Entanglement and nonlocality in diagonal symmetric states of N qubits

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We analyze entanglement and nonlocal properties of the convex set of symmetric N-qubit states which are diagonal in the Dicke basis. First, we demonstrate that within this set, semidefinite positivity of partial transposition (PPT) is necessary and sufficient for separability—which has also been reported recently by Yu [Phys. Rev. A **94**, 060101(R) (2016)]. Furthermore, we show which states among the entangled diagonal symmetric are nonlocal under two-body Bell inequalities. The diagonal symmetric convex set contains a simple and extended family of states that violate the weak Peres conjecture, being PPT with respect to one partition but violating a Bell inequality in such partition. Our method opens directions to address entanglement and nonlocality on higher dimensional symmetric states, where presently very few results are available.

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I. INTRODUCTION

The characterization of entanglement, which has been recognized in the last two decades as the crucial resource for many quantum tasks, remains a challenging open problem when the state under scrutiny is genuinely multipartite. Although assessing if a generic multipartite state is entangled is known to be an NP-hard problem [1,2], the situation may be different if the quantum state possesses some symmetries. Those can be explored to provide novel separability criteria that fail in the general case [3–8].

Symmetric states, ρ_S, are defined as the states of N qubits that lie in the subspace $\mathcal{S}_N \subset \mathcal{B}(\mathcal{H})$, $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ and fulfill that $V_\sigma \rho_S = \rho_S V_{\sigma'}^\dagger$, where V_σ is the operator representing the permutation σ over the N-element set. Due to such symmetry, $\mathcal{S}_N \simeq \mathcal{B}(\mathbb{C}^{N+1})$. Thus, the rank of ρ_S is bounded, $1 \leqslant r(\rho_S) \leqslant N+1$. These states thus describe identical bosons, i.e., invariant under permutations. Symmetric states are, by construction, either fully separable or genuine multipartite entangled [9]. The experimental realization of symmetric states with photons and atoms has paved the way to seminal quantum information test beds such as the verification of truly multipartite entanglement [10,11]. Also, it has been demonstrated that symmetric states outperform other states for metrological tasks [12]. Symmetric states also arise in the interaction of a quantized electromagnetic field in a cavity with a set of two-level atoms, as well as the ground state of many-body Hamiltonians, e.g., the Lipkin-Meshkov-Glick model [13]. Thus, their relevance, and in particular, the characterization of their quantum correlations, impinges in several domains. A natural representation of such states is given by means of the Dicke basis as

$$\rho_S = \sum p_{kj} \left|D_k^N\right\rangle\left\langle D_j^N\right|, \qquad \sum p_{kj} = 1, \tag{1}$$

where $|D_k^N\rangle$ denotes a Dicke state, i.e., states of N qubits invariant under the permutation of their elements

$$\left|D_k^N\right\rangle = \left(C_k^N\right)^{-1/2} \sum_\sigma |\sigma(1^k 0^{N-k})\rangle, \tag{2}$$

where k denotes the number of qubits on the state $|1\rangle$ and $C_k^N = N!/[(N-k)!k!]$ is a normalization factor. Within the symmetric states, the diagonal symmetric (DS) is the convex subset formed by those states which are diagonal in the Dicke basis:

$$\rho_{\rm DS} = \sum_{k=0}^{N} p_k \left|D_k^N\right\rangle\left\langle D_k^N\right|, \quad \sum p_k = 1, \tag{3}$$

and which possess an even larger symmetry due to the fact that they remain invariant under twirling unitary operations [14].

Here we analyze the properties of the DS subset and present the necessary and sufficient conditions to certify entanglement. We have also investigated the nonlocal character of this family by means of two-body Bell inequalities. We show that, within the DS set, there is a large number of states that violate the weak Peres conjecture [15], meaning that they are PPT bound entangled with respect to one partition but nevertheless they violate a Bell inequality.

Our article is organized as follows. In Sec. II, we prove that all N-qubit DS states that are PPT (semidefinite positive under partial transposition) with respect to the largest bipartition are necessarily separable (see also [16] for a different proof based on extremal witnesses). In Sec. III, for the sake of simplicity, we focus on the simplest nontrivial case, $N=4$ qubits. For such a case, we are able to give several geometrical representations of the set of separable states which helps one to understand the structure of DS space. In Sec. IV, we ask which DS-entangled states are nonlocal under two-body Bell inequalities. To this aim, we use the recently introduced two-body Bell inequalities for many-body systems [17] and show that not all entangled DS states violate such inequalities while providing also one of the simplest counterexamples of the weak Peres conjecture [15]. We recall here that the weak Peres conjecture states that if a given density matrix ρ is PPT with respect to a given partition $A|B$, it does not violate a Bell inequality, whereas the strong Peres conjecture refers to the impossibility of violating a Bell inequality if ρ is PPT with respect to all partitions. Recently, several counterexamples to both the weak and the strong Peres conjecture have appeared [18–20]. After a brief summary of the obtained results, we conclude in Sec. V.

