Attainability of the quantum information bound in pure-state models

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(Received 30 January 2017; published 20 April 2017)

The attainability of the quantum Cramér-Rao bound (QCR), the ultimate limit in the precision of the estimation of a physical parameter, requires the saturation of the quantum information bound (QIB). This occurs when the Fisher information associated to a given measurement on the quantum state of a system which encodes the information about the parameter coincides with the quantum Fisher information associated to that quantum state. Braunstein and Caves [Phys. Rev. Lett. **72**, 3439 (1994)] have shown that the QIB can always be achieved via a projective measurement in the eigenvectors basis of an observable called the symmetric logarithmic derivative. However, such projective measurement depends, in general, on the value of the parameter to be estimated, therefore requiring previous knowledge of the quantity one is trying to estimate. For this reason, it is important to investigate under which situation it is possible to saturate the QCR without previous information about the parameter to be estimated. Here, we show the complete solution to the problem of which are the initial pure states and which projective measurements allow the global saturation of the QIB, without the knowledge of the true value of the parameter, when the information about the parameter is encoded in the system by a unitary process.

DOI: 10.1103/PhysRevA.95.042125

I. INTRODUCTION

The aim of quantum statistical estimation theory is to estimate the true value of a real parameter x through suitable measurements on a quantum system of interest. It is assumed that the state of the quantum system belongs to a family $\hat{\rho}(x)$ of density operators, defined on a Hilbert space $\mathcal{H}$ and parametrized by the parameter x. The practical implementation of the estimation process comprises two steps: The first one consists in the acquisition of experimental data from specific quantum measurements on the system of interest while the second one consists in data manipulation in order to obtain an estimative of the true value of the parameter [1]. The first step is implemented via a positive-operator valued measure (POVM), described by a set of positive Hermitian operators $\{\hat{E}_j\}$, which add up to the identity operator ($\sum_{j=1}^{N}\hat{E}_j=\hat{\mathbb{1}}$). The probability of obtaining the measurement result j, if the value of the parameter is x, is then given by $p_j(x)=\mathrm{Tr}[\hat{\rho}(x)\hat{E}_j]$. The second step is implemented by using an estimator to process the data and produce an estimate of the true value of the parameter.

It is well known that there is a fundamental limit for the minimum reachable uncertainty in the estimative of the value of a parameter x, produced by any estimator. When this uncertainty is quantified by the variance δ^2x of the estimates of x, this ultimate lower bound is known as the quantum Cramér-Rao (QCR) bound and is given by $\delta^2x\geqslant 1/\nu\mathcal{F}_Q(x_v)$, where $\mathcal{F}_Q(x_v)$ is the quantum Fisher information (QFI) of the state $\hat{\rho}(x_v)$, ν is the number of repetitions of the measurement on the system, and x_v is the true value of the parameter. The QFI is defined as $\mathcal{F}_Q(x_v)\equiv\max_{\{\hat{E}_j\}}\{\mathcal{F}(x_v,\{\hat{E}_j\})\}$, where $\mathcal{F}(x_v,\{\hat{E}_j\})$ is the Fisher information (FI) associated to the probability distributions $p_j(x_v)=\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j]$. In this regard, $\mathcal{F}_Q(x_v)$ is a measure of the maximum information on the parameter x_v contained in the quantum state $\hat{\rho}(x_v)$. Determining the exact conditions necessary for the saturation of the fundamental limit of precision plays a central role in quantum statistical estimation theory.

Braunstein and Caves [2] (see also Ref. [3]) have investigated and demonstrated the attainability of the QCR bound by separating it in two steps, which are represented by the two inequalities $\delta^2x\geqslant 1/\nu\mathcal{F}(x_v,\{\hat{E}_j\})\geqslant 1/\nu\mathcal{F}_Q(x_v)$. The first inequality corresponds to the classical Cramér-Rao (CCR) bound associated with the particular quantum measurement $\{\hat{E}_j\}$ performed on the system, where $\mathcal{F}(x_v,\{\hat{E}_j\})$ is the Fisher information about the parameter x_v associated to the set of probabilities $\{p_j(x_v)\}$. The saturation of the CCR bound depends on the nature of the estimator used to process the data drawn from the set of probabilities $\{p_j(x_v)\}$ in order to estimate the true value of the parameter. Those estimators that saturates the CCR bound are called *efficient* estimators or *asymptotically efficient* estimators [4] when the saturation only occurs in the limit of a very large number ν of measured data. A typical example of an *asymptotically efficient* estimator is the maximum likelihood estimator [4]. Only special families of probability distributions $\{p_j(x_v)\}$ allow the construction of an *efficient* estimator for finite ν.

The second inequality applies to all quantum measurements $\{\hat{E}_j\}$ and establishes the bound $\mathcal{F}(x_v,\{\hat{E}_j\})\leqslant\mathcal{F}_Q(x_v)$. Saturation of this bound corresponds to finding *optimal measurements* $\{\hat{E}_j\}$, such that

$$\mathcal{F}(x_v,\{\hat{E}_j\})=\mathcal{F}_Q(x_v). \tag{1}$$

These are quantum measurements that would allow one to retrieve all the information about the parameter encoded in the quantum state of the system. The saturation of this bound is also known as the saturation of the *quantum information bound* (QIB) in quantum statistical estimation theory [1]. The quest for determining the *optimal measurements* for any metrological configuration has a long history, going back to the pioneering works of Helstrom [5] and Holevo [6], and has been subject of interest of recent work [1–3,7,8]. In order to prove the attainability of the QCR bound, the authors of

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Ref. [2] have shown that an upper bound to the QFI, based on the so-called symmetric logarithmic derivative (SLD) operator $\hat{L}(x)$, was indeed equal to the QFI. This upper bound was first discovered by Helstrom [5] and Holevo [6] and is given by

$$\mathcal{F}_Q(x_v) \leqslant \mathrm{Tr}[\hat{\rho}(x_v)\hat{L}^2(x_v)].$$

The proof consists in showing that a sufficient condition for achieving the equalities $\mathcal{F}(x_v,\{\hat{E}_j\}) = \mathcal{F}_Q(x_v) = \mathrm{Tr}[\hat{\rho}(x_v)\hat{L}^2(x_v)]$ is given by the use of a POVM $\{\hat{E}_j\}$ such that the operators $\hat{E}_j$ are one-dimensional projection operators onto the eigenstates of the SLD operator $\hat{L}(x_v)$. That is $\{\hat{E}_j(x_v)\} \equiv \{|l_j(x_v)\rangle\langle l_j(x_v)|\}$, where $|l_j(x_v)\rangle$ is an eigenstate of $\hat{L}(x_v)$. At this point it is important to notice that although the use of this optimal POVM is sufficient to saturate the QIB, it depends, in general, on the true value of the parameter one wants to estimate, i.e., $\{\hat{E}_j\} = \{\hat{E}_j(x_v)\}$, which is why this type of saturation is called local. Also, it is important to note that there is no general proof that local saturation of the QIB can be obtained with POVMs corresponding to nonprojective measurements.

Mainly two approaches have been adopted in order to deal with the fact that the optimal POVM depends on the true value x_v of the parameter. The first one relies on adaptive quantum estimation schemes that could, in principle, asymptotically achieve the QCR bound [9–16]. Such approach is valid for any arbitrary state $\hat{\rho}(x)$. The second one looks for the families of density operators $\{\hat{\rho}(x)\}$, for which the use of a specific POVM $\{\hat{E}_j\}$ that does not depend on the true value of the parameter leads to the saturation of the QIB. Our work follows this approach.

Within the second approach, when the family $\{\hat{\rho}(x)\}$ corresponds to operators with no null eigenvalues (full rank), the analysis of the saturation of the QIB is simplified because, given $\hat{\rho}(x)$, there is only one solution for the SLD operator equation:

$$\frac{d\hat{\rho}(x)}{dx} = \frac{1}{2}[\hat{\rho}(x)\hat{L}(x) + \hat{L}^\dagger(x)\hat{\rho}(x)], \tag{2}$$

with $\hat{L}^\dagger(x) = \hat{L}(x)$ [17]. For full-rank operators, Nagaoka [17] showed that saturation of the quantum information bound by using a POVM that does not depend on the true value of the parameter is only possible for the so-called *quasiclassical family* of density operators. He also presented complete characterization of the quantum measurements that guarantee the saturation for this family. Therefore, the problem of finding the states and the corresponding optimal measurements that lead to the saturation of the QIB, independently of the true value of the parameter, in the case of one-parameter families of full-rank density operators has been solved already.

However, for the opposite case of pure states (rank-one density operators), the complete characterization of the families of states and the corresponding measurements that lead to the saturation of the QIB, independently of the true value of the parameter, are still open questions in the case of arbitrary Hilbert spaces. It is important to remark that inside the families of pure states the QFI reaches its largest values. Among these families, the most important ones are those unitarily generated from an initial state $\hat{\rho}_0 = |\phi_+\rangle\langle\phi_+|$ as

$$\hat{\rho}(x) = e^{-i\hat{A}x}\,\hat{\rho}_0\,e^{i\hat{A}x}, \tag{3}$$

where the Hermitian generator $\hat{A}$ does not depend on the parameter x to be estimated. In this case the QFI is given by [3]

$$\mathcal{F}_Q = 4\langle(\Delta\hat{A})^2\rangle_+, \tag{4}$$

where $\langle(\Delta\hat{A})^2\rangle_+ = \mathrm{Tr}[\hat{\rho}_0(\hat{A} - \langle\hat{A}\rangle_+)^2]$ and $\langle\hat{A}\rangle_+ \equiv \mathrm{Tr}[\hat{\rho}_0\hat{A}]$.

For these kinds of families, Ref. [3] considered the situations where the Hermitain operators $\hat{A}$ generate displacements on a Hilbert space basis $\{|x\rangle\}$, i.e., $e^{-i\hat{A}x}|0\rangle = |0 + x\rangle$, where $|0\rangle$ is an arbitrary state. For these situations, the authors could find all the initial states $|\phi_+\rangle$ and the corresponding global optimal POVMs that saturate the QIB, independently of the true value of the parameter x. In Ref. [8], the authors investigated under which conditions a global saturation of QIB can happen for two-level quantum systems.

Here, we present the complete solution to the problem of which are the initial states $|\phi_+\rangle$ and the corresponding families of global projective measurements that allow the saturation of the QIB, within the quantum state family given in Eq. (3), for arbitrary generators $\hat{A}$, that do not depend on the parameter to be estimated, and with discrete spectrum. Since there is no proof that the QIB can be, in general, locally attained via nonprojective measurements, our search for global saturation of the QIB is restricted to the set of projective measurements. We put together a catalog of the initial states $|\phi_+\rangle$ that allow global saturation of the QIB according to the number of eigenstates $|A_{k_l}\rangle$ ($l = 1,\ldots,M$) of the generator $\hat{A}$ which are present in their expansion in the eigenbasis of $\hat{A}$. For a fixed value of the mean $\langle\hat{A}\rangle_{\phi_+}$, each member of this catalog can be expanded in terms of a subset $\{k_l\}$ of eigenstates $|A_{k_l}\rangle$ whose corresponding eigenvalues are equidistant from the mean, provided the coefficients of that expansion satisfy certain symmetry conditions. We show that the global saturation of the QIB requires specific projective measurements within the subspace $\{|A_{k_l}\rangle\}_{l=1,\ldots,M}$, determined by the initial state $|\phi_+\rangle$, and give the full characterization of these projective measurements. We also identify, among all the initial states $|\phi_+\rangle$ that lead to global saturation of the QIB, for a fixed value of the mean $\langle\hat{A}\rangle_{\phi_+}$, which one has the largest QFI. When the spectrum of the generator $\hat{A}$ is lower bounded, such state is a balanced linear superposition of the lowest eigenstate of $\hat{A}$ and the eigenstate symmetric to it in relation to the mean. Interestingly, the QCR bound associated to that state corresponds to the well-known Heisenberg limit in quantum metrology [18]. This shows that, for the situations considered in this paper, the states that lead to the Heisenberg limit saturate the QIB via projective measurements which do not depend on the true value of the parameter.

The paper is organized as follows. In Sec. II we reformulate the conditions for the saturation of the QIB, first settled in Ref. [2], in a way appropriate to treat the one-parameter quantum state families in (3). Next, in Sec. III, we find the solutions for these conditions that give the structure of all the initial states and all the projective measurements that allow the saturation of the QIB without the knowledge of the true value of the parameter. In Sec. IV, we applied our results in two contexts: phase estimation in a two-path interferometry using the Schwinger representation and phase estimation with one bosonic mode. Section V is devoted to showing that our solutions for the saturation of the QIB include the initial states whose quantum Fisher information correspond to the so-called Heisenberg limit and showing that these are the initial states that allow the maximum retrieval of information about the parameter, among all initial states that saturate the QIB. Finally, we give in Sec. VI a summary of our results.

