu$ defined as $\xi \sim |g - g_c|^{-\nu}$, i.e., ($\nu_t$ for triangular lattice, $\nu_s$ for Sierpiński lattice)

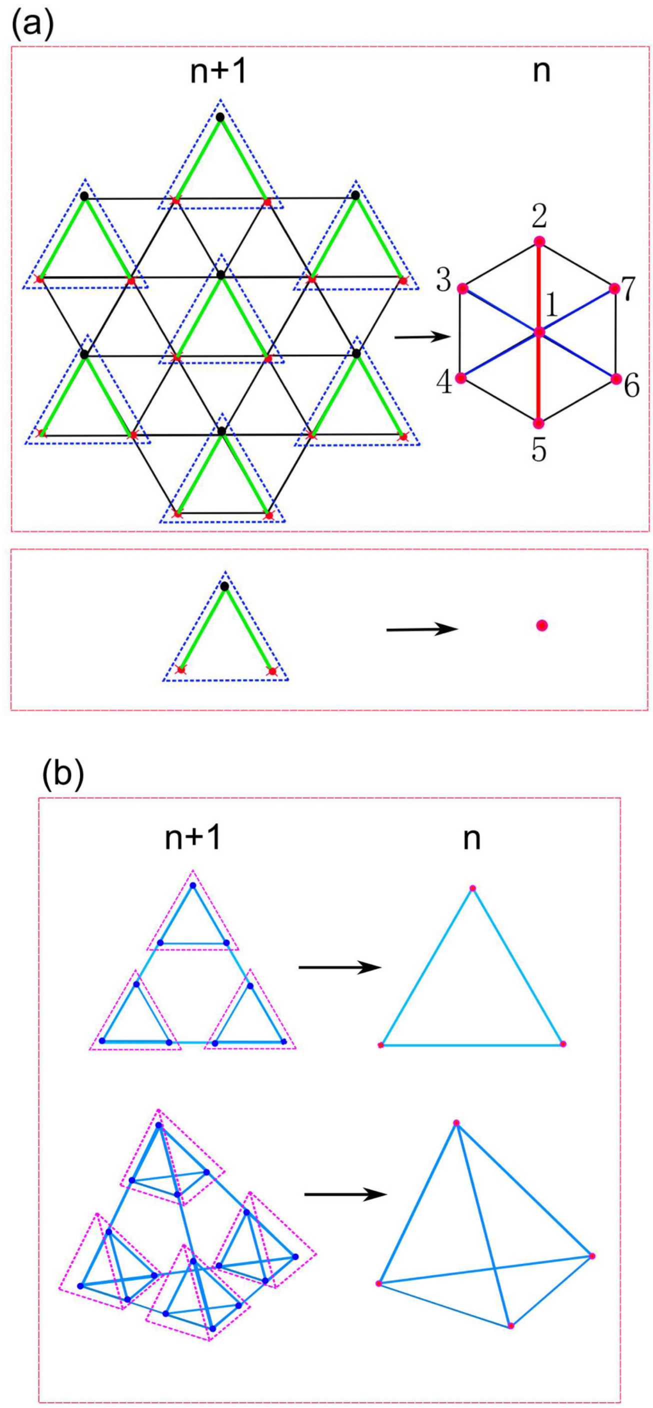


FIG. 1. The procedure of quantum RG transformation for the triangular (a) and fractal lattices (b). The basic cluster of the lattices are shown as the $n$ structure.

$$\nu_t^{-1} = \log_{\sqrt{3}} \frac{dg_t'}{dg}\bigg|_{g=g_c}, \tag{5}$$

$$\nu_s^{-1} = \log_2 \frac{dg_s'}{dg}\bigg|_{g=g_c}. \tag{6}$$

For triangular, Sierpiński triangular, and pyramid lattices, the critical points are respectively $g_c^t = 1.85$, $g_c^{st} = 1.15$

and $g_c^{sp} = 1.27$. And the corresponding correlation length critical exponents are $\nu_t = 0.63$, $\nu_{st} = 0.72$, and $\nu_{sp} = 0.62$, respectively. The critical divergent behavior of correlation length means that the RG method can capture the long-distance critical properties which are independent of the system details.