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II. SEPARABILITY OF DIAGONAL SYMMETRIC STATES

To study the separability of ρ_{DS} for N qubits, one should determine first whether the state is PPT with respect to all possible partitions. Denoting by $\rho_{\mathrm{DS}}^{\Gamma_t}$ the partial transpose with respect to the partition $t|N-t$, permutational symmetry implies $\rho_{\mathrm{DS}}^{\Gamma_t} \in \mathcal{B}(\mathbb{C}^{t+1} \otimes \mathbb{C}^{N-t+1})$ and, accordingly, its rank is bounded by $r(\rho_S^{\Gamma_t}) \leqslant (t+1)(N-t+1)$. The lowest nontrivial case corresponds to $N=4$, since for $N<4$, $\dim(\mathcal{S}_N) \leqslant 6$, and PPT is a necessary and sufficient condition for separability [21]. Recently it has also been demonstrated, using the extremality of witness operators, that if $\rho_{\mathrm{DS}}^{\Gamma_{\lfloor N/2 \rfloor}} \geqslant 0$, then ρ_{DS} is separable [16] (see also [22] for a numerical proof of the $N=4$ case). Our proof, however, relies only on the symmetry displayed by the states without knowing the extremal points of such set. Notice that these results exclude the possibility of finding bound entangled states among the DS set. In contrast, symmetric bound entangled states ρ_S exist even for $N=4$ [23]. To focus our study we start with the following theorem.

Theorem 1. A nontrivial separable DS state ρ_{DS} must be of full rank, i.e., $p_k \neq 0$ for all $k=0,1,\ldots,N$.

By trivial separable DS states we mean states of the form $\rho_{\mathrm{DS}} = \sum p_k |D_k^N\rangle\langle D_k^N|$ with either $k=0$ or $k=N$, as well as any mixture of them. Theorem 1 states that any ρ_{DS} whose rank $r(\rho_{\mathrm{DS}}) \leqslant N$ is entangled unless it is trivially separable. Therefore, from now on we restrict our study to states such that $r(\rho_{\mathrm{DS}}) = N+1$. The proof of the theorem follows directly from Lemma 1 below.

Lemma 1. ρ_{DS} is PPT with respect to each possible partition if and only if $\rho_{\mathrm{DS}}^{\Gamma_{\lfloor N/2 \rfloor}} \geqslant 0$.

Proof. Let us rewrite ρ_{DS} in the canonical basis, e.g., $\{i_1,i_2,\ldots,i_N\}$, where $i_k = \{0,1\}\ \forall k$.

$$\rho_{\mathrm{DS}} = \sum_{\mu,\nu=1}^{2^N} p_{\mu\nu} |i_1^\mu,i_2^\mu,\ldots,i_N^\mu\rangle\langle i_1^\nu,i_2^\nu,\ldots,i_N^\nu|, \qquad (4)$$

and $p_{\mu\nu} = p_m \neq 0$ if and only if $\sum_{s=1}^N i_s^\mu = \sum_{s=1}^N i_s^\nu = m$, since diagonal symmetry imposes that all matrix elements are zero except those containing the same number m of excitations, i.e., number of $i_s = 1$. Since $0 \leqslant \sum_{s=1}^N i_s^\mu \leqslant N$, exactly $\binom{N}{m}$ number of rows (and hence columns by symmetry) in the matrix representation (4) will have p_m as the only nonzero entries. Equivalently, the only nonvanishing matrix elements of any partial transposed matrix $\rho_{\mathrm{DS}}^{\Gamma_t}$ are those fulfilling $\sum_{s=1}^N i_s^\mu + \sum_{s=1}^N i_s^\nu = 2m$, i.e., $p_m \neq 0$ if and only if $\sum_{s=1}^N i_s^\mu = m+t,\ldots,m,\ldots,m-t$ and $\sum_{s=1}^N i_s^\nu = m-t,\ldots,m,\ldots,m+t$.