II. CONDITION FOR GLOBAL SATURATION OF THE QIB IN PURE STATE MODELS

Let us begin with an arbitrary quantum-state family and consider the set of inequalities, first established in Ref. [2], that the Fisher information associated with a POVM $\{\hat{E}_j\}$ must satisfy:

$$\begin{aligned}\mathcal{F}(x_v,\{\hat{E}_j\}) &= \sum_j \frac{1}{\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j]}\left(\mathrm{Tr}\left[\frac{d\hat{\rho}(x)}{dx}\bigg|_{x=x_v}\hat{E}_j\right]\right)^2\\ &= \sum_j \frac{(\mathrm{Re}\{\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j\hat{L}^\dagger(x_v)]\})^2}{\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j]} \qquad (5a)\\ &\leqslant \sum_j \frac{|\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j\hat{L}^\dagger(x_v)]|^2}{\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j]}\\ &= \sum_j \left|\mathrm{Tr}\left[\left(\frac{\hat{\rho}^{1/2}(x_v)\hat{E}_j^{1/2}}{\{\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j]\}^{1/2}}\right)\right.\right.\\ &\qquad \left.\left.\times\left[\hat{E}_j^{1/2}\hat{L}^\dagger(x_v)\hat{\rho}^{1/2}(x_v)\right]\right]\right|^2 \qquad (5b)\\ &\leqslant \mathrm{Tr}[\hat{\rho}(x_v)\hat{L}(x_v)\hat{L}^\dagger(x_v)]\\ &= \mathrm{Tr}[\hat{\rho}(x_v)\hat{L}^2(x_v)] \equiv \mathcal{F}_Q(x_v), \qquad (5c)\end{aligned}$$

where in Eq. (5a) we used the Sylvester equation (2), in Eq. (5b) the inequality $\mathrm{Re}^2(z) \leqslant |z|^2$, and in Eq. (5c) the Cauchy-Schwarz inequality $|\mathrm{Tr}[\hat{A}\hat{B}^\dagger]|^2 \leqslant \mathrm{Tr}[\hat{A}\hat{A}^\dagger]\mathrm{Tr}[\hat{B}\hat{B}^\dagger]$ and the fact that $\hat{L}(x_v)$ is an Hermitian operator. The necessary and sufficient conditions for the saturation of the QIB given in (5) can be condensed into the requirement that the quantities

$$\lambda_j(x_v) = \frac{\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j\hat{L}(x_v)]}{\mathrm{Tr}[\hat{\rho}(x_v)\hat{E}_j]} \qquad (6)$$

be real numbers for all values of j and possible values of x_v.

Let us restrict our attention to the pure quantum state family given in (3), where the generator $\hat{A}$ of the unitary transformation has a discrete spectrum. In that case, if the system is initially in the state $|\phi_+\rangle$, after the unitary transformation it will be in the state

$$|\phi_+(x_v)\rangle = e^{-i\{\hat{A}-\langle\hat{A}\rangle_+\}x_v}\,|\phi_+\rangle\,, \qquad (7)$$

where x_v is the true value of the parameter to be estimated and the phase $e^{-ix_v\langle\hat{A}\rangle_+}$ guarantees that the QFI is just the variance of the generator $\hat{A}$ in the initial state $|\phi_+\rangle$ [see Eq. (4)]. We consider now projective quantum measurements on the system,

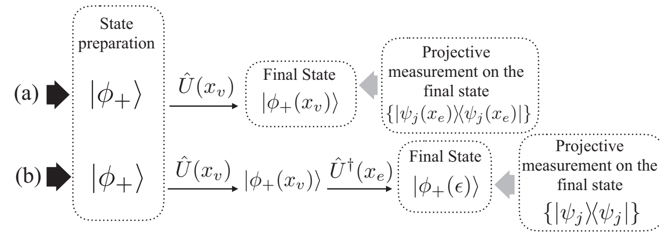

FIG. 1. (a) Quantum estimation process of the parameter x_v corresponding to the laboratory's setup. In this case, the projective measurement on the final state depends on an estimated value x_e for the parameter, where $|\psi_j(x_e)\rangle$ is given in Eq. (8). (b) Equivalent quantum estimation process appropriate for theoretical analysis. In this case, the parameter to be estimated is $\epsilon \equiv x_v - x_e$, which is imprinted on the final state given in Eq. (10), and the projective measurement on that state does not depend on ϵ.

described by the projectors

$$\begin{aligned}\hat{E}_j(x_e) &= |\psi_j(x_e)\rangle\langle\psi_j(x_e)|\\ &= e^{-i\{\hat{A}-\langle\hat{A}\rangle_+\}x_e}|\psi_j\rangle\langle\psi_j|e^{i\{\hat{A}-\langle\hat{A}\rangle_+\}x_e}, \qquad (8)\end{aligned}$$

which may depend on a guess x_e at the true value of the parameter, based, for example, on some prior information about that value. Here, $\{|\psi_j\rangle\}$ is a countable basis of the Hilbert space of the system. The probability of getting the result j in the projective measurement $\{|\psi_j(x_e)\rangle\langle\psi_j(x_e)|\}$ can then be written as

$$\begin{aligned}p_j(x_e,x_v) &= \mathrm{Tr}[|\phi_+(x_v)\rangle\langle\phi_+(x_v)|\hat{E}_j(x_e)]\\ &= \mathrm{Tr}[|\phi_+(\epsilon)\rangle\langle\phi_+(\epsilon)|\,|\psi_j\rangle\langle\psi_j|] = p_j(\epsilon),\end{aligned}$$

where we define

$$\epsilon = x_v - x_e. \qquad (9)$$

Notice that $p_j(\epsilon)$ corresponds equivalently to the probability of getting the result j in the projective measurement $\{|\psi_j\rangle\langle\psi_j|\}$ on the final state

$$|\phi_+(\epsilon)\rangle = \hat{U}(\epsilon)\,|\phi_+\rangle = e^{-i\{\hat{A}-\langle\hat{A}\rangle_+\}\epsilon}\,|\phi_+\rangle\,. \qquad (10)$$

The relation between the Fisher information associated to the measurement $\{|\psi_j(x_e)\rangle\langle\psi_j(x_e)|\}$ on the state $|\phi_+(x_v)\rangle$ and the Fisher information associated to the measurement $\{|\psi_j\rangle\langle\psi_j|\}$ on the state $|\phi_+(\epsilon)\rangle$ is

$$\mathcal{F}(x_v,\{|\psi_j(x_e)\rangle\}) = \mathcal{F}(\epsilon,\{|\psi_j\rangle\}) \equiv \mathcal{F}(\epsilon),$$

where we use $\partial p_j(x_e,x)/\partial x|_{x=x_v} = \mathrm{d}p_j(\epsilon')/\mathrm{d}\epsilon'|_{\epsilon'=\epsilon}$. Therefore, the estimation of the true value x_v of the parameter x in the pure state family given in Eq. (7) via the projective measurement $\{|\psi_j(x_e)\rangle\langle\psi_j(x_e)|\}$, which depends on the estimated value x_e, is equivalent to the estimation of the parameter ϵ in the pure state family given in Eq. (10) via the projective measurement $\{|\psi_j\rangle\langle\psi_j|\}$, which does not depend on the values x_e and x_v (see Fig. 1).

The Sylvester equation that define the SLD operator associated with the states $|\phi_+(\epsilon)\rangle$ can be written as an algebraic equation between operators:

$$\begin{aligned}\hat{L}_0' &= 2\hat{U}^\dagger(\epsilon)\frac{\mathrm{d}\hat{\rho}(\epsilon')}{\mathrm{d}\epsilon'}\bigg|_{\epsilon'=\epsilon}\hat{U}(\epsilon)\\ &= \hat{\rho}_0\hat{L}_0(\epsilon) + \hat{L}_0(\epsilon)\hat{\rho}_0,\end{aligned} \tag{11}$$

where $\hat{\rho}_0 \equiv |\phi_+\rangle\langle\phi_+|$, and

$$\begin{aligned}\hat{L}_0' &\equiv 2i[\hat{\rho}_0,(\hat{A} - \langle\hat{A}\rangle_+)],\\ \hat{L}_0(\epsilon) &\equiv \hat{U}^\dagger(\epsilon)\hat{L}(\epsilon)\hat{U}(\epsilon).\end{aligned}$$

Given an initial state $\hat{\rho}_0$, the structure of the infinite solutions $\hat{L}_0(\epsilon)$ of Eq. (11) can be better displayed if one defines the auxiliary state

$$|\phi_-\rangle \equiv \frac{-2i}{\sqrt{\mathcal{F}_Q}}(\hat{A} - \langle\hat{A}\rangle_+)\,|\phi_+\rangle\,, \tag{12}$$

orthogonal to the initial state $|\phi_+\rangle$. In this case, one can rewrite the operator $\hat{L}_0'$ as

$$\hat{L}_0' = \sqrt{\mathcal{F}_Q}(|\phi_+\rangle\,\langle\phi_-| + |\phi_-\rangle\,\langle\phi_+|),$$

with $\mathcal{F}_Q = 4\langle(\Delta\hat{A})^2\rangle_+$. Let us introduce now a countable basis $\{|\phi_k\rangle\}$ of the Hilbert space of the system, with $|\phi_1\rangle = |\phi_+\rangle$ and $|\phi_2\rangle = |\phi_-\rangle$. In this basis, all the solutions $\hat{L}_0(\epsilon)$ have the matrix structure

$$\begin{array}{c} \\ \langle\phi_+| \\ \langle\phi_-| \\ \langle\phi_3| \\ \vdots \\ \langle\phi_k| \\ \vdots \end{array}
\begin{array}{c} \begin{array}{cccccc} |\phi_+\rangle & |\phi_-\rangle & |\phi_3\rangle & \cdots & |\phi_k\rangle & \cdots \end{array} \\
\left(\begin{array}{c|ccccc} 0 & \sqrt{\mathcal{F}_Q} & 0 & \cdots & 0 & \cdots \\ \hline \sqrt{\mathcal{F}_Q} & & & & & \\ 0 & & & & & \\ \vdots & & & \mathbb{L}(\epsilon) & & \\ 0 & & & & & \\ \vdots & & & & & \end{array}\right)\end{array}, \tag{13}$$

where $\mathbb{L}(\epsilon)$ is an arbitrary Hermitian matrix. When the matrix $\mathbb{L}(\epsilon)$ is the null matrix, we recover the particular solution $\hat{L}_0'$. We stress that Eq. (11) has an infinite number of solutions even if $\epsilon = 0$ (i.e., when the guess value x_e coincides with the true value x_v) because $\mathbb{L}(0)$ is not necessarily the null matrix.

We are now able to rewrite the saturation conditions of the QIB in Eq. (6) for our pure quantum state family models as the requirement that

$$\begin{aligned}\lambda_j(\epsilon) &= \frac{\mathrm{Tr}[\hat{\rho}(\epsilon)\,|\psi_j\rangle\langle\psi_j|\,\hat{L}(\epsilon)]}{\mathrm{Tr}[\hat{\rho}(\epsilon)\,|\psi_j\rangle\langle\psi_j|]} &\text{(14a)}\\ &= \frac{\langle\psi_j|\hat{U}(\epsilon)\hat{L}_0(\epsilon)|\phi_+\rangle}{\langle\psi_j|\phi_+(\epsilon)\rangle} &\text{(14b)}\\ &= \sqrt{\mathcal{F}_Q}\frac{\langle\psi_j|\,\phi_-(\epsilon)\rangle}{\langle\psi_j|\,\phi_+(\epsilon)\rangle} &\text{(14c)}\end{aligned}$$

be real numbers. Here we define

$$|\phi_-(\epsilon)\rangle = \hat{U}(\epsilon)\,|\phi_-\rangle = e^{-i\{\hat{A}-\langle\hat{A}\rangle_+\}\epsilon}\,|\phi_-\rangle\,. \tag{15}$$

In Eq. (14b), we used the fact that, according to Eq. (13), all SLD operators $\hat{L}_0(\epsilon)$ verify $\hat{L}_0(\epsilon)\,|\phi_+\rangle = \hat{L}_0'\,|\phi_+\rangle = \sqrt{F_Q}\,|\phi_-\rangle$ for all values of ϵ.

