Confluent Crum-Darboux transformations in Dirac Hamiltonians with PT-symmetric Bragg gratings

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We consider optical systems where propagation of light can be described by a Dirac-like equation with PT-symmetric Hamiltonian. In order to construct exactly solvable configurations, we extend the confluent Crum-Darboux transformation for the one-dimensional Dirac equation. The properties of the associated intertwining operators are discussed and the explicit form for higher-order transformations is presented. We utilize the results to derive a multiparametric class of exactly solvable systems where the balanced gain and loss represented by the PT-symmetric refractive index can imply localization of the electric field in the material.

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I. INTRODUCTION

In specific situations, the propagation of light can be described by equations that are at home in quantum mechanics. Indeed, the Helmholtz equation in paraxial approximation acquires the form of a Schrödinger-like equation [1]. One deals with coupled differential equations of the Dirac-type within the context of the coupled-wave theory of distributed feedback lasers [2,3]. The interaction present in the associated Hamiltonians depends on the optical properties of the material the light is propagating in. These optical characteristics can be described by the refractive index. It can be position-dependent in an optically inhomogeneous material; it can be also complex valued when gain and loss occur in the system [4–6].

The description of optical systems with complex refractive index departs from the concept of standard quantum mechanics, where the operators are required to be Hermitian. However, a link can be still established within the realm of PT-symmetric quantum mechanics, where the requirement of Hermiticity is relaxed and replaced by PT symmetry, where P is the space inversion and T is the time reversal operator [7–12]. The relation between optics and PT-symmetric quantum mechanics has been exploited extensively in recent years. Optical systems described by a Schrödinger equation with complex PT-symmetric potential were analyzed, e.g., in [13–20]. Those described by Dirac equation were discussed in [21] where the attention was paid to the spectral singularities and the PT-symmetry breaking.

The light propagating through a nonuniform Bragg grating with small fluctuations of the refractive index can be well described by the coupled mode theory [3,22]. In the case of a monochromatic electromagnetic field of the form

$$\mathbf{E} = \mathbf{e_1}[E(x_3)e^{-i\omega t} + E^*(x_3)e^{i\omega t}],$$
$$\mathbf{H} = \mathbf{e_2}[H(x_3)e^{-i\omega t} + H^*(x_3)e^{i\omega t}],$$

the Maxwell equations reduce to $\frac{\partial}{\partial x_3}E(x_3) = i\omega\mu_0 H(x_3)$ and $\frac{\partial}{\partial x_3}H = i\omega\epsilon_0 n^2(x_3)E(x_3)$ where $n(x_3)$ is the refractive index. Combining the two equations, we find that the electric field has to satisfy

$$\frac{\partial^2}{\partial x_3^2}E(x_3) + k^2\left(\frac{n(x_3)}{n_0}\right)^2 E(x_3) = 0. \quad (1)$$

Here, $k = \omega n_0/c$, $\mu_0\epsilon_0 = c^{-1}$, and n_0 is the reference refractive index. Fixing the electric field in the form of two counterpropagating waves,

$$E(x) = u(x)\exp\left(ix + \frac{i}{2}\phi(x)\right) + v(x)\exp\left(-ix - \frac{i}{2}\phi(x)\right), \quad x = k_0 x_3, \quad (2)$$

where k_0 is defined as the wave number of light at the Bragg scattering resonance frequency $\omega_0 = ck_0/n_0$, the equation (1) can be brought into the form of two coupled equations [22],

$$\partial_x u = i[\rho(x)u(x) + \kappa(x)v(x)],$$
$$\partial_x v = -i[\rho(x)v(x) + \kappa(x)u(x)], \quad (3)$$

where $\rho(x) = \sigma(\xi) + \Delta - \frac{1}{2}\partial_x\phi(x)$ and $\Delta = (\omega - \omega_0)/\omega_0$. The functions $\sigma(x), \kappa(x)$ determine the profile of the refractive index $n(x) = n_0\{1 + \sigma(x) + 2\kappa(x)\cos[2x + \phi(x)]\}$. In order to keep fluctuations of $n(x)$ small, we require $|\sigma(x)| << 1$ and $|\kappa| << 1$. With the use of $F = (u,v)^T$, we can rewrite (3) as

$$HF(x) = [-i\sigma_3\partial_x + i\sigma_2\rho(x) - \sigma_1\kappa(x)]F(x) = 0, \quad (4)$$

which coincides with the one-dimensional *non-Hermitian* stationary Dirac equation at zero energy.[1] The systems described by Eq. (4) will be of our main interest.