## III. ENTANGLEMENT OF SPIN SYSTEM ON TRIANGULAR LATTICE

### A. Pairwise entanglement on basic cluster of the triangular lattice

We measure the quantum entanglement between two nearest-neighbor spins by using the concurrence [37]. After choosing the basic cluster (which can be simple manifestation of this triangular lattice) as shown $n$ structure in Fig. 1(a), one can find that the concurrence between the pairs (1−3, 1−4, 1−5, 1−6, and 1−7) is equal to that of the spin dimer including 1 and 2. The Hamiltonian of this basic cluster (in which all couplings are the same) is

$$H = -J\left(\sum_{i=2}^{7}\sigma_1^z\sigma_i^z + \sum_{i=2}^{6}\sigma_i^z\sigma_{i+1}^z + \sigma_2^z\sigma_7^z\right) - h\sum_{i=1}^{7}\sigma_i^x. \quad (7)$$

Its ground state $|\varphi_0\rangle$ can be calculated by solving $H|\varphi_0\rangle = E_0|\varphi_0\rangle$, and then the corresponding density matrix is

$$\rho = |\varphi_0\rangle\langle\varphi_0|. \quad (8)$$

By tracing over sites 3, 4, 5, 6, and 7, one can obtained the reduced density matrix of sites 1 and 2 (i.e., $\rho_{12} = \mathrm{Tr}_{34567}\rho$). The concurrence provided in Ref. [37], as a measure of entanglement between spins 1 and 2, can be defined as

$$C = \mathrm{Max}\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (9)$$

where $\lambda_k (k = 1,2,3,4)$ are the square roots of the eigenvalues of the operator $R$ (in descending order), and the operator $R$ reads

$$R = \rho_{12}\tilde{\rho}_{12}, \quad (10)$$

with the spin-flipped matrix $\tilde{\rho}_{12} = (\sigma^y \otimes \sigma^y)\rho_{12}^*(\sigma^y \otimes \sigma^y)$.

By using the definition of concurrence, i.e., Eq. (9), we can directly calculate the pairwise entanglement between two neighbor spins on the triangular lattice. It is found that the entanglement is a function of the exchange coupling and the external field, i.e., $C(J,h)$. It can be transformed into a function of $g$, i.e., $C(J,h)$ can be written as $C(g)$, where the effective field $g = h/J$. According to the idea of RG method, it is well known that the global properties of some physical quantities can depict the real phase-transition or critical characteristics. It is natural to combine the entanglement with the effective coupling or the effective field obtained in the RG transformation to obtain the information of global quantum correlation.

### B. Critical block entanglement on the triangular lattice

The entanglements between two blocks (or spins) are plotted in Fig. 2, which can capture the quantum correlation between the single block (or site) and its heat bath (the rest of the system). According to the site-block transformation, the large system with $N = 7\times3^n$ spins can be described by a cluster with $N = 7$ effective spins after the $n$th RG step. Some similar properties of entanglements with the increasing external field $g$ perform after different RG transformation steps. The entanglements first increase from zero to maxima and then decrease to zero monotonically. When the external field does not exist, the exchange couplings keep the system staying at the ferromagnetic phase ($|\uparrow\uparrow\uparrow\cdots\rangle$ or $|\downarrow\downarrow\downarrow\cdots\rangle$), i.e., the unentangled matrix-product state. As the external magnetic field along the $x$ axis increases, some spins may be polarized to the direction of the field. That is to say, the state $|\uparrow\rangle$ (or $|\downarrow\rangle$) changes into $|\rightarrow\rangle = 1/\sqrt{2}(|\uparrow\rangle + |\downarrow\rangle)$). Because there is the quantum fluctuation, the probability for the spin pair staying at the entangled state [like $|\psi_\pm\rangle = 1/\sqrt{|c_1|^2 + |c_2|^2}(c_1|\uparrow\downarrow\rangle \pm c_2|\downarrow\uparrow\rangle)$ or $|\varphi_\pm\rangle = 1/\sqrt{|c_3|^2 + |c_4|^2}(c_3|\uparrow\uparrow\rangle \pm c_4|\downarrow\downarrow\rangle)$] may increase with the increasing field $g$. As the size of the system becomes large, the range of the field in which entanglement can exist shrinks and focuses to the vicinity of the QCP $g_c$. In the thermodynamic limit, the critical behavior can be described by the entanglement; i.e., only large quantum fluctuation can produce and enhance the entanglement while the effective field is approaching $g_c$. The monogamy of entanglement in the triangular lattice can been found. Besides, we also found that the maximum of entanglement in triangular lattice is smaller than those in spin chain and square lattice [28].