From Eq. (14b) one can see that, when $\epsilon = 0$ ($x_e = x_v$), if the states $|\psi_j\rangle$ are eigenstates of $\hat{L}_0(0)$, then the conditions in Eqs. (14) are automatically satisfied for all values of j and we recover in our formalism the conditions for the saturation of the QIB first stated in Ref. [2]. We are, however, interested in finding the conditions for global saturation of the QIB, which correspond to all the initial states $|\phi_+\rangle$ and all the projective measurements $\{|\psi_j\rangle\langle\psi_j|\}$ that allow the saturation of the QIB for all values of ϵ. This is equivalent to finding projective measurements on the final state $|\phi_+(x_v)\rangle$ that, regardless of the true value x_v of the parameter, lead to the saturation of the QIB. For this sake, it is convenient to rewrite the inequalities that must be satisfied by the Fisher information $\mathcal{F}(\epsilon)$ as

$$\begin{aligned}\mathcal{F}(\epsilon) &= \mathcal{F}_Q\left(1 - \sum_j \frac{\{\mathrm{Im}[w_j(\epsilon)z_j^*(\epsilon)]\}^2}{p_j(\epsilon)}\right)\\ &\leqslant \mathcal{F}_Q = 4\langle(\Delta\hat{A})^2\rangle_{\phi_+},\end{aligned}$$

where

$$\langle\psi_j\,|\phi_+(\epsilon)\rangle \equiv |z_j(\epsilon)|e^{i\alpha_{j,+}(\epsilon)}, \tag{16a}$$

$$\langle\psi_j|\phi_-(\epsilon)\rangle \equiv w_j(\epsilon), \tag{16b}$$

with $\alpha_{j,+}(\epsilon) = \arg(z_j(\epsilon))$.

This yields conditions which are equivalent to those in Eq. (14) and can be written as

$$\mathrm{Im}[w_j(\epsilon)z_j^*(\epsilon)] = 0. \tag{17}$$

Notice that the relation above must apply for any value of ϵ and for all j. For future use, we rewrite the conditions in Eq. (17) as

$$\begin{aligned}&\sum_{j'\neq j} |v_{j,j'}|\frac{|z_{j'}(\epsilon)|}{|z_j(\epsilon)|}\cos[\alpha_{j',+}(\epsilon) - \alpha_{j,+}(\epsilon) + \phi_{j,j'}]\\ &\quad = \langle\hat{A}\rangle_+ - v_{j,j},\end{aligned} \tag{18}$$

where we define $\langle\psi_j|\,\hat{A}\,|\psi_{j'}\rangle \equiv |v_{j,j'}|e^{i\phi_{j,j'}}$.

III. PROJECTIVE MEASUREMENTS AND STATES FOR A GLOBAL SATURATION OF THE QIB

Any initial state $|\phi_+\rangle$ of the quantum-state family given in Eq. (10) can be written in the basis of eigenstates of the generator $\hat{A}$. In order to find the initial states that allow global saturation of the QIB, we consider states $|\phi_+\rangle$ that are finite linear combinations of the eigenstates $|A_{k_l}\rangle$ of $\hat{A}$:

$$|\phi_+\rangle = \sum_{l=1}^{M}|c_{k_l}|\,e^{i\theta_{k_l}}\,|A_{k_l}\rangle\,, \tag{19}$$

with $\theta_{k_l} = \arg(c_{k_l})$ and $c_{k_l} \neq 0$. The set of integers $\{k_l\}_{l=1,\dots,M}$ with $k_1 < k_2 < \cdots < k_M$ are the labels of the eigenstates that define the subspace $\{|A_{k_l}\rangle\}_{l=1,\dots,M}$ of the Hilbert space of the system. When the spectrum of $\hat{A}$ is unbounded, the initial states $|\phi_+\rangle$ may have an infinite number of terms in an expansion like in Eq. (19). In this case, as it will be shown, depending on the class of projective measurements one uses, one either takes the limit $M \to \infty$ or has to consider instead approximated states, which correspond to a truncation up to sufficiently large M terms in Eq. (19). It is also important to notice that the evolved state $|\phi_+(\epsilon)\rangle$ remains in the subspace $\{|A_{k_l}\rangle\}_{l=1,\dots,M}$ for all values of ϵ. We also assume that the mean value $\langle\hat{A}\rangle_+$ is a predetermined fixed quantity and therefore all the considered initial states have to satisfy this constraint.

A global saturation of the QIB for a initial state $|\phi_+\rangle$ and a projective measurement $\{|\psi_j\rangle\langle\psi_j|\}$ means that

$$\mathcal{F}(\epsilon) = \sum_j p_j(\epsilon)\lambda_j^2(\epsilon) \tag{20a}$$

$$= \mathcal{F}_Q \sum_j \langle\phi_+(\epsilon)|\psi_j\rangle \frac{\langle\psi_j|\phi_-(\epsilon)\rangle^2}{\langle\psi_j|\phi_+(\epsilon)\rangle} \tag{20b}$$

$$= \mathcal{F}_Q \langle\phi_-(\epsilon)|\left(\sum_j |\psi_j\rangle\langle\psi_j|\right)|\phi_-(\epsilon)\rangle \tag{20c}$$

$$= \mathcal{F}_Q, \tag{20d}$$

for all values of ϵ. From Eqs. (20a) to (20b) we use the definition of $\lambda_j(\epsilon)$ given in Eq. (14) and from (20b) to (20c) we use that $\lambda_j^*(\epsilon) = \lambda_j(\epsilon)$. Therefore, the last equality in Eqs. (20) holds only if projectors $\{|\psi_j\rangle\langle\psi_j|\}$ span the subspace wherein the evolved state $|\phi_+(\epsilon)\rangle$ lives, i.e.,

$$\hat{\mathbb{1}}_M \equiv \sum_{l=1}^{M} |A_{k_l}\rangle\langle A_{k_l}| = \sum_{j=1}^{M} |\psi_j\rangle\langle\psi_j|, \tag{21}$$

where we used the fact that the projectors $\{|\psi_j\rangle\langle\psi_j|\}$ are linearly independent. For this reason, one can write

$$|\psi_j\rangle = \sum_{l=1}^{M} |b_{j,k_l}|\, e^{i\theta_{j,k_l}}\ |A_{k_l}\rangle, \tag{22}$$

where $\theta_{j,k_l} = \arg(b_{j,k_l})$.

Now, using the expansions in Eqs. (19) and (22), and the definition of the state $|\phi_-\rangle$ in (12), we arrive to

$$z_j(\epsilon) = \sum_{l=1}^{M} |c_{k_l}|\,|b_{j,k_l}| e^{i\{-(A_{k_l} - \langle\hat{A}\rangle_{\phi_+})\epsilon - \theta_{j,k_l} + \theta_{k_l}\}}, \tag{23a}$$

$$w_j(\epsilon) = \frac{-2i}{\sqrt{\mathcal{F}_Q}} \sum_{l=1}^{M} |c_{k_l}|\,|b_{j,k_l}|\,(A_{k_l} - \langle\hat{A}\rangle) \times e^{i\{-(A_{k_l} - \langle\hat{A}\rangle_{\phi_+})\epsilon - \theta_{j,k_l} + \theta_{k_l}\}}. \tag{23b}$$

In order to obtain the structure of the initials states $|\phi_+\rangle$ and the projective measurements $\{|\psi_j\rangle\langle\psi_j|\}_{j=1,\dots,M}$ that allow for a global saturation of the QIB, we substitute Eqs. (23) in Eqs. (17) and analyze which are the conditions that the sets $\{A_{k_l}\}$, $\{c_{k_l}\}$, and $\{b_{j,k_l}\}$ with $j,l = 1,\dots,M$ must satisfy in order to be solutions of these equations. This is done in the following section.

A. Structure of the initial states and the projective measurements

In Appendix A, we show that if the set of eigenstates $\{|A_{k_l}\rangle\}_{l=1,\dots,M}$ present in the decomposition of $|\phi_+\rangle$ does not contain two eigenstates $|A_{k_l}\rangle$ corresponding to the same eigenvalue of $\hat{A}$, Eqs. (17) are satisfied if and only if the sets $\{A_{k_l}\}$, $\{c_{k_l}\}$ and $\{b_{j,k_l}\}$ with $j,l = 1,\dots,M$, verify the conditions

$$A_{k_l} - \langle\hat{A}\rangle_+ = -(A_{k_{\delta(l)}} - \langle\hat{A}\rangle_+), \tag{24a}$$

$$|c_{k_l}||b_{j,k_l}| = |c_{k_{\delta(l)}}||b_{j,k_{\delta(l)}}|, \tag{24b}$$

$$(\theta_{k_{\delta(l)}} - \theta_{j,k_{\delta(l)}}) + (\theta_{k_l} - \theta_{j,k_l}) = \xi_j, \tag{24c}$$

where ξ_j are arbitrary real numbers. When $\xi_j = n_j\pi$, where n_j is an even integer, the solutions correspond to real wave functions $z_j(\epsilon) = \langle\psi_j|\phi_+(\epsilon)\rangle$ and $w_j(\epsilon) = \langle\psi_j|\phi_-(\epsilon)\rangle$, and when n_j is odd, the solutions correspond to pure imaginary wave functions. Here, $\delta(l) \equiv M - (l-1)$, for $l = 1,2,\dots,\lceil M/2\rceil$, where $\lceil\cdots\rceil$ is the ceiling function. It is interesting to note that when $M = 2$, Eq. (24c) does not constitute a restriction on the two phases, θ_{k_1} and θ_{k_2}, which appear in the expansion of the initial state $|\phi_+\rangle$ in Eq. (19). In this case, using only the conditions in Eqs. (24a) and (24b), one can show that $w_j(\epsilon)z_j^*(\epsilon)$ is given by

$$w_j(\epsilon)z_j^*(\epsilon) = 4|c_{k_1}|^2|b_{k_2}|^2(A_{k_1} - \langle\hat{A}\rangle_+) \times \sin\left(2(A_{k_1} - \langle\hat{A}\rangle_+)\epsilon + \theta_{k_2} - \theta_{k_1} + \theta_{j,k_1} - \theta_{j,k_2}\right), \tag{25}$$

and it is always real. Therefore, the condition of the saturation of the QIB in Eq. (17) is fulfilled independently of the values of the phases θ_{k_1} and θ_{k_2}. Notice also that $w_j(\epsilon)z_j^*(\epsilon)$, in Eq. (25) [$j = 1,2$], is real independently of the values θ_{j,k_1} and θ_{j,k_2} of the phases that appear in the expansion of the states $|\psi_j\rangle$ of the projective measurement basis in Eq. (22). This means, in particular, that if an initial state $|\phi_+\rangle$ saturates the QIB with a projective measurement basis $\{|\psi_j\rangle\}_{j=1,2}$, then it also saturates the QIB with any projective measurement basis $\{|\tilde{\psi}_j\rangle = e^{ih(\hat{A})}|\psi_j\rangle\}_{j=1,2}$, where $h(\hat{A})$ is real function of the operator $\hat{A}$.

When $M > 2$, Eq. (24c) fixes the relations between the phases θ_{k_l} and $\theta_{k_{\delta(l)}}$, of the initial state $|\phi_+\rangle$, and the phases θ_{j,k_l} and $\theta_{j,k_{\delta(l)}}$, of the states $|\psi_j\rangle$ of the projective measurement basis, that indeed are crucial for the saturation of the QIB.

If there are some eigenstates $|A_{k_l}\rangle$ in the decomposition of $|\phi_+\rangle$ corresponding to the same eigenvalue of $\hat{A}$, then Eqs. (24) are only sufficient conditions to get equality in Eqs. (17). However, in this case we cannot guarantee that they are also necessary conditions.