Rewriting the above conditions in a matrix form, $\rho_{\mathrm{DS}}^{\Gamma_{\lfloor N/2 \rfloor}} \geqslant 0$, corresponds to

$$M_k = \begin{bmatrix} p_{0+k} & p_{1+k} & p_{2+k} & \cdots & p_{m+k} \\ p_{1+k} & p_{2+k} & p_{3+k} & \cdots & p_{m+1+k} \\ p_{2+k} & p_{3+k} & p_{4+k} & \cdots & p_{m+2+k} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ p_{m+k} & p_{m+1+k} & p_{m+2+k} & \cdots & p_{2m+k} \end{bmatrix} \geqslant 0, \qquad (5)$$

where $m = \lfloor \frac{N}{2} \rfloor$ and $k=0,1$. Note that the M_k's are Hankel matrices [24]. The PPT condition with respect to any other partition t, $\rho_{\mathrm{DS}}^{\Gamma_t}$, is included in the positive semidefinite of the principal minors of M_k. Finally, notice that a necessary condition for $M_{0,1} \geqslant 0$ is that $p_k \neq 0$ for all $k=0,1,\ldots,N+1$. Therefore, this proves that a generic diagonal symmetric state must be of full rank $\rho_{\mathrm{DS}} = N+1$, otherwise it is entangled or trivially separable. This proves Theorem 1. An alternative proof of the theorem not relying on the properties of Hankel matrices is given in the Appendix.

With the above lemma, we proceed now to the main theorem and its demonstration.

Theorem 2. All diagonal symmetric states of N qubits that are PPT with respect to the largest possible bipartition $\lfloor N/2 \rfloor | N - \lfloor N/2 \rfloor$ are separable (see also [16]).

Proof. The proof of Theorem 2 is as follows:

(1) We have proven in Lemma 1 that if $\rho_{\mathrm{DS}}^{\Gamma_{\lfloor N/2 \rfloor}} \geqslant 0$, then $\rho_{\mathrm{DS}}^{\Gamma_t} \geqslant 0$ for $t=1,\ldots,\lfloor N/2 \rfloor$, which means that ρ_{DS} is full PPT if and only if $\rho_{\mathrm{DS}}^{\Gamma_{\lfloor N/2 \rfloor}} \geqslant 0$.

(2) We construct an extended symmetric mixed state, ρ_{EXT}, which fulfills $\rho_{\mathrm{EXT}} = \rho_{\mathrm{EXT}}^{\Gamma_1} = \rho_{\mathrm{EXT}}^{\Gamma_{\lfloor N/2 \rfloor}}$ and show that $\rho_{\mathrm{EXT}} \geqslant 0$ if and only if $\rho_{\mathrm{DS}}^{\Gamma_{\lfloor N/2 \rfloor}} \geqslant 0$. This ρ_{EXT} is actually obtained by adding some coherence (off-diagonals) terms to the ρ_{DS}. Since the rank of ρ_{DS} is maximal, adding coherences cannot make it larger.

(3) Using the fact that any $\rho \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^N)$ such that $\rho = \rho^{\Gamma_1}$ is separable [25], if $\rho_{\mathrm{EXT}} \geqslant 0$, then it is separable.

(4) Finally, we prove that ρ_{DS} can be obtained from ρ_{EXT} by local unitary transformations. Therefore, if $\rho_{\mathrm{DS}}^{\Gamma_{\lfloor N/2 \rfloor}} \geqslant 0$, then ρ_{DS} is separable. The details of steps (2) and (3) are given below.

Step (2). We extend the original matrix $\rho_{\mathrm{DS}} = \sum p_i |D_i^N\rangle\langle D_i^N|$ as follows:

$$\rho_{\mathrm{DS}} \rightarrow \rho_{\mathrm{EXT}} = \sum p_i |D_m^N\rangle\langle D_l^N|, \quad m+l = 2i \qquad (6)$$

$\forall N \geqslant m,l \geqslant 0$. Notice that by adding the coherences $|D_m^N\rangle\langle D_l^N|$, the extended matrix fulfills $\rho_{\mathrm{EXT}} = \rho_{\mathrm{EXT}}^{\Gamma_t}$, for every $t=1,\ldots,\lfloor N/2 \rfloor$. Notice that PPT has to be performed in the canonical basis. However, by expressing ρ_{EXT} in the Dicke basis,

$$\rho_{\mathrm{EXT}} = \begin{bmatrix} p_0 & 0 & p_1 & 0 & p_2 & 0 & \ldots & p_{n/2} \\ 0 & p_1 & 0 & p_2 & 0 & p_3 & \ldots & 0 \\ p_1 & 0 & p_2 & 0 & p_3 & 0 & \ldots & p_{n/2+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{n/2} & 0 & p_{n/2+1} & \cdot & \cdot & \cdot & \cdot & p_n \end{bmatrix}, \qquad (7)$$

it is straightforward to see that $\rho_{\mathrm{EXT}} \geqslant 0$ if and only if the associated Hankel matrices $M_{0,1} \geqslant 0$.