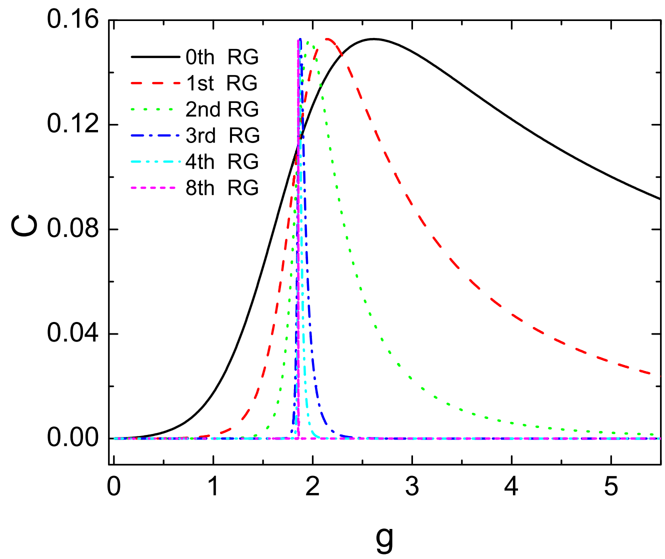


FIG. 2. The evolution of concurrence C vs the effective field ($g = h/J$) in terms of the RG iteration on the triangular lattice.

Nonanalytic and scaling behaviors of physical quantities are the distinguishing features of the critical phenomena accompanying with continuous phase transitions. As mentioned above, the entanglement changes continuously when the external field increases across the QCP, but it changes sharply as the size of the system becomes large. Its derivative may exhibit more details on how the entanglement varies with the field at the vicinity of the QCP $g_c$. The results of the derivative of concurrence $dC/dg$ with the effective field $g$ at different RG steps are shown in Fig. 3. Figure 3 exhibits that in the thermodynamic limit the sudden change of the quantum entanglement occurring around the critical point

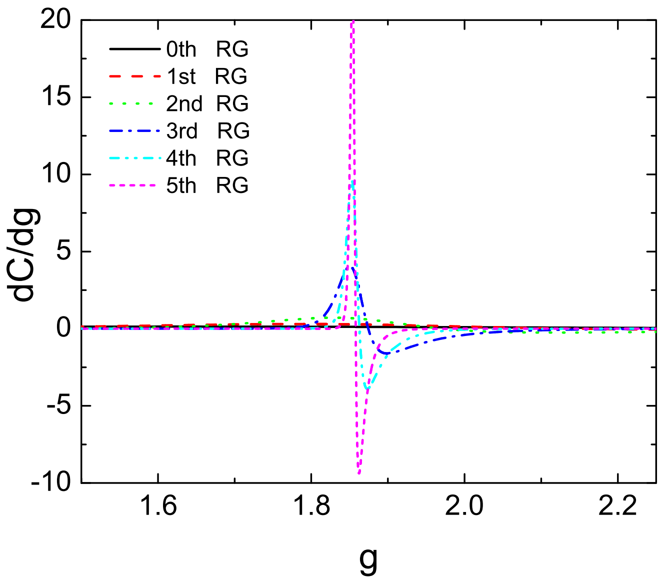


FIG. 3. The evolution of the first derivative of the concurrence under different RG steps on the triangular lattice.

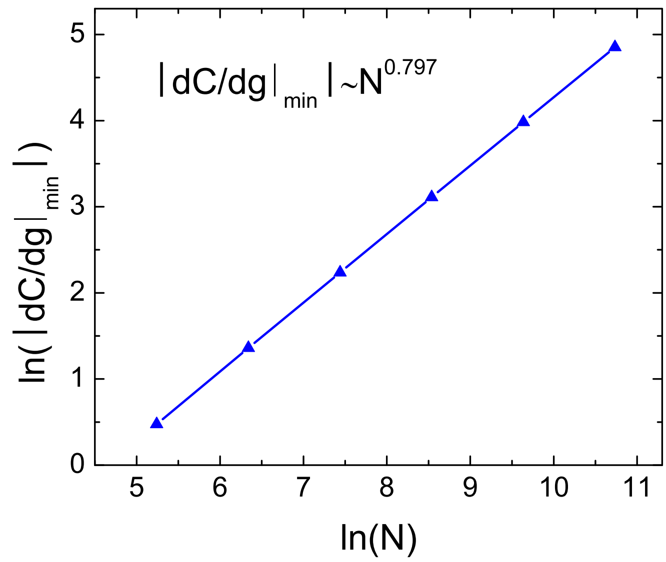


FIG. 4. The scaling behavior of the minimum of the first derivative of concurrence for various system sizes on the triangular lattice. The RG steps show that the minimum diverges as $|(\frac{dC}{dg})|_{g_m}| \sim N^{\theta}$ ($\theta = 0.797$).