Inserting the conditions given in Eqs. (24) into the expression for $z_j(\epsilon)$, given in Eq. (23a), one gets

$$z_j(\epsilon) = e^{i\left(\frac{\xi_j}{2} + s_j(\epsilon)\pi\right)}|\eta_j^{(M)}(\epsilon)|, \tag{26}$$

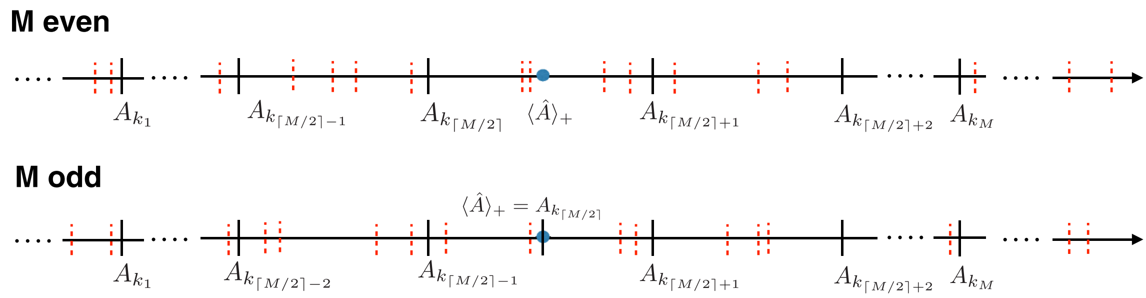

FIG. 2. The vertical dashed (red) and full (black) lines represent the position of the eigenvalues of the Hermitian generator $\hat{A}$ along the real line. The blue dot indicates the location of the fixed mean value $\langle\hat{A}\rangle_+$. The full (black) vertical lines correspond to the subset of eigenvalues $\{A_{k_l}\}_{l=1,\dots,M}$, whose corresponding eigenstates were used to construct the initial state $|\phi_+\rangle$ and, therefore, verify the symmetry in Eq. (24a). The dashed (red) vertical lines correspond to the rest of the spectrum of $\hat{A}$ that do not enter in the construction of the initial state $|\phi_+\rangle$.

with the integer $s_j(\epsilon)$ defined as $e^{is_j(\epsilon)\pi} = \text{sgn}(\eta_j^{(M)}(\epsilon))$ and where we also define

$$\eta_j^{(M)}(\epsilon) = \begin{cases} 2\sum_{l=1}^{\lceil M/2\rceil} |c_{k_l}||b_{j,k_l}|\cos\left((A_{k_l} - \langle\hat{A}\rangle_+)\epsilon + (\theta_{k_l} - \theta_{j,k_l})\right), & \text{for even M,} \\ 2\sum_{l=1}^{\lceil M/2\rceil-1} |c_{k_l}||b_{j,k_l}|\cos\left((A_{k_l} - \langle\hat{A}\rangle_+)\epsilon + (\theta_{k_l} - \theta_{j,k_l})\right) + |c_{k_{\lceil M/2\rceil}}||b_{j,k_{\lceil M/2\rceil}}| & \text{for odd M.} \end{cases} \tag{27}$$

Therefore, the phase of the wave function $z_j(\epsilon) = \langle\psi_j|\phi_+(\epsilon)\rangle$ is

$$\alpha_{j,+}(\epsilon) = \arg(z_j(\epsilon)) = \left(\frac{\xi_j}{2} + s_j(\epsilon)\pi\right), \tag{28}$$

for all values of ϵ.

B. Interpretations of the conditions for a global saturation of the QIB

The condition given in Eq. (24a) establishes the symmetry that the subsets of eigenvalues $\{A_{k_l}\}_{l=1,\dots,M}$ of the generator $\hat{A}$, whose respective eigenstates enter in the decomposition of the initial state $|\phi_+\rangle$, must exhibit. This symmetry is sketched in Fig. 2. It requires that, given a fixed value for $\langle\hat{A}\rangle_+$, the expansion of the initial state $|\phi_+\rangle$ in the eigenbasis of $\hat{A}$ contains $\lceil M/2\rceil$ (M even) or $\lceil M/2\rceil - 1$ (M odd) pairs of eigenstates of $\hat{A}$, each pair corresponding to symmetric eigenvalues, A_{k_l} and $A_{k_{\delta(l)}}$, with respect to the mean $\langle\hat{A}\rangle_+$. Notice that for an arbitrary generator $\hat{A}$, such a expansion with $M > 2$ may not exist. This is not the case if the spectrum of $\hat{A}$ is equally spaced. On the other hand, it is always possible to find initial states $|\phi_+\rangle$ whose expansion in the eigenbasis of $\hat{A}$ contains $M = 2$ eigenstates and that satisfy condition (24a) for arbitrary generators $\hat{A}$.

If we now use in Eq. (24b) the orthonormality of the measurement basis vectors $\{|\psi_j\rangle\}$

$$\sum_{j=1}^{M} b_{j,k_l} b^*_{j,k_{l'}} = \delta_{ll'}, \tag{29}$$

we get conditions for the moduli of the expansion coefficients of the initial state $|\phi_+\rangle$ and of the states $|\psi_j\rangle$ of the measurement basis in terms of the eigenstates of the generator $\hat{A}$:

$$|c_{k_l}| = |c_{k_{\delta(l)}}|, \ \ (l = 1,\dots,M) \tag{30}$$

and

$$|b_{j,k_l}| = |b_{j,k_{\delta(l)}}|, \ \ (j,l = 1,\dots,M), \tag{31}$$

respectively. These conditions imply that the eigenstates $|A_{k_l}\rangle$ and $|A_{k_{\delta(l)}}\rangle$ appear with equal weights in the expansion of the initial state and of the measurement basis states in terms of the eigenbasis of $\hat{A}$. They also imply that, in order to allow a saturation of the QIB for all values of ϵ, both the initial state $|\phi_+\rangle$ and the states $|\psi_j\rangle$ of the projective measurement basis must have zero *skewness* relative to the operator $\hat{A}$. For example, it is straightforward to verify that, for the initial state $\hat{\rho}_0 \equiv |\phi_+\rangle\langle\phi_+|$, the condition in Eq. (30) leads to

$$S \equiv \text{Tr}\left[\hat{\rho}_0\left(\frac{\hat{A} - \langle\hat{A}\rangle_+}{\sqrt{\langle\Delta^2\hat{A}\rangle}}\right)^3\right] = \frac{\langle\hat{A}^3\rangle - \langle\hat{A}\rangle^3 - 3\langle\hat{A}\rangle\langle\Delta^2\hat{A}\rangle}{(\langle\Delta^2\hat{A}\rangle)^{3/2}} = 0, \tag{32}$$

where S is the *skewness* of the state $\hat{\rho}_0$ relative to the generator $\hat{A}$.

Using Eq. (24a), we see that

$$\langle\hat{A}\rangle_+ = \frac{A_{k_l} + A_{k_{\delta(l)}}}{2}, \ \ \text{for all } l,\dots,M. \tag{33}$$

It is easy to check that, for all the initial states $|\phi_+\rangle$ in Eq. (19) with A_{k_l} verifying the symmetry in Eq. (24a) and also the balance condition in Eq. (30), the mean value of the generator

$\hat{A}$ coincides with the prefixed value $\langle\hat{A}\rangle_+$:

$$\langle\phi_+|\,\hat{A}\,|\phi_+\rangle \equiv \sum_{l=1}^{M} |c_{k_l}|^2\; A_{k_l} = \left\{\begin{array}{ll} \sum_{l=1}^{\lceil M/2\rceil-1} 2\,|c_{k_l}|^2\;\frac{A_{k_l}+A_{k_{\delta(l)}}}{2} + |c_{\lceil M/2\rceil}|^2 A_{k_{\lceil M/2\rceil}} & \text{,if M is odd} \\ \sum_{l=1}^{M/2} 2\,|c_{k_l}|^2\;\frac{A_{k_l}+A_{k_{\delta(l)}}}{2} & \text{,if M is even} \end{array}\right\} = \langle\hat{A}\rangle_{\phi_+}, \tag{34}$$

where we used Eq. (33), the normalization condition, $\sum_{l=1}^{M}|c_{k_l}|^2 = 1$, of the state $|\phi_+\rangle$, and, if M is odd, that $\langle\hat{A}\rangle_+ = A_{\lceil M/2\rceil}$.

We can also check that all the states $\{|\psi_j\rangle\}$ of a projective measurement basis that satisfy the balance condition in Eq. (31) and the condition on the phases in Eq. (24c) satisfy the conditions for the a global saturation of the QIB, given in Eq. (18). Indeed, using (21), (24c), and (31) we get, for $j \neq j'$,

$$\begin{aligned} |v_{j,j'}|e^{i\phi_{j,j'}} &\equiv \langle\psi_j|\,\hat{A}\,|\psi_{j'}\rangle = \langle\psi_j|\,(\hat{A}-\langle\hat{A}\rangle_+)\,|\psi_{j'}\rangle \\ &= 2\,e^{i\left(\frac{\xi_j-\xi_{j'}}{2}+\frac{\pi}{2}\right)}\sum_{l=1}^{\lceil M/2\rceil}|b_{j,k_l}||b_{j',k_l}| \\ &\quad\times(A_{k_l}-\langle\hat{A}\rangle)\sin\left(\theta_{j',k_l}-\theta_{j,k_l}-(\xi_j-\xi_{j'})/2\right) \\ &\equiv 2\,e^{i\left(\frac{\xi_j-\xi_{j'}}{2}+\frac{\pi}{2}+s'_{j,j'}\pi\right)}|\eta'_{j,j'}|, \end{aligned} \tag{35}$$

with $e^{is'_{j,j'}\pi} = \text{sgn}(\eta'_{j,j'})$, so that

$$\phi_{j,j'} = \pi/2 + (\xi_j-\xi_{j'})/2 + s'_{j,j'}\pi. \tag{36}$$

Gathering together the results in Eqs. (28) and (36), we obtain

$$\alpha_{j',+}(\epsilon) - \alpha_{j,+}(\epsilon) + \phi_{j,j'} = \frac{\pi}{2} + [s_{j'}(\epsilon) - s_j(\epsilon) + s'_{j,j'}]\pi.$$

If we insert the above relation into the saturation condition of Eq. (18), the left-hand side of that equation turns equal to zero. On the other hand, in an analogous way to that used in Eq. (34), we can show that

$$\langle\psi_j|\,\hat{A}\,|\psi_j\rangle = \langle\hat{A}\rangle_+ \quad \text{for} \quad j = 1,\ldots,M. \tag{37}$$

Therefore, the right-hand side of Eq. (18) is also null by virtue of the balance condition on the coefficients in Eq. (31) and the symmetry of the spectrum $\{A_{k_l}\}_{l=1,\ldots,M}$, given in (24a).

C. Projective measurements for a global saturation of the QIB

In the previous section, we have shown that the states of a projective measurement basis $\{|\psi_j\rangle\}_{j=1,\ldots,M}$ that leads to a global saturation of the QIB must have a balanced decomposition in terms of the subset $\{|A_{k_l}\rangle\}_{l=1,\ldots,M}$ of eigenstates of the generator $\hat{A}$. That is, the coefficients $b_{j,k_l} = \langle A_{k_l}|\psi_j\rangle$ of the decomposition must verify the conditions in (31) and (24c). However, the orthonormality of the measurement basis states $|\psi_j\rangle$ places supplementary conditions on the coefficients b_{j,k_l}. In what follows, we will show two examples of families of projective measurements that fulfill all the requirements for allowing a global saturation of the QIB.

1. *First family of projective measurements*

We arrive at the first family of projective measurements when, based on Eq. (37), we investigate the structure of the measurement basis $\{|\psi_j\rangle\}$ that satisfies the condition $\langle\psi_j|\,\hat{A}\,|\psi_j\rangle = \alpha$ $(j = 1,\ldots,M)$, where the constant α does not depend on the value of j and is not necessarily equal to $\langle\hat{A}\rangle_+$. In Appendix B we show that one solution to this condition corresponds to a decomposition of the states $|\psi_j\rangle$ in terms of the eigenstates $\{|A_{k_l}\rangle\}_{l=1,\ldots,M}$ with coefficients

$$b_{j,k_l} = \frac{1}{\sqrt{M}}e^{i\theta_{j,k_l}}, \tag{38}$$

where the phases are

$$\theta_{j,k_l} = (j\pi/M)f_l + j\beta/M + \phi_{k_l}, \tag{39}$$

with

$$f_l = \begin{cases} (l-1)+[(-1)^l+1](M-1)/2, & \text{for even M,} \\ (l-1)(1-M), & \text{for odd M,} \end{cases} \tag{40}$$

and β and ϕ_{k_l} arbitrary real numbers.