Step (3). Finally, it is straightforward to notice that ρ_{DS} can be obtained by applying local unitaries on ρ_{EXT} as follows:

$$\rho_{\mathrm{DS}} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, U^{\otimes N}\rho_{\mathrm{EXT}}(U^{\dagger})^{\otimes N}, \tag{8}$$

with

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}.$$

Since local unitaries cannot change the entanglement properties of the matrix, this ends the demonstration of the theorem stating that given a diagonal symmetric matrix, ρ_{DS}, the state is separable if and only if the state ρ_{DS} is PPT with respect to the largest bipartition. □

III. GEOMETRY OF $N = 4$ DS SPACE

To understand the geometry of the space of DS states, we focus now on the $N = 4$ case where only two partitions exist, 1|3 and 2|2. The ranks of interest are $r(\rho_{\mathrm{DS}}) \leqslant 5$, $r(\rho_{\mathrm{DS}}^{\Gamma_1}) \leqslant 8$, and $r(\rho_{\mathrm{DS}}^{\Gamma_2}) \leqslant 9$. As we shall see, generic separable states are of maximal tri-rank (5,8,9), while extremal separable states can have a tri-rank as low as (5,6,6). We use Theorem 2 to determine the geometry of the DS PPT states. We write an un-normalized generic ρ_{DS} in the canonical basis:

$$\rho_{\mathrm{DS}} = \begin{bmatrix}
p_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p_1 & p_1 & \cdot & p_1 & \cdot & \cdot & \cdot & p_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p_1 & p_1 & \cdot & p_1 & \cdot & \cdot & \cdot & p_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & p_2 & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & p_2 & \cdot & \cdot & \cdot \\
\cdot & p_1 & p_1 & \cdot & p_1 & \cdot & \cdot & \cdot & p_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & p_2 & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & p_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & p_2 & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & p_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_3 & \cdot & \cdot & \cdot & p_3 & \cdot & p_3 & p_3 & \cdot \\
\cdot & p_1 & p_1 & \cdot & p_1 & \cdot & \cdot & \cdot & p_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & p_2 & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & p_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & p_2 & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & p_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_3 & \cdot & \cdot & \cdot & p_3 & \cdot & p_3 & p_3 & \cdot \\
\cdot & \cdot & \cdot & p_2 & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & p_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_3 & \cdot & \cdot & \cdot & p_3 & \cdot & p_3 & p_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_3 & \cdot & \cdot & \cdot & p_3 & \cdot & p_3 & p_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_4
\end{bmatrix}. \tag{9}$$

The PPT region is given by the inequalities which arose by imposing $\rho_{\mathrm{DS}}^{\Gamma_2} \geqslant 0$. Explicitly,

$$\rho_{\mathrm{DS}}^{\Gamma_2} = \begin{bmatrix}
p_0 & \cdot & \cdot & \cdot & \cdot & p_1 & p_1 & \cdot & \cdot & p_1 & p_1 & \cdot & \cdot & \cdot & \cdot & p_2 \\
\cdot & p_1 & p_1 & \cdot & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & p_1 & p_1 & \cdot & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & p_1 & \cdot & \cdot & \cdot & p_1 & \cdot & \cdot & \cdot & \cdot & p_2 & p_2 & \cdot \\
p_1 & \cdot & \cdot & \cdot & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & \cdot & \cdot & \cdot & p_3 \\
p_1 & \cdot & \cdot & \cdot & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & \cdot & \cdot & \cdot & p_3 \\
\cdot & p_2 & p_2 & \cdot & \cdot & \cdot & \cdot & p_3 & \cdot & \cdot & \cdot & p_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & p_1 & \cdot & \cdot & \cdot & p_1 & \cdot & \cdot & \cdot & \cdot & p_2 & p_2 & \cdot \\
p_1 & \cdot & \cdot & \cdot & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & \cdot & \cdot & \cdot & p_3 \\
p_1 & \cdot & \cdot & \cdot & \cdot & p_2 & p_2 & \cdot & \cdot & p_2 & p_2 & \cdot & \cdot & \cdot & \cdot & p_3 \\
\cdot & p_2 & p_2 & \cdot & \cdot & \cdot & \cdot & p_3 & \cdot & \cdot & \cdot & p_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & \cdot & p_3 & p_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & p_2 & \cdot & \cdot & \cdot & \cdot & p_3 & p_3 & \cdot \\
p_2 & \cdot & \cdot & \cdot & \cdot & p_3 & p_3 & \cdot & \cdot & p_3 & p_3 & \cdot & \cdot & \cdot & \cdot & p_4
\end{bmatrix}. \tag{10}$$

It is straightforward to see that $\rho_{\mathrm{DS}}^{\Gamma_2} \geqslant 0$ corresponds to $M_0 \geqslant 0$ and $M_1 \geqslant 0$ [Eq. (5)] where

$$M_0 = \begin{vmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{vmatrix}, \qquad M_1 = \begin{vmatrix} p_1 & p_2 \\ p_2 & p_3 \end{vmatrix}. \tag{11}$$

To give a geometrical picture of the set of separable DS states, notice that the conditions arising from $M_k \geqslant 0$ given by Eq. (11) reduce to $p_i p_{i+2} \geqslant p_{i+1}^2$ for $i = 0,1,2$ plus the condition $\det(M_0) \geqslant 0$. After imposing the proper normalization of the states, the above inequalities