becomes more distinct. However, there are some different properties between the singular behaviors of the maxima and minima for $dC/dg$. The position for the maximum of $dC/dg$ approaches the critical point and its maximum increases with the increasing size of system. By combining this point with the phenomenon shown in Fig. 2, one may find that the ferromagnetic phase can restrain the entanglement and this restraint effect will become stronger as the size of system becomes large. But the fluctuation caused by the external field will enhance the entanglement more quickly, because when $g$ is close to $g_c$ the quantum fluctuation will become stronger at the thermodynamic limit. In contrast, the magnetic field destroys the entanglement quickly in the paramagnetic phase. It is shown by the minima of the entanglement when $g$ is larger than $g_c$. The nonanalytic behavior is the result of the change of concurrence at $g = g_c$, which can also give some insights into quantum phase transitions, especially for the relation between quantum fluctuation and entanglement.

The effects of the increase of size of triangular lattice on the entanglement have been studied qualitatively, and the related quantitative research should proceed further. After a more detailed analysis, it is found that the position of the minimum of $dC/dg$, i.e., $g_m$, also trends towards the critical point $g_c$ as the size of system increases. Figure 4 shows the scaling behavior of $y \equiv |(\frac{dC}{dg})|_{g_m}|$ versus the size of the system $N$. It is clear that there is a linear behavior of $\ln y$ versus $\ln N$. Naturally, the scaling behavior may be expressed by $|(\frac{dC}{dg})|_{g_m}| \sim N^{\theta}$ with the entanglement exponent $\theta \doteq 0.797$. For the spin chains, it was found that the entanglement exponent $\theta$ is related to the correlation length exponent $\nu$, by $\theta = 1/\nu$ [16–18]. In this work, we find their relation on the triangular lattice in another expression. The correlation length exhibits exponentially divergent behavior in the vicinity of $g_c$, i.e., $\xi \sim |g - g_c|^{-\nu}$. Along with the RG procedure, the correlation length performs scaling behavior in the $n$th step as $\xi_n \sim |g_n - g_c|^{-\nu}$, and $\xi_n = \xi/l_B^n$, with $l_B = \sqrt{3}$ ($l_B$ is the length of the side in each block, which is related to $N$, i.e., $7\times(l_B^n)^d = N$). Thus, it leads to an expression for $|\frac{dg_n}{dg}|_{g_c}$ in terms of $l_B$ and $\nu$ as

$$\left|\frac{dg_n}{dg}\right|_{g_c} = \left|\frac{g_n - g_c}{g - g_c}\right| \sim \left(l_B^n\right)^{\frac{1}{\nu}} = (N/7)^{\frac{1}{\nu d}}. \quad (11)$$

Equation (11) implies that $\theta = 1/(\nu d)$, since $|(\frac{dC}{dg})|_{g_m}| \sim |\frac{dg_n}{dg}|_{g_c}$ at the QCP. As mentioned above, we find that $\theta \doteq 0.797$, which is consistent with the exact result $\theta = 0.794$. The universal class of Ising model on the triangular lattice can be recovered by this entanglement exponent. The aforementioned discussion implies that the entanglement with renormalization coupling constants can effectively capture the global long-distance critical behaviors of the transverse-field Ising model on 2D lattices [28].