[1]Using the ansatz $E(x,t) = u(x,t)\exp[ix + \frac{i}{2}\phi(x)] + v(x,t)\exp[-ix - \frac{i}{2}\phi(x)]$, there would appear $i\partial_t F$ on the right-hand side of (4); see [3]. Let us notice that the Dirac-like equation appears also in the description of another optical system where two identical coupled PT wave guides are considered; see [25].

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The Crum-Darboux transformation is a differential operator that annihilates fixed eigenstates of the initial, known,[2] Hamiltonian [23,24]. There exists an unambiguous prescription for the new Hamiltonian such that it is intertwined with the initial one by this transformation. The intertwining relation implies that the exact solutions for the stationary equation of the new system are obtained by acting on the eigenstates of the initial Hamiltonian with the intertwining operator. Besides being very fruitful for quantum mechanics, see, e.g., Ref. [26] for a review, it proves to be an effective tool for the analysis of optical systems [27–31].

When the Crum-Darboux transformation annihilates eigenstates and Jordan states associated with a single eigenvalue it is called *confluent*. In the literature, it was discussed mostly for Schrödinger Hamiltonians in the context of quantum systems [32–38], as well as for optical settings with complex refractive index [39,40]. Recently, it was also used for the construction of a limited class of Dirac Hamiltonians in [41].

In the current article, we will find $\rho(x)$ and $\kappa(x)$ with the help of the Crum-Darboux transformations such that (4) is exactly solvable. We modify the confluent Crum-Darboux transformation for the use with Dirac Hamiltonians with a generic potential. We consider chains of confluent Crum-Darboux transformations and show explicit formulas for both the higher-order intertwining operator and the new Hamiltonian. In particular, we focus on Crum-Darboux transformations that render the new Hamiltonian PT-symmetric. Finally, we construct a class of PT-symmetric Dirac operators whose relevance for the description of optical systems is discussed.

II. FIRST-ORDER TRANSFORMATIONS AND THEIR CHAINS

In order to introduce the confluent Crum-Darboux transformations, let us start with the simplest case, the transformation of the first order. We set the initial, one-dimensional Dirac Hamiltonian in the following form:

$$H_0 = -i\sigma_3\partial_x + V_0, \tag{5}$$

where V_0 is an arbitrary matrix valued function. Following [42], we make the following ansatz for the intertwining operator

$$L_1 = \partial_x - U_xU^{-1}. \tag{6}$$

Since we assume U is an invertible matrix and $U_x \equiv \partial_x U$, by construction, the above equation means that $L_1U = 0$. We want L_1 to be an intertwining operator between H_0 and another Hamiltonian $H_1 = -i\sigma_3\partial_x + V_1$. Hence the explicit form of both L_1 and V_1 should be fixed such that

$$L_1H_0 = H_1L_1. \tag{7}$$

The intertwining relation captures the essence[3] of the Crum-Darboux transformation L_1; it allows one to map the eigenstates of the initial Hamiltonian H_0 into those of H_1.

When comparing the coefficients of the corresponding derivatives in Eq. (7), we get two equations for the unknown matrices V_1 and U,

$$V_1 = V_0 - i[U_xU^{-1},\sigma_3], \quad (U^{-1}H_0U)_x = 0. \tag{8}$$

The first one fixes V_1. The second one is a differential equation for U. It is satisfied whenever $H_0U = U\Lambda$ with Λ being a constant matrix. As any matrix with complex elements can be transformed into a Jordan form, we can fix U in such a way that

$$H_0U = U\Lambda, \quad \text{for } \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad \Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{9}$$

Hence we can write down U as

$$U = (\Omega_1,\Omega_2). \tag{10}$$

The spinors Ω_1 and Ω_2 either satisfy $H_0\Omega_a = \lambda_a\Omega_a$, in the case when Λ is diagonal,[4] or $(H_0 - \lambda)\Omega_1 = 0$ and $(H_0 - \lambda)\Omega_2 = \Omega_1$ when Λ has the form of a Jordan block. The latter choice represents a direct generalization of [42], where only a diagonal Λ was considered.