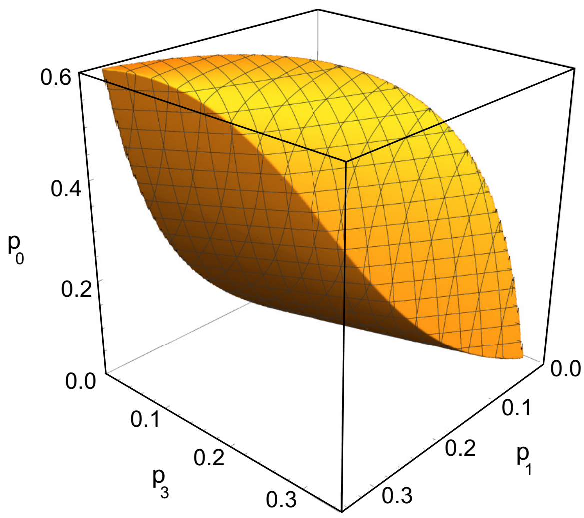

FIG. 1. Representation of the PPT-DS region for $N = 4$. The picture corresponds to $s = p_0 + p_4 = 0.6$. Notice the shape of the region of separable states is independent of the value of s.

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$$E_1 \equiv 8p_0p_2 - 3p_1^2 \geqslant 0, \tag{12a}$$

$$E_2 \equiv 9p_1p_3 - 4p_2^2 \geqslant 0, \tag{12b}$$

$$E_3 \equiv 8p_2p_4 - 3p_3^2 \geqslant 0, \tag{12c}$$

$$F_2 \equiv p_4(72p_0p_2 - 27p_1^2) - 2p_2^3 - 9p_3(p_1p_2 + 3p_0p_3) \geqslant 0. \tag{12d}$$

Normalization, $\sum_i p_i = 1$, reduces the number of free parameters to just four. The region bounded by Eqs. (12) in the (p_0,p_1,p_3) space can be easily depicted by imposing the constraint $p_0 + p_4 = s$, for $s \in (0,1)$ as shown in Fig. 1. While the parameter k varies continuously in the range $(0,1)$, the shape of the PPT volume remains invariant making its properties independent of s.

Taking "slices" of constant p_0 in the above representation makes the geometry of PPT diagonal symmetric states even simpler, as depicted in Fig. 2. For each slice, the PPT region is bounded just from two surfaces, $F_2 = 0$ and $E_1 = 0$, and thus $\text{PPT}_{\Gamma_2} \subset \text{PPT}_{\Gamma_1}$. In Fig. 2 we also represent all the possible

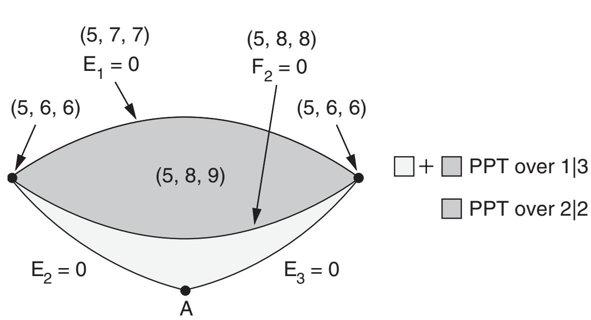

FIG. 2. The generic PPT region given by Eqs. (12). Generic separable states are of rank (5,8,9) (dark area), while extremal separable states are of rank (5,6,6), (5,8,8), and (5,7,7). Point A is an edge entangled state that satisfies $\text{PPT}_{\Gamma_1} > 0$ but $\text{PPT}_{\Gamma_2} < 0$.

tri-ranks $(r(\rho),r(\rho^{\Gamma_1}),r(\rho^{\Gamma_2}))$ a PPT DS state can have, (5,8,9) being the generic one. Point A in Fig. 2 corresponds to the state ρ_{JC}, given below in Eq. (13), which is derived from the Jaynes-Cumming model describing a two-level atom interacting with the first M levels of an electromagnetic field in a cavity. Notice that $\rho_{\text{JC}} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^M)$ but when expressed in the Dicke basis they can be mapped onto a diagonal symmetric state; $\rho_{\text{JC}} \in \mathcal{B}[(\mathbb{C}^2)^{\otimes M}]$. For the particular case of $M = 4$, these states read [26]

$$\rho_{\text{JC}} = \frac{a}{4}|D_0^4\rangle\langle D_0^4| + |D_1^4\rangle\langle D_1^4| + \frac{3}{2a}|D_2^4\rangle\langle D_2^4| + \frac{1}{a^2}|D_3^4\rangle\langle D_3^4| + \frac{1}{4a^2b}|D_4^4\rangle\langle D_4^4|, \tag{13}$$

with $a,b \in \mathbb{R} \geqslant 0$ and $a > b$. Interestingly enough, this family of states fulfill $E_1 = 0$ and $E_3 = 0$ and they are extremal points in the subset of states satisfying $PPT_{\Gamma_1} \geqslant 0$ but they do not fulfill that $PPT_{\Gamma_2} > 0$. Thus, they are PPT-edge states with respect to the partition $\rho_{\text{JC}}^{\Gamma_1}$. Furthermore, these states have been shown to be bound entangled in $\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^4)$ [26] where there is only a possible partition.