## IV. ENTANGLEMENT OF SPIN SYSTEM ON SIERPIŃSKI FRACTAL LATTICE

In this section, we turn to studying the entanglement between two vertex spins on Sierpiński fractal lattice with fractal dimension $d_f = 1.585$ and $d_f = 2$, respectively. Their related numerical results of concurrence versus $g$ were respectively plotted in Figs. 5(a) and 5(b). By comparing Figs. 5 with Fig. 2, one can easily finds that the entanglements on the fractal lattices exhibit similar properties to those on the triangular lattice. The entanglement first increases and then decreases with increasing $g$. The effective coupling constants between two spins can be obtained by the RG procedure. Therefore, the entanglement of a large system can be calculated by combining the concurrence with the effective coupling constants. Using this method and the cluster chosen in Fig. 1(b), a large system with $N$ can be depicted by a cluster with 3 (or 4) effective spins for the $n$th RG step. The concurrence $C$ versus the transverse field $g$ for different RG iterations is plotted in Fig. 5. As the

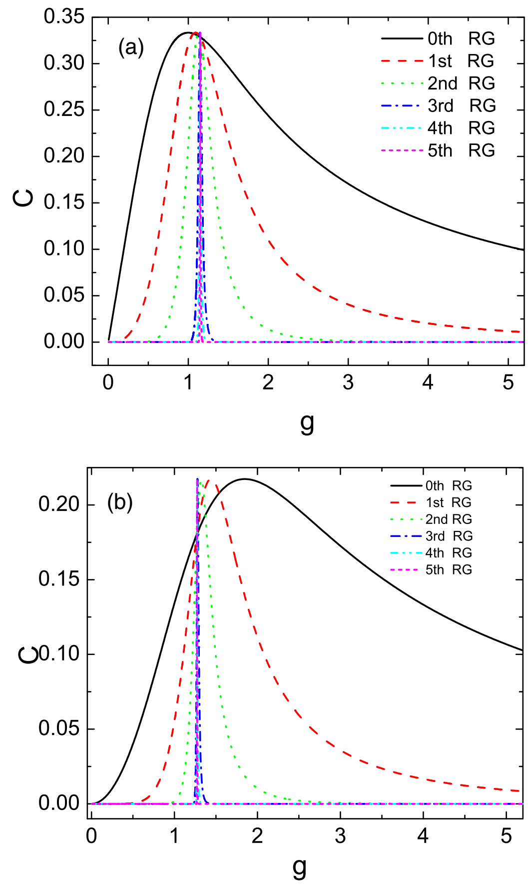


FIG. 5. The evolution of concurrence C vs the effective field ($g = h/J$) in terms of the RG iteration on the Sierpiński triangular lattice (a) and pyramid lattice (b).

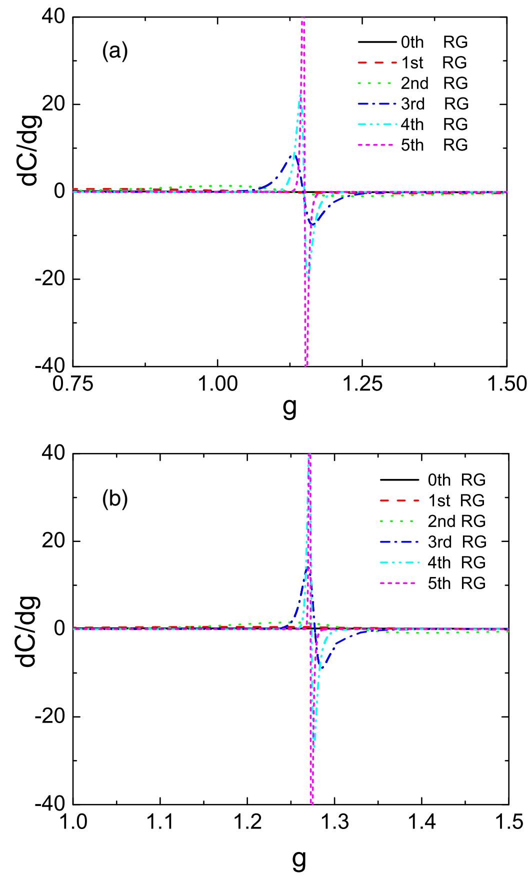


FIG. 6. The evolution of the first derivative of the concurrence under different RG steps on the Sierpiński triangular lattice (a) and pyramid lattice (b).

size of the system increases, the parameter space in which the concurrence can exist contracts. Meanwhile, the position of the concurrence maximum is close to the QCP $g_c$. The ground state of the system has a ferromagnetic order at $g = 0$, and as $g$ increases the ground state was mixed into some factor of the paramagnetic order, which induces some quantum correlation (entanglement). As $g$ becomes large enough, the paramagnetic order will destroy the quantum entanglement finally. After sufficient RG transformations, the concurrence develops two different features: At the critical point (or very near the critical point) it reaches its maximum, and when $g \neq g_c$ it almost vanishes. The physical origin of two different behaviors of the entanglement is as follows: The quantum fluctuation extends over the whole system at the QCP, and it spreads to very finite length scales when $g \neq g_c$. This result makes it clear that the QCP can be detected by the position of entanglement maximum, when the system is large enough.