Now, when $M > 2$, using Eq. (39) in Eq. (24c), we get for the phases of the initial state $|\phi_+\rangle$

$$\begin{aligned}&\theta_{k_l}+\theta_{k_{\delta(l)}} \\ &\quad= \begin{cases} \frac{j}{M}[2\pi(M-1)+2\beta]+\xi_j+\phi_{k_l}+\phi_{k_{\delta(l)}}, & \text{M even,} \\ \frac{j}{M}[-\pi(M-1)^2+2\beta]+\xi_j+\phi_{k_l}+\phi_{k_{\delta(l)}}, & \text{M odd,} \end{cases}\end{aligned}$$

with $\delta(l) = M-(l-1)$. If we choose in Eq. (41) $\beta = -\pi(M-1)$, if M is even, or $\beta = \pi(M-1)^2/2$, if M is odd, then we can choose $\xi_j = 0\,[j = 1,\ldots,M]$, to get

$$\theta_{k_l}+\theta_{k_{\delta(l)}} = \phi_{k_l}+\phi_{k_{\delta(l)}}. \tag{41}$$

Notice that the phases ϕ_{k_l} can always be interpreted as the result of the mapping $|\psi_j\rangle \equiv e^{ih(\hat{A})}|\tilde{\psi}_j\rangle$, with h being a real function, where $\phi_{k_l} = h(A_{k_l})$ and the states $|\tilde{\psi}_j\rangle$ of the projective measurement basis have the coefficients $\tilde{b}_{j,k_l} \equiv \langle\tilde{\psi}_j|\phi_+\rangle$, given in Eq. (38), with the phases $\tilde{\theta}_{j,k_l} = (j\pi/M)f_l + j\beta/M$. Therefore, once we arbitrarily fix the phases θ_{k_l} of the initial state $|\phi_+\rangle$, the states $|\psi_j\rangle$ of the projective measurement basis must have the phases θ_{j,k_l} given in Eq. (39), with $\phi_{k_l} = h(A_{k_l})$ for any real function h. This shows that the phases θ_{k_l} can be chosen arbitrarily, since the phases ϕ_{k_l} are arbitrary. Furthermore, we see that, for this example of projective measurement, there are no conditions on the real numbers ξ_j, so they can be chosen equal to zero.

The family of projective measurements defined in Eq.(38) and Eq.(39) verify the balance condition in Eq. (31) regardless of the subset $\{\hat{A}_{k_l}\}_{l=1,\ldots,M}$ of eigenstates of $\hat{A}$ present in the decomposition of the initial state $|\phi_+\rangle$. However, Eq. (31) and the symmetry imposed by Eq. (33) on the eigenvalues $\{A_{k_l}\}_{l=1,\ldots,M}$ guarantee that $\langle\psi_j|\,\hat{A}\,|\psi_j\rangle = \langle\hat{A}\rangle_+$ for $j = 1,\ldots,M$.

2. *Second type of projective measurements*

The second example of a projective measurement basis $\{|\psi_j\rangle\}_{j=1,\ldots,M}$ that allows a global saturation of the QIB is the one whose coefficients b_{j,k_l} are given by

$$b_{j,k_l} \equiv \langle A_{k_l}|\psi_j\rangle = \sqrt{\frac{(M-l)!(l-1)!}{(j-1)!(M-j)!}}\left(\frac{e^{i\vartheta}}{2}\right)^{l-\frac{M+1}{2}} \times P_{M-l}^{(l-j,l+j-(M+1))}(0), \quad (42)$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials [19], and ϑ is an arbitrary real number. These coefficients can be connected to the matrix elements

$$d^{\mathrm{j}}_{m_z',m_z}(\pi/2) \equiv \langle \mathrm{j},m_z'|e^{i\frac{\pi}{2\hbar}\hat{J}_y}|\mathrm{j},m_z\rangle \quad (43)$$

in the theory of angular momentum [19], where $|\mathrm{j},m_z\rangle$ are eigenstates of the component $\hat{J}_z$ of the angular momentum operator $\hat{\boldsymbol{J}}$, if the respective indexes are identified as $M = 2\mathrm{j}+1$, $l = m_z' + (M+1)/2$ and $j = m_z + (M+1)/2$. The condition $1 \leqslant l,j \leqslant M$ corresponds here to the constraint $-\mathrm{j} \leqslant m_z',m_z \leqslant \mathrm{j}$. Notice that even values of M correspond to half-integrals values of j, while odd values of M correspond to integral values. More specifically, this mapping of indexes leads to the correspondence

$$b_{j,k_l} \to e^{i(m_z'\vartheta)} d^{\mathrm{j}}_{m_z',m_z}(\pi/2). \quad (44)$$

Using the properties of the matrix elements $d^{\mathrm{j}}_{m_z',m_z}(\beta)$ [19], it is easy to show that $|d^{\mathrm{j}}_{m_z',m_z}(\pi/2)| = |d^{\mathrm{j}}_{-m_z',m_z}(\pi/2)|$, which is exactly the balance condition $|b_{j,k_l}| = |b_{j,k_{\delta(l)}}|$, with $\delta(l) = M-(l-1)$. Since the real numbers $d^{\mathrm{j}}_{m_z',m_z}(\pi/2)$ are elements of an orthogonal matrix (real unitary matrix), the orthonormality of the states $|\psi_j\rangle$ is guaranteed. Because the matrix elements $d^{\mathrm{j}}_{m_z',m_z}(\pi/2)$ are real numbers, we have for the phases of the coefficients b_{j,k_l}

$$\theta_{j,k_l} = [l-(M+1)/2]\vartheta + s''_{l,j,M}\pi, \quad (45)$$

where the integer $s''_{l,j,M}$ is such that $e^{is''_{l,j,M}\pi} = \mathrm{sgn}(P_{M-l}^{(l-j,l+j-(M+1))}(0))$. Now, when $M > 2$, using Eq. (24c), it is easy to see that, in this case, the phases θ_{k_l} of the initial state $|\phi_+\rangle$ must satisfy

$$\theta_{k_l} + \theta_{k_{\delta(l)}} = 0 \quad \mathrm{mod}\ 2\pi, \quad (46a)$$

$$(s''_{l,j,M} + s''_{\delta(l),j,M})\pi + \xi_j = 0 \quad \mathrm{mod}\ 2\pi. \quad (46b)$$

The set of Eqs. (46b) determines the values of ξ_j, mod 2π. This implies that, in contrast to the use of the first family of projective measurements, here the phases θ_{k_l} of the coefficients c_{k_l}, in the decomposition of the initial state $|\phi_+\rangle$ in the eigenbasis of the generator $\hat{A}$ [cf. Eq. (19)], are no longer completely arbitrary.

Notice that the subset $\{A_{k_l}\}_{l=1,\ldots,M}$ of eigenvalues of $\hat{A}$ that obey the symmetry in Eq. (24a) (see also Fig. 2), required for a global saturation of the QIB are not necessarily equally spaced. Thus, the states $|\psi_j\rangle = \sum_{j=1}^{M} b_{j,k_l}|A_{k_l}\rangle$, with the coefficients b_{j,k_l} given in (42), are not necessarily equivalent to eigenstates of an angular momentum operator. However, when the eigenvalues $\{A_{k_l}\}_{l=1,\ldots,M}$ of the operator $\hat{A}$ are equally spaced, the operator $\hat{A}$, restricted to the subspace $\{|A_{k_l}\rangle\}_{l=1,\ldots,M}$, is itself equivalent to an angular momentum operator, and if we use the basis $\{|\psi_j\rangle\}_{j=1,\ldots,M}$ with the coefficient b_{j,k_l} given in (42), then the states $|\psi_j\rangle$ are also eigenstates of an angular momentum operator.

IV. SOME EXAMPLES OF GLOBAL SATURATION OF THE QIB

The case in which the generator $\hat{A}$ is indeed an angular momentum component, let us say $\hat{A} = \hat{J}_z/\hbar$, was studied in Ref. [20] in the context of phase estimation in two-path interferometry, using the Schwinger representation. In this case, the parameter to be estimated, $x_v = \Delta\varphi_v$, is the phase difference between the two paths. Our complete characterization of the structure of the initial states $|\phi_+\rangle$ and the projective measurements $\{|\psi_j\rangle\}$ that lead to a global saturation of the QIB contains the results presented in Ref. [20] as special cases. Indeed, if we use Eq. (30) together with Eq. (46a), we see that the initial states that permit a global saturation of the QIB, for phase estimation in two-path interferometry, satisfy

$$\begin{aligned}\langle \mathrm{j},m_z|\phi_+\rangle &= |\langle \mathrm{j},m_z|\phi_+\rangle|e^{i\theta_{m_z}} \\ &= |\langle \mathrm{j},-m_z|\phi_+\rangle|e^{-i\theta_{-m_z}} \\ &= \langle \mathrm{j},-m_z|\phi_+{}^*\rangle,\end{aligned} \quad (47)$$

with $-\mathrm{j} \leqslant m_z \leqslant \mathrm{j}$, $\theta_{m_z} \equiv \theta_{k_l}$, where the index m_z is connected with $k_l = l$ by a suitable map. Equation (47) is exactly the condition given in Eq. (8) of Ref. [20] for initial states $|\phi_+\rangle$ with a fixed photon number $N = 2\mathrm{j}$. The projective measurement for a global saturation of the QIB in this case is $\{|\psi_j\rangle = |\mathrm{j},m_x\rangle\}$, where $\{|\mathrm{j},m_x\rangle\}$ are eigenstates of the $\hat{J}_x$ component of an angular momentum and the index m_x is connected with j by a suitable map. Notice that this is exactly the projective measurement basis given by the coefficients in Eq. (43), since $e^{i\frac{\pi}{2\hbar}\hat{J}_y}|\mathrm{j},m_z\rangle = |\mathrm{j},m_x\rangle$, and coincides with the projective measurement basis used in Ref. [20]. The number M_T of coefficients $\langle \mathrm{j},m_z\rangle\,\phi_+$ different from zero could be such that $M_T < M = 2\mathrm{j}+1$, the total number of possible values of m_z. However, there is no difference for the saturation of the QIB if we consider the subspace $\{|\mathrm{j},m_z\rangle\}$, with $m_z = l-(M+1)/2$ and $l = 1,\ldots,M$, as the subspace where the initial state $|\phi_+\rangle$ lives. This subspace is equally spanned by the projective measurement $\{|\mathrm{j},m_x\rangle\}$, with $m_x = j-(M+1)/2$ and $j = 1,\ldots,M$.

Our results show that all measurement basis of the family $\{e^{-i\varphi\hat{J}_z/\hbar}|\mathrm{j},m_x\rangle\}$, where φ is an arbitrary phase, lead to the saturation of QIB for the initial states that satisfy Eq. (47). That is,

$$\begin{aligned}\mathcal{F}(\Delta\varphi_v,\{e^{-i\varphi\hat{J}_z/\hbar}|\mathrm{j},m_x\rangle\}) &= \mathcal{F}_Q \equiv 4\langle(\Delta\hat{J}_z)^2\rangle_+ \\ &= 4\big\langle\hat{J}_z^2\big\rangle_+\end{aligned} \quad (48)$$

for all values φ [see Eq. (1)]. Notice that the initial states $|\phi_+\rangle$ that satisfy (47) have $\langle\hat{J}_z\rangle_+ = 0$.

The formalism used here assumes that the spectrum $\{A_{k_l}\}_{l=1,\ldots,M}$ corresponding to the subspace $\{|A_{k_l}\rangle\}_{l=1,\ldots,M}$ where the initial state lives is not degenerate. This is not the case if the initial states has a fluctuating photon number, i.e.,

$$|\phi_+\rangle = \sum_{\mathrm{j}}\sum_{m_z^{(\mathrm{j})}} c_{m_z^{(\mathrm{j})}}\big|\mathrm{j},m_z^{(\mathrm{j})}\big\rangle, \quad (49)$$

with $c_{m_z^{(j)}} \equiv \langle j, m_z^{(j)} | \phi_+ \rangle$. Since $j = N/2$ is no longer fixed, eigenstates $|j, m_z^{(j)}\rangle$ with equal values of $m_z^{(j)}$ but different values of j could enter in the decomposition of $|\phi_+\rangle$. Such states, however, are eigenstates of $\hat{J}_z$ corresponding to the same eigenvalue $\hbar m_z^{(j)}$. Nevertheless, if the state in Eq. (49) verifies the conditions in Eq. (47) for all values of j, then it can be shown that global saturation of the QIB can be reached via the projective measurement basis $\{|\Psi_j\rangle = |j, m_x\rangle\}$, with

$$\sum_{j} \sum_{m_x^{(j)}} |j, m_x^{(j)}\rangle\langle j, m_x^{(j)}| = \oplus_{j=0}^{j_{\max}} \hat{\mathbb{1}}_j,$$

where $j_{\max}$ is the largest value of j in the expansion in Eq. (49). However, one cannot guarantee, in this case, that those are the only states that permit a global saturation of the QIB. The coefficients $b_{j,k_l} \equiv \langle A_{k_l} | \Psi_j \rangle$ are

$$\begin{aligned} b_{j,k_l} &\to \langle j', m_z^{(j')} | j, m_x^{(j)} \rangle \\ &= \delta_{j'j}\, e^{i(m_z' \vartheta)} d^{j}_{m_z', m_z}(\pi/2), \end{aligned} \tag{50}$$

with $j = 0, \ldots, j_{\max}$ and $-j \leqslant m_z^{(j)} \leqslant j$. Therefore, in each invariant subspace $\hat{\mathbb{1}}_j$, the corresponding coefficients b_{j,k_l} are $e^{i(m_z'\vartheta)} d^{j}_{m_z', m_z}(\pi/2)$. Notice that it is allowed to consider states with $j_{\max} \to \infty$.