IV. NONLOCALITY OF DIAGONAL SYMMETRIC STATES

All entangled DS are, by construction, genuine multipartite entangled. It is, therefore, natural to ask about their properties with respect to nonlocality. Recently, it has been shown that it is possible to detect nonlocality on symmetric N-qubit states just involving only one- and two-body correlations [17]. Such Bell inequalities provide an experimentally accessible setup to test nonlocality in many-body systems without relying on N-body correlations. The Bell inequality reads

$$\mathcal{B}(\theta,\phi) \equiv \alpha S_0 + \beta S_1 + \frac{\gamma}{2}S_{00} + \delta S_{01} + \frac{\epsilon}{2}S_{11} + \beta_C \geqslant 0, \tag{14}$$

with

$$S_l \equiv \sum\nolimits_{i=1}^{N} \langle \mathcal{M}_l^{(i)} \rangle, \tag{15}$$

$$S_{lr} \equiv \sum\nolimits_{i \neq j=1}^{N} \langle \mathcal{M}_l^{(i)} \mathcal{M}_r^{(j)} \rangle, \tag{16}$$

for $l,r = 0,1$, where $\mathcal{M}_0 = \cos\phi\,\sigma_z + \sin\phi\,\sigma_x$ and $\mathcal{M}_1 = \cos\theta\,\sigma_z + \sin\theta\,\sigma_x$ are the measurements and θ, ϕ the devices orientation angles.

In particular, Eq. (14) is violated by all entangled Dicke states $|D_k^N\rangle$ (i.e., $k \neq 0,N$) for the following set of specific parameters [27]:

$$\begin{aligned} \nu &= \left\lfloor \frac{N}{2} \right\rfloor - k, \\ \alpha &= \upsilon\ 2\nu N(N-1), \\ \beta &= \alpha/N, \\ \gamma &= N(N-1), \\ \delta &= N, \\ \epsilon &= -2, \\ \beta_C &= \binom{N}{2}[N + 2(2\nu^2 + 1)]. \end{aligned} \tag{17}$$

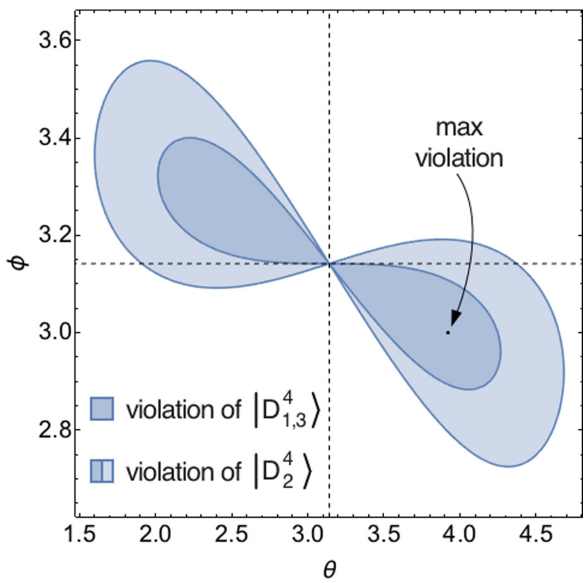

FIG. 3. Device angles for which there is Bell violation of Eq. (14) for the Dicke states $|D^4_{1,3}\rangle$ (dark) and $|D^4_2\rangle$ (dark + light). The device angle that corresponds to the maximal Bell violation is indicated.