Once U is fixed, the intertwining operator L_1 as well as H_1 are uniquely defined. The operator (6) annihilates both vectors Ω_a, $L\Omega_a = 0$, $a = 1,2$. It is worth noticing that, at the moment, the present construction is formal and additional care is needed to obtain a physically relevant result. In particular, the requirement of regularity has to be imposed on the intertwining operator such that the new Hamiltonian H_1 does not contain any new singularities. This is equivalent to the requirement that $\det U \neq 0$ for the considered domain of x. We will discuss this issue later on.

Let us continue our discussion with the case of a chain of two consecutive first order Crum-Darboux transformations. We focus on the situation where the intertwining operators are systematically constructed from the Jordan states associated with a fixed eigenvalue λ_*. First, we shall fix the notation. Given a Hamiltonian H_0, we denote the two independent (not necessarily physical) eigenvectors of H_0 corresponding to an eigenvalue λ as Ψ_0 and $\widetilde{\Psi}_0$,

$$(H_0 - \lambda)\Psi_0 = 0, \quad (H_0 - \lambda)\widetilde{\Psi}_0 = 0. \tag{11}$$

The Jordan states $\chi_0^{(n)}$ and $\widetilde{\chi}_0^{(n)}$ associated with Ψ_0 and $\widetilde{\Psi}_0$, respectively, can be defined as the solutions of the $(n+1)$th iterated Dirac equation, $(H_0 - \lambda)^{n+1}\chi_0^{(n)} = (H_0 - \lambda)^{n+1}\widetilde{\chi}_0^{(n)} = 0$, in the following form [37,45]:

$$\chi_0^{(n)} = \frac{\partial^n \Psi_0}{\partial\lambda^n} + \sum_{k=0}^{n-1}\left(c_k^{(n)}\chi_0^{(k)} + d_k^{(n)}\tilde{\chi}_0^{(k)}\right),$$
$$(H_0 - \lambda)^n\chi_0^{(n)} = \Psi_0, \tag{12}$$

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[2] The solutions of the associated initial stationary equation are supposed to be known.

[3] In quantum mechanics, the energy spectra of the Hamiltonians H_0 and H_1 are typically identical up to a single discrete energy level [24,42]. The transformation also serves for the analysis of integrable systems; see e.g. [43,44].

[4] Let us notice that H_1 reduces to H_0 when $\lambda_1 = \lambda_2$; see (9).

$$\widetilde{\chi}_0^{(n)} = \frac{\partial^n \widetilde{\Psi}}{\partial \lambda^n} + \sum_{k=0}^{n-1} \left(\tilde{c}_k^{(n)} \chi_0^{(k)} + \tilde{d}_k^{(n)} \widetilde{\chi}_0^{(k)}\right),$$

$$(H_0 - \lambda)^n \widetilde{\chi}_0^{(n)} = \widetilde{\Psi}_0, \tag{13}$$

where $c_k^{(n)}$, $\tilde{c}_k^{(n)}$, $d_k^{(n)}$, and $\tilde{d}_k^{(n)}$ are some complex numbers and $\chi_0^{(0)} = \Psi_0$ and $\widetilde{\chi}_0^{(0)} = \widetilde{\Psi}_0$. In particular, the Jordan states satisfy

$$(H_0 - \lambda)\chi_0^{(n)} = \chi_0^{(n-1)}, \quad (H_0 - \lambda)\widetilde{\chi}_0^{(n)} = \widetilde{\chi}_0^{(n-1)}. \tag{14}$$

The fact that the functions $\chi_0^{(n)}$ and $\widetilde{\chi}_0^{(n)}$ are solutions of the equations in (12) and (13) can be understood when the nth derivative of the stationary equation (11) with respect to λ is calculated. Considering $n = 1$, we get

$$\frac{\partial}{\partial \lambda}(H_0 - \lambda)\Psi_0 = (H_0 - \lambda)\frac{\partial \Psi_0}{\partial \lambda} - \Psi_0 = 0. \tag{15}$$

Hence the function $\chi_0^{(1)} \equiv \frac{\partial \Psi_0}{\partial \lambda} + c_0^{(1)}\Psi_0 + d_0^{(1)}\widetilde{\Psi}_0$, where Ψ_0 and $\widetilde{\Psi}_0$ are solutions of the homogeneous equation (11), solves the inhomogeneous equation in (12) for $n = 1$.