It is worth studying further whether the fractal structure will influence the nonanalytic behaviors of the entanglement at zero-temperature quantum phase transition. By using the RG method and the concurrence, we will focus on some related features of quantum phase transitions for the Ising model on Sierpiński fractal lattice. The first derivative of the concurrence with the effective field $g$ at different RG steps is given in Figs. 6(a) and 6(b). As the size of the system becomes large, the derivative of the concurrence $dC/dg$ tends to diverge close to the QCP. Both the maxima and minima of the derivative of the concurrence show nonanalytic (or singular) behavior and have asymmetrical shapes as shown in Fig. 6. Comparing curves in Figs. 6 and 3, one can directly arrive at the following conclusion: $dC/dg$ on the Sierpiński fractal lattices exhibits similar characteristics to that on the triangular lattice. There are some different properties between the singular behavior of the maxima and minima as a function of the external field. The positions for the maxima approach the QCP and the maxima get bigger with increasing size of the system. The physics picture given by the concurrence is also similar to that given in Sec. III B. The nonanalytic behavior is the result of the change of concurrence at $g = g_c$, which can reflect the effect of quantum fluctuation on the entanglement during quantum phase transitions.

After some RG steps, the position of the minimum of $dC/dg$, i.e., $g_m$, is close to the critical point $g_c$ (as shown in Fig. 6). The relation between the scaling behavior of entanglement and the divergence of correlation length on the fractal lattice is interesting but not clear. Using a similar analysis on the finite-size scaling behavior on triangular lattice, the relevant issue on fractal lattice was investigated in the following content. The functional dependence between $y \equiv |(\frac{dC}{dg})|_{g_m}|$ and the size of the system $N$ is shown in Fig. 7.

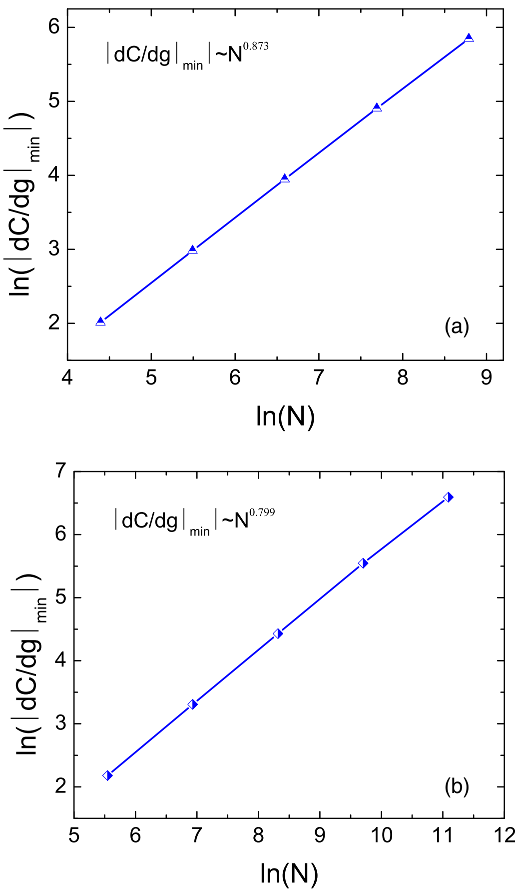


FIG. 7. The scaling behavior of the minimum of the first derivative of concurrence for various system sizes on the Sierpiński triangular lattice (a) and pyramid lattice (b). The RG steps show that the minimum diverges as $|(\frac{dC}{dg})|_{g_m}| \sim N^{\theta}$.