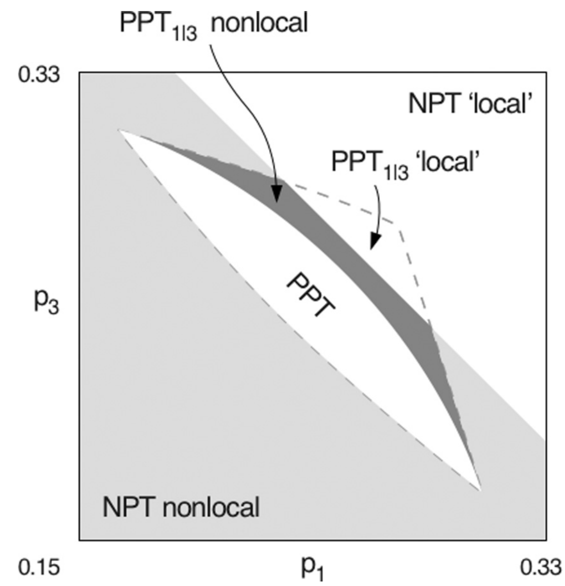

FIG. 4. Nonlocality of DS states for $p_0 = p_4 = 0.1$. Gray areas correspond to $\rho_{\rm DS}$ which are nonlocal, while white areas depict local states. PPT denotes separable states. By $\text{PPT}_{1|3}$ we refer to $\rho_{\rm DS}$ such that it is PPT with respect to partition 1|3 but NPT with respect to partition 2|2. The boundaries of the PPT region obtained by numerically evaluating Bell inequalities are in one-to-one correspondence with the regions bounded by Eqs. (12).

Defining $Q(|\psi\rangle) := \langle\psi|\mathcal{B}|\psi\rangle$, if $Q(|\psi\rangle) < 0$ the state $|\psi\rangle$ violates Bell inequality (14) thereby the state $|\psi\rangle$ is entangled and nonlocal. For simplicity, we keep on in the $N = 4$ case and consider that $r(\rho_{\rm DS}) = 5$. With the parametrization given in Eq. (17), the Bell inequalities for the entangled Dicke states are given by the expressions

$$Q(|D^4_{1,3}\rangle) = 75 + 12\cos\theta + 3\cos^2\theta + 48\cos\phi - 18\cos^2\phi - 3\sin^2\theta + 24\sin\theta\,\sin\phi + 18\sin^2\phi, \quad (18)$$

$$Q(|D^4_2\rangle) = 46 + 6\cos^2\theta - 16\cos\theta\,\cos\phi - 36\cos^2\phi - 6\sin^2\theta + 32\sin\theta\,\sin\phi + 36\sin^2\phi. \quad (19)$$

We first search the optimal device orientation, i.e., the one which provides maximal violation for $|D^{N=4}_i\rangle$, according to (18) and (19). It corresponds to $\theta = 3.916$ and $\phi = 3.002$ yielding the values $Q(|D^4_{1,3}\rangle) = -0.683$ and $Q(|D^4_2\rangle) = -2.913$. The dependence of the Bell inequality on the device orientation is depicted in Fig. 3, for all entangled Dicke states $|D^4_{1,2,3}\rangle$. Given the permutational invariance of DS states, nonlocal DS states, i.e., states for which $Q(\rho_{\rm DS}) \equiv \sum p_k Q(|D^4_k\rangle) < 0$, are given by the following inequality:

$$(p_1 + p_3)Q_1 + p_2 Q_2 > (p_0 + p_4)Q_0, \quad (20)$$

where $Q_k \equiv |Q(|D^4_k\rangle)|$. For $N = 4$, it is easy to check which states are nonlocal in terms of the negativity of partial transposition (NPT) conditions. Our results are schematically summarized in Fig. 4, where we fix the values of p_0 and p_4 and evaluate numerically the Bell inequality (14) for any possible value of p_i $(i = 1,2,3)$ of the DS mixture. Gray regions correspond to DS states that violate the two-body Bell inequality.

The boundaries between local and nonlocal states arise from the condition $(p_1 + p_3)Q_1 + p_2 Q_2 = (p_0 + p_4)Q_0$. In Fig. 4, first we notice that, as expected, not all NPT states are nonlocal under two-body Bell inequalities. Secondly, that such inequality provides exactly the boundaries of the separable states. That is, for this values of p_0 and p_4, Eqs. (12) can be derived independently from the violation of inequality given by Eq. (14). Finally, the dark-gray area in Fig. 4 indicates that there exist states which are PPT with respect to 1|3, but nonlocal—hence violating the weak Peres conjecture. For other values of p_0 and p_4 we can see that also the Jaynes-Cumming states as described in Eq. (13) do violate a Bell inequality. Thus, despite the fact that there are no bound entangled states in the diagonal symmetric convex set of N qubits, as demonstrated in Theorem2 (see also [16]), there are states $\rho \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^4)$ that can be mapped onto the diagonal symmetric states of four qubits and which are PPT with respect to the only possible partition $(\mathbb{C}^2|\mathbb{C}^4)$ and nonlocal, i.e., bound entangled states that violate Bell inequalities. These PPT states are the simplest counterexample to the Peres conjecture [15,18,19], as they have a very simple structure and their nonlocality is revealed using only two-body Bell inequalities.