It is interesting to show how the global saturation of the QIB in the context of phase estimation with one bosonic mode may happen. In this case, the generator $\hat{A} = \hat{n} = \hat{a}^\dagger \hat{a}$ is the number operator associated with the bosonic mode, described by the annihilation operator $\hat{a}$. Since the generator $\hat{n}$ has a nondegenerate spectrum, our results provide all the initial states $|\phi_+\rangle$ that allow a global saturation of the QIB under projective measurements. Let us see how these states can be constructed. Given a fixed value for $\langle \hat{n} \rangle_+$, since the spectrum of $\hat{n}$ is equally spaced, state $|\phi_+\rangle$ satisfies the symmetry condition in Eq. (24a) only if $\langle \hat{n} \rangle_+$ coincides with some eigenvalue of $\hat{n}$ or is the arithmetic mean of any two eigenvalues. Then, all the eigenstates $|n\rangle$ with eigenvalues $0 \leqslant n \leqslant \langle \hat{n} \rangle_+$ and the eigenstates symmetric to them with respect to the mean $\langle \hat{n} \rangle_+$, can be used to construct an initial state according to Eqs. (19) and (30). It is, then, easy to see that because the spectrum of $\hat{n}$ is lower bounded, the number of terms in Eq. (19) must be finite. This means, for example, that coherent states

$$|\phi_+\rangle = |\alpha\rangle \equiv \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

with $\langle \hat{n} \rangle_+ = |\alpha|^2$, are not among the initial states that allow a global saturation of the QIB under projective measurements.

However, if we consider coherent states with large values of $\langle \hat{n} \rangle_+ = |\alpha|^2$, we can approximate the Poisson distribution by a Gaussian [21], i.e.,

$$p_n \equiv e^{-\langle \hat{n} \rangle_+} \frac{\langle \hat{n} \rangle_+^n}{n!} \approx \frac{e^{-\frac{1}{2\langle \hat{n} \rangle_+}(n - \langle \hat{n} \rangle_+)^2}}{\sqrt{2\pi \langle \hat{n} \rangle_+}} \equiv g_n,$$

yielding

$$|\phi_+\rangle = |\alpha\rangle \approx \sum_{n=0}^{\infty} \sqrt{g_n} e^{i\theta n} |n\rangle \approx \sum_{n=0}^{M-1} \sqrt{g_n} e^{i n \theta} |n\rangle, \tag{51}$$

with $M = 2\langle \hat{n} \rangle_+ + 1$. Clearly this state verifies the balance condition $\sqrt{g_n} = \sqrt{g_{2\langle \hat{n} \rangle_+ + 2 - n}}$ in Eq. (30) so that it can saturate the QIB if we use the projective measurement basis

$$|\psi_j\rangle = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} e^{i\theta_{j,n}} |n\rangle,$$

where the phases $\theta_{j,n}$ are given in Eqs. (39) and (40) with $k_l = l = n - 1$. It is interesting to notice that, because the phases in the state Eq. (51) do not satisfy the conditions in Eq. (46a), it is not possible, in this case, to use the projective measurement basis defined in Eq. (42).

V. GLOBAL SATURATION OF THE QIB AND THE HEISENBERG LIMIT

A very relevant problem in quantum metrology consists in determining, for fixed resources, which are the states that reach the largest possible QIB. Such states lead to the lowest possible quantum Cramér-Rao bound, using those resources. For the pure state families given in Eq. (3), one can consider $\langle \hat{A} \rangle_+$ as the fixed resource. We show now that, for those families, the largest QIB among all the initial states $|\phi_+\rangle$ that allow a global saturation of that bound corresponds to

$$\mathcal{F}_Q^{HL} = 4(\langle \hat{A} \rangle_+ - A_0)^2, \tag{52}$$

when the generator $\hat{A}$ has a lower bounded spectrum. Here, A_0 is the lowest eigenvalue of $\hat{A}$. The quantum Cramér-Rao bound $1/\nu\mathcal{F}_Q^{HL}$ is known in the literature as the Heisenberg limit [18]. This implies that the Heisenberg limit can be attained with projective measurements, without any previous information about the true value of the parameter and without the use of any adaptive estimation scheme. It also implies that the Heisenberg limit cannot be surpassed under these conditions.

The initial states that permit a global saturation of the QIB and have a quantum Fisher information equal to $\mathcal{F}_Q^{HL}$ are written as

$$|\phi_+^{HL}\rangle = \frac{1}{\sqrt{2}}(|A_0\rangle + e^{i\theta_k} |A_k\rangle),$$

where $A_k \equiv 2\langle \hat{A} \rangle_+ - A_0$ and θ_k is an arbitrary phase. The states in the projective measurements basis that lead to the saturation of the QIB, for the initial states above, have the structure

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|A_0\rangle + e^{i\theta_{1,k}} |A_k\rangle),$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|A_0\rangle - e^{i\theta_{1,k}} |A_k\rangle),$$

with $\theta_{1,k}$ an arbitrary phase.

In order to show that $\mathcal{F}_Q^{HL}$ is the largest quantum information associated with the states that may globally saturate the QIB, notice that, for a fixed value of $\langle \hat{A} \rangle_+$, there are several initial states $|\phi_+^M\rangle$ that can be decomposed in the form given in Eq. (19), for $M \geqslant 2$, which satisfy condition (30). All these states allow a global saturation of the QIB, that is

$$\mathcal{F}_M(x_\upsilon, \{|\psi_j\rangle\}) = \mathcal{F}_Q^M = 4\langle (\Delta \hat{A})^2 \rangle_{\phi_+^M}, \tag{53}$$

regardless of the value of x_υ, where $\mathcal{F}_M(x_\upsilon, \{|\psi_j\rangle\})$ is the Fisher information associated with the projective measurement $\{|\psi_j\rangle\langle\psi_j|\}$ on the states $|\phi_+^M\rangle$. Here, $\langle(\Delta\hat{A})^2\rangle_{\phi_+^M} = \mathrm{Tr}[|\phi_+^M\rangle\langle\phi_+^M|(\hat{A}-\langle\hat{A}\rangle_+)^2]$.

Using condition (30), that is, $|c_{k_l}| = |c_{k_{\delta(l)}}|$, we can write

$$\begin{aligned}\mathcal{F}_Q^M &\equiv 4\langle(\Delta\hat{A})^2\rangle_{\phi_+^M} = 8\sum_{l=1}^{\lceil M/2\rceil}|c_{k_l}|^2(\langle\hat{A}\rangle_+ - A_{k_l})^2\\ &= 4(\langle\hat{A}\rangle_+ - A_{k_1})^2 - 4c(\langle\hat{A}\rangle_+ - A_{k_1})^2\\ &\quad - 8\sum_{l=2}^{\lceil M/2\rceil}|c_{k_l}|^2[(\langle\hat{A}\rangle_+ - A_{k_1})^2 - (\langle\hat{A}\rangle_+ - A_{k_l})^2]\\ &\leqslant 4(\langle\hat{A}\rangle_+ - A_{k_1})^2 \equiv \mathcal{F}_Q^{M=2}\\ &\leqslant 4(\langle\hat{A}\rangle_+ - A_0)^2 \equiv \mathcal{F}_Q^{HL}.\end{aligned} \tag{54}$$

Here, we used $|c_{k_1}|^2 = 1/2 - \sum_{l=2}^{s(M)}|c_{k_l}|^2 - c/2$, where $s(M)$ is equal to $\lceil M/2\rceil$ if M is even and equal to $\lceil M/2\rceil - 1$ if M is odd. We also set $c = 0$ if M is even and $c = |c_{k_{\lceil M/2\rceil}}|^2 = \langle\hat{A}\rangle_+$ if M is odd, and we use that $|\langle\hat{A}\rangle_+ - A_0| \geqslant |\langle\hat{A}\rangle_+ - A_{k_1}| \geqslant |\langle\hat{A}\rangle_+ - A_{k_l}|$, for $l = 2,\dots,M$. This shows that $\mathcal{F}_Q^{HL}$ is the largest quantum Fisher information associated to the initial states that allow a global saturation of the QIB.

VI. CONCLUSION

In conclusion, we have considered the long-standing quest to find all the initial states, together with the corresponding projective measurements, that allow a saturation of the quantum information bound (QIB) without any previous information about the true value of the parameter to be estimated and without the use of any adaptive estimation scheme. We have been able to completely solve this problem for the important situation where information about the parameter is imprinted on an initial pure probe state via an unitary process whose generator does not depend explicitly on the parameter to be estimated.

We have fully characterized all the initial states and corresponding projective measurements that allow a global saturation of the QIB under such conditions. We have also shown that, for a fixed mean value $\langle\hat{A}\rangle_+$ of the generator of the unitary transformation, the largest quantum Fisher information associated to those states leads to the so-called Heisenberg limit. This implies that the Heisenberg limit can be attained with projective measurements, without any previous information about the true value of the parameter and without the use of any adaptive estimation scheme.

ACKNOWLEDGMENTS

We are grateful to Marcus Venicius Cougo Pinto for fruitful discussions. We acknowledge financial support from the Brazilian agencies FAPERJ, CNPq, CAPES and the INCT-Informação Quântica.

APPENDIX A

Here we show that Eqs. (17) are satisfied if and only if the sets $\{A_{k_l}\}$, $\{c_{k_l}\}$, and $\{b_{j,k_l}\}$ with $j,l = 1,\dots,M$, verify the conditions in Eqs. (24), assuming that the set of eigenstates $\{|A_{k_l}\rangle\}_{l=1,\dots,M}$ present in the decomposition of $|\phi_+\rangle$ does not contain two eigenstates $|A_{k_l}\rangle$ corresponding to the same eigenvalue of $\hat{A}$. We start writing

$$\begin{aligned}\mathrm{Im}[w_j(\epsilon)z_j^*(\epsilon)] &= -\frac{2}{\sqrt{\mathcal{F}_Q}}\sum_{l=1}^{M}\sum_{l'=1}^{M}|c_{k_l}|c_{k_{l'}}||b_{j,k_l}||b_{j,k_{l'}}|(A_{k_l} - \langle\hat{A}\rangle_+)\cos((A_{k_{l'}} - A_{k_l})\epsilon + \theta_{k_l} - \theta_{j,k_l} - \theta_{k_{l'}} + \theta_{j,k_{l'}})\\ &= \sum_{l=1}^{M}\sum_{l'=1}^{M}(A_{k_l} - \langle\hat{A}\rangle_+)[h_{j,l,l'}\cos((A_{k_{l'}} - A_{k_l})\epsilon) - g_{j,l,l'}\sin((A_{k_{l'}} - A_{k_l})\epsilon)],\end{aligned} \tag{A1}$$

where we use the expressions for $z_j(\epsilon)$ and $w_j(\epsilon)$ in Eqs. (23). Furthermore, we use the identity $\cos(x+y) = \cos x\cos y - \sin x\sin y$ and define

$$\begin{aligned}h_{j,l,l'} &\equiv |c_{k_l}|c_{k_{l'}}||b_{j,k_l}||b_{j,k_{l'}}|\\ &\qquad\times\cos(\theta_{k_l} - \theta_{j,k_l} - \theta_{k_{l'}} + \theta_{j,k_{l'}}),\end{aligned} \tag{A2a}$$

$$\begin{aligned}g_{j,l,l'} &\equiv |c_{k_l}|c_{k_{l'}}||b_{j,k_l}||b_{j,k_{l'}}|\\ &\qquad\times\sin(\theta_{k_l} - \theta_{j,k_l} - \theta_{k_{l'}} + \theta_{j,k_{l'}}).\end{aligned} \tag{A2b}$$