It is obvious that there is a linear relation between $\ln y$ and $\ln N$, i.e., $\ln y \sim \theta \ln N$. The finite-size scaling behavior of entanglement can be expressed as $|(\frac{dC}{dg})|_{g_m}| \sim N^{\theta}$, where $\theta$ is the so-called entanglement exponent. The divergent behavior of correlation length exhibits exponential behavior near the critical point $g_c$, i.e., $\xi \sim |g - g_c|^{-\nu}$. According to the idea of the RG method, the correlation length in the $n$th RG step performs as $\xi^n \sim |g_n - g_c|^{-\nu}$, and $\xi^n = \xi / l_B^n$, with $l_B = 2$. $l_B$ is the length of the side in each block, and the relation between $l_B$ and $N$ can be obtain by the definition of fractal dimension $d_f$, i.e., $N = N_0 * l_B^{nd_f}$, where $N_0 = 3$ or 4. One can obtain the expression for $|\frac{dg_n}{dg}|_{g_c}$ in terms of $l_B$ and $\nu$ as follows:

$$\left|\frac{dg_n}{dg}\right|_{g_c} = \left|\frac{g_n - g_c}{g - g_c}\right| \sim (l_B^n)^{\frac{1}{\nu}} = \left(\frac{1}{N_0}N\right)^{\frac{1}{\nu d_f}} \sim N^{\frac{1}{\nu d_f}}. \quad (12)$$

Under the RG method, the relation $|(\frac{dC}{dg})|_{g_m}| \sim |\frac{dg_n}{dg}|_{g_c}$ holds near the QCP. Thus, one can obtain the scaling law relation of the entanglement exponent, $\theta = 1/(\nu d_f)$. This result means that it is the fractal dimensionality but not the space dimensionality to determine this relation. As shown in Figs. 7(a) and 7(b), the numerical result is $\theta \doteq 0.873$ (or 0.799) for the Sierpiński triangular (or Sierpiński pyramid) lattice, which is consistent with the exact result $\theta = 0.877$ (or 0.809). The previous results of phase transitions on the fractal lattices have shown that fractal dimension determines the types and the university of phase transitions. In this work, which is different from the the translation symmetry lattices like the triangle lattice, the sparse structure of the fractal lattices will influence the property of quantum correlation and the few nearest neighbors inside of fractal lattices may influence the quantum entanglement between the vertices (blocks) of the system. Therefore, the scaling law relation of the entanglement exponent, $\theta = 1/(\nu d_f)$, tells us that there is a natural relation between the critical entanglement and the correlation of quantum fluctuation. The fractal dimensionality or space dimensionality, which can be used to defined the relation between the line and volume (or area) of system, may bridge between the correlation length of quantum fluctuation and the global entanglement of the critical systems. The entanglement exponent is an effective symbol to depict the universal class of Ising model on Sierpiński fractal lattices. The entanglement can also give important information about the global long-distance critical behaviors of the transverse-field Ising model on the fractal lattices, which is useful to design the robust and extensible quantum communication networks.

## V. DISCUSSION AND SUMMARY

Combining the quantum renormalization group method with the concurrence, we have studied the relationship between quantum entanglement and quantum phase transition of the spin systems on the triangular lattice and Sierpiński fractal lattices, respectively. The results indicate that the properties of the bipartite ground-state entanglement rely on the magnetic field, the exchange coupling, the system size, and the structure of lattice. As the external field increases from zero, the concurrence increases first and then decreases to zero. Its peak is approaching the quantum critical point when the size of the system becomes large. Meanwhile, the range of the magnetic field in which the entanglement can exist shrinks further. In the thermodynamic limit, the entanglement, especially its first derivative, may depict the system's critical behavior clearly. When the size of the system becomes large, the first derivative of entanglement exhibits singular behavior, and its maximum or minimum value is approaching the quantum critical point. The singularity of the entanglement between spin blocks is a good indicator to reflect the quantum phase transition.

Finally, there is the finite-size scaling behavior of entanglement around the quantum critical point. Moreover, these scaling relations on the different lattice structures are different. For the triangular lattice, the space dimensionality determines the scaling law relationship between the critical exponent of the entanglement and the critical exponent of correlation length. However, for fractal lattices, it is the fractal dimensionality but not the space dimensionality to determine this relationship. It is clear that the concurrence (a kind of quantum entanglement) involves some information about the structure of lattices. The reason is that the concurrence is determined by the parameters given by the RG equations, which are different for different lattices. It is well known that the types and the university of phase transition are related with the space or fractal dimensionality. The fractal dimensionality (or space dimensionality), which can be used to defined the relation between the line and volume (or area) of system, may bridge between the correlation length of quantum fluctuation and the global entanglement of the critical systems.

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