V. CONCLUSIONS

We have analyzed the entanglement of diagonal symmetric states of N qubits which describe bosonic states of identical particles. First, we have proven that separable diagonal symmetric states of N qubits are necessarily of full rank, i.e., $N + 1$. Secondly, we have demonstrated, exploting the symmetries of the state, that for such family, PPT is a sufficient and necessary condition for separability. Third, for $N = 4$, we have provided a complete geometrical description of the set of separable DS states. Finally, we have shown that there is not a one-to-one correspondence between entanglement and nonlocality using two-body Bell inequalities, and that there exist an extended family of diagonal symmetric states that are PPT with respect to a partition but nevertheless violate a Bell inequality in this partition. We conclude by remarking that some of our methods can be extended to symmetric states in higher dimensions where, so far, very few results are known.

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APPENDIX: AN ALTERNATIVE PROOF OF THEOREM 1

This proof is based on the separability criteria from Ref. [28]. The criterion given there is both necessary and sufficient, but for our purpose only a part of the necessary conditions would suffice.

Let $\rho = \sum_{j=1}^{m} \lambda_j |\psi_j\rangle\langle\psi_j|$ be the spectral decomposition of a state with $\lambda_i > 0$. Consider the normalized pure state $|\chi\rangle = \sum_{j=1}^{m} x_j |\psi_j\rangle$, $x_j \in \mathbb{C}$, and let its single party marginals be σ_K, $K = 1,2,\dots,N$. Then the state ρ is separable, which implies that there are $n \geqslant m$ number of distinct solutions $\boldsymbol{x}^{(i)}$ (two solutions $\boldsymbol{x}^{(i)}$ and $\boldsymbol{x}^{(j)}$ are distinct if and only if $\boldsymbol{x}^{(i)} \neq c\boldsymbol{x}^{(j)}$) to the system of equations

$$\det(\sigma_K - I) = 0, \quad K = 1,2,\dots,N. \tag{A1}$$

The normalized state $|\chi\rangle$ in our case is given by $|\chi\rangle = \sum_{k=0}^{N} x_k |D_k^N\rangle$. Splitting each $|D_k^N\rangle$ into $1|23\cdots N$,

$$|\chi\rangle = \sum_{k=0}^{N} x_k \left(\sqrt{\frac{N-k}{N}} |0\rangle|\bar{k}\rangle + \sqrt{\frac{k}{N}} |1\rangle|\overline{k-1}\rangle \right) \tag{A2a}$$

$$= |0\rangle \sum_{k=0}^{N-1} a_k |\bar{k}\rangle + |1\rangle \sum_{k=0}^{N-1} b_k |\bar{k}\rangle, \tag{A2b}$$

where $|\bar{k}\rangle := |D_k^{N-1}\rangle$ for $0 \leqslant k \leqslant N-1$, otherwise 0; $a_k := [(N-k)/N]^{1/2} x_k$, $b_k := [(k+1)/N)]^{1/2} x_{k+1}$. The normalization is simply given by $\sum_{k=0}^{N-1} (|a_k|^2 + |b_k|^2) = 1$. Any single-qubit marginal of $|\chi\rangle$ is given by

$$\sigma = \begin{pmatrix} \sum_{k=0}^{N-1} |a_k|^2 & \sum_{k=0}^{N-1} a_k \bar{b}_k \\ \sum_{k=0}^{N-1} \bar{a}_k b_k & \sum_{k=0}^{N-1} |b_k|^2 \end{pmatrix}. \tag{A3}$$

Hence the separability condition of Eq. (A1) reads

$$\left(\sum_{k=0}^{N-1} |a_k|^2 - 1 \right) \left(\sum_{k=0}^{N-1} |b_k|^2 - 1 \right) - \left(\sum_{k=0}^{N-1} a_k \bar{b}_k \right) \left(\sum_{k=0}^{N-1} \bar{a}_k b_k \right) = 0$$

$$\Rightarrow (a_k)_{k=0}^{N-1} = c (b_k)_{k=0}^{N-1}, \quad \text{where } c \in \mathbb{C}, \tag{A4}$$

by Cauchy-Schwarz inequality and using the normalization. So, the general solution of Eq. (A1) is given by

$$x_k = c \sqrt{\frac{k+1}{N-k}} x_{k+1}, \qquad k = 0,1,\dots,N-1.$$

By definition of $|\chi\rangle$, if $p_k = 0$ for some k, the corresponding $x_k = 0$. If $p_0 = 0$, there is one unique solution $\boldsymbol{x} = (0,0,\dots,0,1)$. Since the number of (distinct) solutions has to be at least the number of nonzero p_k's, $p_N = 1$ is the only nonzero p_k. Similarly, if $p_N = 0$, $p_0 = 1$ is the only nonzero p_k and there is no solution if $p_0 = 0 = p_N$. For any other $p_k = 0$ with $p_0 p_N \neq 0$ there are exactly two solutions $\{(1,0,\dots,0), (0,\dots,0,1)\}$ to Eq. (A1), and hence for ρ to be separable at most two of the p_k's could be nonzero, which are p_0 and p_N. □

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