Because the equality in Eqs. (A1) must hold for any value of ϵ, we can write those equations for $-\epsilon$ and combine the two cases in order to arrive to the equivalent equations

$$\begin{aligned}&\sum_{l=1}^{M}h_{j,l,l}(A_{k_l} - \langle\hat{A}\rangle_+) + \sum_{l=1_{l\neq l'}}^{M}\sum_{l'=1}^{M}h_{j,l,l'}\\ &\qquad\times(A_{k_l} - \langle\hat{A}\rangle_+)\cos((A_{k_{l'}} - A_{k_l})\epsilon) = 0\end{aligned} \tag{A3a}$$

$$\begin{aligned}&\sum_{l=1_{l\neq l'}}^{M}\sum_{l'=1}^{M}g_{j,l,l'}\\ &\qquad\times(A_{k_l} - \langle\hat{A}\rangle_+)\sin((A_{k_{l'}} - A_{k_l})\epsilon) = 0,\end{aligned} \tag{A3b}$$

that must be valid for all values of ϵ and $j = 1,\dots,M$. It is more convenient to rewrite Eqs. (A3) summing over indexes such that $l < l'$:

$$\sum_{l=1}^{M}\sum_{l'=l+1}^{M}\tilde{h}_{j,l,l'}(\cos(\omega_{l'l}\epsilon) - 1) = 0, \tag{A4a}$$

$$\sum_{l=1}^{M}\sum_{l'=l+1}^{M}\tilde{g}_{j,l,l'}\sin(\omega_{l'l}\epsilon) = 0, \tag{A4b}$$

where we define $\tilde{g}_{j,l,l'} \equiv g_{j,l,l'}(A_{k_l} - \langle\hat{A}\rangle_+ + A_{k_{l'}} - \langle\hat{A}\rangle_+)$ and the frequencies $\omega_{l'l} \equiv A_{k_{l'}} - A_{k_l}$. We also use that when we evaluate Eq. (A3a) for $\epsilon = 0$ we have $\sum_{l=1}^{M} h_{j,l,l}(A_{k_l} - \langle\hat{A}\rangle_+) = -\sum_{l=1_{l\neq l'}}^{M}\sum_{l'=1}^{M} h_{j,l,l'}(A_{k_l} - \langle\hat{A}\rangle_+)$.

Note that, in principle, the frequencies $\omega_{l'l}$ can be degenerate or nondegenerate. So, we can divide the sum in Eq. (A4b) as

$$\sum_{l}\sum_{l'} \tilde{g}_{j,l,l'} \sin(\omega_{l'l}\epsilon) \tag{A5a}$$

$$+ \sum_{l''}\sum_{l'''}\left[\sum_{l=l''}\sum_{l'=l'''}\tilde{g}_{j,l,l'}\right]\sin(\omega_{l'''l''}\epsilon) = 0, \tag{A5b}$$

where the sums over the indexes l,l' correspond to the nondegenerate frequencies and the sums over the indexes l'',l''' over the degenerate ones. Note that we consider in Eq. (A4b) that $l < l'$, then $A_{k_l} < A_{k_{l'}}$, because $k_1 < \cdots < k_M$. Analogously, $l'' < l'''$, so that $A_{k_{l''}} < A_{k_{l'''}}$.

Because the functions $\sin(\omega_{l'l}\epsilon)$ and $\sin(\omega_{l'''l''}\epsilon)$ are linearly independent, the coefficient $\tilde{g}_{j,l,l'}$ and $\sum_{l=l''}\sum_{l'=l'''}\tilde{g}_{j,l,l'}$, in Eqs. (A5), must be equal to zero for all values of j. Now, note that the coefficients of the expansion in Eq. (19) of the initial state $|\phi_+\rangle$ are such that $c_{k_l} \neq 0$ for $l = 1,\ldots,M$. Notice also that the coefficient b_{j,k_l} of the expansion of the measuring basis states $|\psi_j\rangle$, in Eq. (22), can only be zero for specific values of l but not for all values of $j = 1,\ldots,M$. So, from $\tilde{g}_{j,l,l'} = 0$, in Eq. (A5b), we arrive to

$$A_{k_l} - \langle\hat{A}\rangle_+ + A_{k_{l'}} - \langle\hat{A}\rangle_+ = 0. \tag{A6}$$

This condition simply says that, for nondegenerate frequencies $\omega_{l'l} \equiv A_{k_{l'}} - A_{k_l}$, the mean $\langle\hat{A}\rangle_+$ must be the arithmetic mean of the eigenvalues A_{k_l} and $A_{k_{l'}}$, i.e., $(A_{k_{l'}} + A_{k_l})/2 = \langle\hat{A}\rangle_+$. Clearly, the frequency $\omega_{M,1}$ is nondegenerate because $\omega_{l'l} < \omega_{M,1}$ for all values of $l,l' = 1,\ldots,M$ ($A_{k_1} < \cdots < A_{k_M}$). Therefore, we must have $(A_{k_M} + A_{k_1})/2 = \langle\hat{A}\rangle_+$, or equivalently,

$$A_{k_M} - \langle\hat{A}\rangle_+ + A_{k_1} - \langle\hat{A}\rangle_+ = 0. \tag{A7}$$

Now, note that the frequencies $\omega_{M,2}$ and $\omega_{M-1,1}$ must be degenerate. Otherwise, we would arrive to the contradictory results $\langle\hat{A}\rangle_+ = (A_{k_M} + A_{k_2})/2 > (A_{k_M} + A_{k_1})/2 = \langle\hat{A}\rangle_+$ or $\langle\hat{A}\rangle_+ = (A_{k_{M-1}} + A_{k_1})/2 < (A_{k_M} + A_{k_1})/2 = \langle\hat{A}\rangle_+$ by using Eq. (A7) and that $A_{k_2} > A_{k_1}$, and $A_{k_{M-1}} < A_{k_M}$. However, since $\omega_{l',l} < \omega_{M,2}$ and $\omega_{l',l} < \omega_{M-1,1}$ for all values of $l,l' = 2,\ldots,M$ and $\omega_{M,2},\omega_{M-1,1} < \omega_{M,1}$, the only possibility is that $\omega_{M-1,1} = \omega_{M,2}$. This is equivalent to the condition $(A_{k_{M-1}} + A_{k_2})/2 = (A_{k_M} + A_{k_1})/2$, and, using Eq. (A7), it is also equivalent to

$$A_{k_{M-1}} - \langle\hat{A}\rangle_+ + A_{k_2} - \langle\hat{A}\rangle_+ = 0. \tag{A8}$$

We can now repeat the arguments for the frequencies $\omega_{M-1,3}$ and $\omega_{M-2,2}$. Indeed, this frequencies must be degenerate because, otherwise, we arrive at the contradictory results $\langle\hat{A}\rangle_+ = (A_{k_{M-1}} + A_{k_3})/2 > (A_{k_{M-1}} + A_{k_2})/2 = \langle\hat{A}\rangle_+$ or $\langle\hat{A}\rangle_+ = (A_{k_{M-2}} + A_{k_2})/2 < (A_{k_{M-1}} + A_{k_2})/2 = \langle\hat{A}\rangle_+$ by using Eq. (A8) and that $A_{k_3} > A_{k_2}$, $A_{k_{M-2}} < A_{k_{M-1}}$. However, since $\omega_{l',l} < \omega_{M-1,3}$ and $\omega_{l',l} < \omega_{M-2,2}$ for all values of $l,l' = 3,\ldots,M$ and $\omega_{M-1,3},\omega_{M-2,2} < \omega_{M-1,2}$, the only possibility is that $\omega_{M-2,2} = \omega_{M-1,3}$. This is equivalent to the condition $(A_{k_{M-2}} + A_{k_3})/2 = (A_{k_{M-1}} + A_{k_2})/2$, and, using Eq. (A8), it is also equivalent to

$$A_{k_{M-2}} - \langle\hat{A}\rangle_+ + A_{k_3} - \langle\hat{A}\rangle_+ = 0. \tag{A9}$$

These two steps illustrate the iterative process to be followed. They show that the frequencies $\omega_{\delta(l),1+l}$ and $\omega_{M-l,l}$, with $\delta(l) = M - (l-1)$ and $l = 1,\ldots,s(M)$, where $s(M) \equiv \lceil M/2\rceil$ if M is even and $s(M) \equiv \lceil M/2\rceil - 1$ if M is odd, are degenerate in such a way that $\omega_{\delta(l),1+l} = \omega_{M-l,l}$ and that they are different from any other frequencies. This is enough to prove that $A_{k_{\delta(l)}} - \langle\hat{A}\rangle_+ + A_{k_l} - \langle\hat{A}\rangle_+ = 0$, that is exactly the condition in Eq. (24a). This symmetry condition for the spectrum of eigenvalues $\{A_{k_l}\}_{l=1,\ldots,M}$, of $\hat{A}$, that enter in the decomposition of the initial state $|\phi_+\rangle$, is illustrated in Fig. 1, where we see that when M is odd necessarily $A_{\lceil M/2\rceil} = \langle\hat{A}\rangle_+$.

Because $\omega_{\delta(l),1+l} = \omega_{M-l,l}$ for $l = 1,\ldots,s(M)$, and because these frequencies are different from any other frequencies, the coefficient of $\sin(\omega_{\delta(l),1+l}) = \sin(\omega_{M-l,l})$ in Eq. (A5b) must be equal to zero, i.e., $(g_{j,\delta(l),l+1} + g_{j,l,M-l})(A_{k_{\delta_l}} - \langle\hat{A}\rangle_+ + A_{k_{l+1}} - \langle\hat{A}\rangle_+) = 0$. Analogously, the coefficient of $\cos(\omega_{\delta(l),1+l}) - 1 = \cos(\omega_{M-l,l}) - 1$ in Eq. (A4a) must be equal to zero, i.e., $(h_{j,\delta(l),l+1} - h_{j,l,M-l})(A_{k_{\delta_l}} - \langle\hat{A}\rangle_+ + A_{k_{l+1}} - \langle\hat{A}\rangle_+) = 0$. This leads to the following sets of equations:

$$|c_{k_{\delta(l)}}||c_{k_{l+1}}||b_{j,k_{\delta(l)}}||b_{j,k_{l+1}}|\sin(\theta_{k_{\delta(l)}} - \theta_{j,k_{\delta(l)}} - \theta_{k_{l+1}} + \theta_{j,k_{l+1}}) = -|c_{k_{M-l}}||c_{k_l}||b_{j,k_{M-l}}||b_{j,k_l}|\sin(\theta_{k_l} - \theta_{j,k_l} - \theta_{k_{M-l}} + \theta_{j,k_{M-l}}), \tag{A10a}$$

$$|c_{k_{\delta(l)}}||c_{k_{l+1}}||b_{j,k_{\delta(l)}}||b_{j,k_{l+1}}|\cos(\theta_{k_{\delta(l)}} - \theta_{j,k_{\delta(l)}} - \theta_{k_{l+1}} + \theta_{j,k_{l+1}}) = |c_{k_{M-l}}||c_{k_l}||b_{j,k_{M-l}}||b_{j,k_l}|\cos(\theta_{k_l} - \theta_{j,k_l} - \theta_{k_{M-l}} + \theta_{j,k_{M-l}}). \tag{A10b}$$

By taking the square in both Eqs. (A10a) and (A10b) and adding them, we get

$$|c_{k_{\delta(l)}}||b_{j,k_{\delta(l)}}||c_{k_{l+1}}||b_{j,k_{l+1}}| = |c_{k_{\delta(l+1)}}||b_{j,k_{\delta(l+1)}}||c_{k_l}||b_{j,k_l}|, \tag{A11}$$

where $l = 1,2,\ldots,s(M)$ and we use that $\delta(l+1) = M - l$. By applying $l = s(M)$ in Eq. (A11), and noting that if M is even $\delta(s(M)+1) = s(M)$ and $\delta(s(M)) = s(M)+1$ and if M is odd $\delta(s(M)+1) = s(M)+1$, we arrive at

$$|c_{k_{s(M)}}||b_{j,k_{s(M)}}| = |c_{k_{\delta(s(M))}}||b_{j,k_{\delta(s(M))}}|. \tag{A12}$$

After that, we substitute $l = s(M) - 1$ in Eq. (A11). In the resulting expression, we apply Eq. (A12) in order to have

$$|c_{k_{s(M)-1}}||b_{j,k_{s(M)-1}}| = |c_{k_{\delta(s(M)-1)}}||b_{j,k_{\delta(s(M)-1)}}|. \tag{A13}$$

Then, we use $l = s(M) - 2$, together with Eq. (A13), in Eq. (A11). Following this iterative procedure, we are able to show that

$$|c_{k_l}||b_{j,k_l}| = |c_{k_{\delta(l)}}||b_{j,k_{\delta(l)}}|, \tag{A14}$$

for $l = 1,2,\ldots,s(M)$, which is the result of Eq. (24b).

Now, we plug Eq. (A14) into Eqs. (A10) and, by solving the resulting system of equations, we get the following solution:

$$(\theta_{k_{\delta(l)}} - \theta_{j,k_{\delta(l)}}) + (\theta_{k_l} - \theta_{j,k_l}) = (\theta_{k_{l+1}} - \theta_{j,k_{l+1}}) + (\theta_{k_{M-l}} - \theta_{j,k_{M-l}}). \quad \text{(A15)}$$

If we apply $l = s(M) - 1$ in Eq. (A15) [$l = s(M)$ does not give any extra information about the phase relation], we have

$$(\theta_{k_{s(M)-1}} - \theta_{j,k_{s(M)-1}}) + (\theta_{k_{\delta(s(M)-1)}} - \theta_{j,k_{\delta(s(M)-1)}}) = (\theta_{k_{s(M)}} - \theta_{j,k_{s(M)}}) + (\theta_{k_{\delta(s(M))}} - \theta_{j,k_{\delta(s(M))}}). \quad \text{(A16)}$$

If we use $l = s(M) - 2$ in Eq. (A15) and then substitute Eq. (A16) in the result, we obtain

$$(\theta_{k_{s(M)-2}} - \theta_{j,k_{s(M)-2}}) + (\theta_{k_{\delta(s(M)-2)}} - \theta_{j,k_{\delta(s(M)-2)}}) = (\theta_{k_{s(M)}} - \theta_{j,k_{s(M)}}) + (\theta_{k_{\delta(s(M))}} - \theta_{j,k_{\delta(s(M))}}). \quad \text{(A17)}$$

If we put $l = s(M) - 3$ in Eq. (A15) and plug Eq. (A17) into the result, we will find the same term of the right-hand side of the Eqs. (A16) and (A17). Repeating these steps iteratively for all the remaining terms, we will see that the terms $(\theta_{k_{\delta(l)}} - \theta_{j,k_{\delta(l)}}) + (\theta_{k_l} - \theta_{j,k_l})$ are equal for all $l = 1,2,\ldots,s(M)$. So, as in principle the phases are all different from each other, this equality among all the expressions only holds if

$$(\theta_{k_{\delta(l)}} - \theta_{j,k_{\delta(l)}}) + (\theta_{k_l} - \theta_{j,k_l}) = \xi_j, \quad \text{(A18)}$$

where ξ_j is a constant depending only on j. Equation (A18) is that one in Eq. (24c).

APPENDIX B

Here we prove that one solution to the conditions

$$\langle\psi_j|\,\hat{A}\,|\psi_j\rangle = \alpha \quad , \quad j = 1,\ldots,M, \quad \text{(B1)}$$

is given by states $|\psi_j\rangle$ whose expansion in the eigenbasis of the generator $\hat{A}$ is of the form shown in Eq. (22), with the coefficients given in Eq. (38) and the phases in (39) and (40). Remember that, since all the coefficients $b_{j,k_l} \neq 0$, the subspace spanned by $\{|\psi_j\rangle\}_{j=1,\ldots,M}$ and the subspace spanned by $\{|A_{k_l}\rangle\}_{l=1,\ldots,M}$ coincide. Let us start defining an auxiliary unitary operator $\hat{V}$ within the subspace $\{|\psi_j\rangle\}_{j=1,\ldots,M}$ of the system Hilbert space, such that

$$\hat{V}\,|\psi_j\rangle = |\psi_{j+1}\rangle\,, \quad \text{(B2)}$$

$$\hat{V}\,|\psi_M\rangle = e^{i\beta}\,|\psi_1\rangle\,, \quad \text{(B3)}$$

where $1 \leqslant j \leqslant (M-1)$ and β is an arbitrary phase. We call $\hat{V}$ the *shift* operator over the basis $\{|\psi\rangle_j\}_{j=1,\ldots,M}$. It is important to notice that, for every finite basis, it is always possibly to define an operator $\hat{V}$ that shift the elements of the basis. The unitary matrix, in the basis $\{|\psi_j\rangle\}_{j=1,\ldots,M}$, that represent $\hat{V}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & e^{i\beta} \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Its eigenvalues are of the form $e^{id_l} = e^{i(f_l\pi+\beta)/M}$ and the eigenvectors of the form

$$|d_l\rangle = \frac{1}{\sqrt{M}}\sum_{j=1}^{M} e^{-i\theta'_{j,l}}\,|\psi_j\rangle\,, \quad \text{(B4)}$$

with

$$\theta'_{j,l} = (j\pi/M)\,f_l + j\beta/M, \quad \text{(B5)}$$

and f_l given in Eq. (40). Therefore, the subspace $\{|\psi_j\rangle\}$ can be equivalently described by the basis $\{|d_l\rangle\}_{l=1,\ldots,M}$ formed by the eigenstates of the *shift* operator. We emphasize here that the states $|d_l\rangle$ belong to the system Hilbert space, which could have an arbitrary dimension. The matrix whose elements are

$$\langle\psi_j\,|d_l\rangle = (1/\sqrt{M})e^{-i\theta'_{j,l}} \quad \text{(B6)}$$

is unitary, so we can invert the relation in Eq. (B4) to write

$$|\psi_j\rangle = \frac{1}{\sqrt{M}}\sum_{l=1}^{M} e^{i\theta'_{j,l}}\,|d_l\rangle\,. \quad \text{(B7)}$$

We can express the unitary *shift* operator as $\hat{V} = e^{i\hat{D}}$, where $\hat{D}$ is a Hermitian operator with eigenvalues

$$d_l \equiv (f_l\pi + \beta)/M, \quad \text{(B8)}$$

and eigenvectors given in Eq. (B4). Notice that $\hat{D}$ has a nondegenerate spectrum and that its diagonal elements in the basis $\{|\psi_j\rangle\}$ are all equals, i.e.,

$$\langle\psi_j|\,\hat{D}\,|\psi_j\rangle = \frac{1}{M}\sum_{l=1}^{M} d_l \equiv \alpha_1, \quad \text{(B9)}$$

where α_1 does not depend on the value of j.

Now, remember that we are looking for the states $|\psi_j\rangle$ that verify (B1). This is a similar condition to the one in (B9) for the generator $\hat{D}$ of the shift operator $\hat{V} = e^{i\hat{D}}$. Let us show that it is possible to consider that $\hat{A}$ is diagonal in the subspace spanned by the basis $\{|d_l\rangle\}_{l=1,\ldots,M}$ of eigenstates of $\hat{D}$. We first note that

$$\langle\psi_{j=M}|\,d_l\rangle = \frac{e^{-i\beta}}{\sqrt{M}}, \quad \text{(B10)}$$

for all values $l = 1,\ldots,M$. Additionally, using Eq. (40) in Eq. (B5), we get for $l \neq l'$ the phases differences

$$\theta'_{j,l} - \theta'_{j,l'} = \begin{cases} \frac{j\pi}{M}\{(l-l')(M+1) + [(-1)^l - (-1)^{l'}]/2\}, \\ \qquad \text{for M even,} \\ \frac{j\pi}{M}(l-l')(M+1), \text{ for M odd.} \end{cases} \quad \text{(B11)}$$

In particular, we have

$$\theta'_{j=M,l} - \theta'_{j=M,l'} = 2n\pi, \quad \text{(B12a)}$$

$$\theta'_{j,l} - \theta'_{j,l'} \neq 2n\pi \quad \text{for } 1 \leqslant j \leqslant (M-1), \quad \text{(B12b)}$$

with n an integer. Now, we obtain

$$\langle\psi_{j=M}|\,\hat{A}\,|\psi_{j=M}\rangle = \frac{1}{M}\left\{\sum_{l=1}^{M}\langle d_l|\,\hat{A}\,|d_l\rangle + \sum_{l\neq l'}^{M}\langle d_l|\,\hat{A}\,|d_{l'}\rangle\right\} = \alpha, \tag{B13}$$

$$\langle\psi_{j\neq M}|\,\hat{A}\,|\psi_{j\neq M}\rangle = \frac{1}{M}\left\{\sum_{l=1}^{M}\langle d_l|\,\hat{A}\,|d_l\rangle + \sum_{l\neq l'}^{M} e^{i(\theta'_{j,l}-\theta'_{j,l'})}\langle d_l|\,\hat{A}\,|d_{l'}\rangle\right\} = \alpha. \tag{B14}$$

Since $(\theta'_{j,m}-\theta'_{j,m'})\neq 2n\pi$ [see Eq. (B12b)], comparing Eq. (B13) with Eq. (B14), we see that one possibility is that

$$\langle d_l|\,\hat{A}\,|d_{l'}\rangle = 0, \tag{B15}$$

for all $l\neq l'$, which means that $\hat{A}$ is diagonal in the subspace spanned by $\{|d_l\rangle\}_{l=1,\ldots,M}$. The other possibility is that the second terms in Eqs. (B13) and (B14) are null. It is interesting to notice that this second possibility is verified if we use the coefficients in Eq. (42) to define the states $|\psi_j\rangle$ through (22) and then use those states in the definition of the eigenstates of the *shift* operator in (B4).

Because the operator $\hat{D}$ is nondegenerate, the result in Eq. (B15) means that we can identify the eigenstates of $\hat{D}$ and the eigenstates of $\hat{A}$ in the subspace $\{|\psi_j\rangle\}_{j=1,\ldots,M}$. The order of this identification is unimportant, so we can set

$$|d_l\rangle = |A_{k_l}\rangle\ ,\ l=1,\ldots,M. \tag{B16}$$

In order to obtain the projective measurement with states given in Eq. (22), with coefficient given in (38) whose phases are given in (39), we observe that if we apply an arbitrary unitary evolution $e^{i(h(\hat{A}))}$ to the states in Eq. (B7) [here $h(\hat{A})$ is any Hermitian operator that depends on $\hat{A}$], we obtain an equivalently admissible projective measurement [one that also fulfills the condition that all the matrix elements $\langle\psi_j|\,\hat{A}\,|\psi_j\rangle$ are equal]. This is the reason why we include the extra phases $\phi_{k_l}\equiv h(A_{k_l})$ in Eq. (39) in comparison with the phases in Eq. (B5).

We can verify the consistency of our results looking at the orthonormality relation:

$$\langle\psi_j|\,\psi_{j'}\rangle = \frac{1}{M}\sum_{l=1}^{M}e^{i(\theta_{j,k_l}-\theta_{j',k_l})} = e^{i(\gamma_j-\gamma_{j'})}e^{i(j-j')\beta/M}\frac{1}{M}\sum_{l=1}^{M}e^{i\pi(j-j')f_l/M} = \delta_{jj'}, \tag{B17}$$

where we use that

$$\frac{1}{M}\sum_{l=1}^{M}e^{i\pi(j-j')f_l/M} = \delta_{jj'}. \tag{B18}$$

For $j=j'$ we can immediately check this equality. In order to check the equality in Eq. (B18) for $j\neq j'$, we proceed as follows. For M even, we have

$$\begin{aligned}\sum_{l=1}^{M}e^{i\pi(j-j')f_l/M} &= \sum_{l\to\text{even}}^{M}e^{i\pi(j-j')(l+M-2)/M}+\sum_{l\to\text{odd}}^{M}e^{i\pi(j-j')(l-1)/M}\\ &= e^{i\pi(j-j')}(1+e^{2i\pi(j-j')/M}+e^{4i\pi(j-j')/M}+\cdots)\\ &\quad +(1+e^{2i\pi(j-j')/M}+e^{4i\pi(j-j')/M}+\cdots)\\ &= \frac{(1+e^{i\pi(j-j')})(1-e^{i\pi(j-j')})}{1+e^{2i\pi(j-j')/M)}} = \frac{(1-e^{2i\pi(j-j')})}{1+e^{2i\pi(j-j')/M)}} = 0,\end{aligned} \tag{B19}$$

and for M odd, we have

$$\begin{aligned}\sum_{l=1}^{M}e^{i\pi(j-j')f_l/M} &= \sum_{l=1}^{M}e^{i\pi(j-j')(l-1)(1-M)/M}\\ &= 1+e^{i\pi(j-j')(1-M)/M}+e^{2i\pi(j-j')(1-M)/M}+\cdots\\ &= \frac{1-e^{-i\pi(j-j')(M-1)}}{1+e^{i\pi(j-j')(1-M)/M}} = 0,\end{aligned} \tag{B20}$$

since $M-1$ is an even number.

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