As the first link in the chain of transformations we are going to discuss, we define L_1 such that it annihilates Ψ_0 and $\chi_0^{(1)}$, $(H_0 - \lambda_*)\Psi_0 = (H_0 - \lambda_*)\chi_0^{(1)} - \Psi_0 = 0$, and intertwines H_0 with the Hamiltonian H_1,

$$L_1\Psi_0 = L_1\chi_0^{(1)} = 0, \quad L_1 H_0 = H_1 L_1. \tag{16}$$

The eigenstates Ψ_1 and $\tilde{\Psi}_1$ of H_1, $(H_1 - \lambda)\Psi_1 = (H_1 - \lambda)\tilde{\Psi}_1 = 0$, can be written for any λ as

$$\widetilde{\Psi}_1 = L_1\widetilde{\Psi}_0, \quad \Psi_1 = \begin{cases} L_1\Psi_0 & \text{for } \lambda \neq \lambda_*, \\ L_1\chi_0^{(2)} & \text{for } \lambda = \lambda_*. \end{cases} \tag{17}$$

Here, we have utilized (14) and (16) which imply $L_1\chi_0^{(1)} = L_1(H_0 - \lambda_*)\chi_0^{(2)} = (H_1 - \lambda_*)L_1\chi_0^{(2)} = 0$. The Jordan states $\chi_1^{(1)}$ or $\widetilde{\chi}_1^{(1)}$ of H_1 associated with either Ψ_1 or $\widetilde{\Psi}_1$, $(H_1 - \lambda)\chi_1^{(1)} = \Psi_1$, $(H_1 - \lambda)\widetilde{\chi}_1^{(1)} = \widetilde{\Psi}_1$, are

$$\widetilde{\chi}_1^{(1)} = L_1\widetilde{\chi}_0^{(1)}, \quad \chi_1^{(1)} = \begin{cases} L_1\chi_0^{(1)} & \text{for } \lambda \neq \lambda_*, \\ L_1\chi_0^{(3)} & \text{for } \lambda = \lambda_*. \end{cases} \tag{18}$$

For the second link of the chain, we fix L_2 such that $L_2\Psi_1 = L_2\chi_1^{(1)} = 0$, $(H_1 - \lambda_*)\Psi_1 = 0$, and $(H_1 - \lambda_*)\chi_1^{(1)} = \Psi_1$. Then there exists H_2 such that the following relations hold true:

$$L_1 H_0 = H_1 L_1, \quad L_2 H_1 = H_2 L_2, \quad L_2 L_1 H_0 = H_1 L_2 L_1. \tag{19}$$

We can identify the chain of the two Darboux transformations with a second-order intertwining operator $\mathcal{L}_2 = L_2 L_1$ that directly intertwines H_0 with H_2. $\mathcal{L}_2$ can be defined directly as the operator that annihilates Ψ_0, $\chi_0^{(1)}$, $\chi_0^{(2)}$, $\chi_0^{(3)}$, see (17) and (18), and the coefficient at the highest derivative is normalized to one. The chain is illustrated in Fig. 1.

The Hamiltonians H_0, H_1 are not Hermitian in general. Hence the construction of the intertwining operator $L_1^\sharp$ which satisfies $H_0 L_1^\sharp = L_1^\sharp H_1$ is less straightforward than in the Hermitian case. Let us define $L_1^\sharp = \partial_x - U_x^\sharp (U^\sharp)^{-1}$ where $U^\sharp = (\widetilde{\Psi}_1, \widetilde{\chi}_1^{(1)})$. Then $L_1^\sharp$ intertwines H_1 and H_0,

$$H_0 L_1^\sharp = L_1^\sharp H_1, \quad L_1^\sharp \widetilde{\Psi}_1 = L_1^\sharp \widetilde{\chi}_1^{(1)} = 0. \tag{20}$$

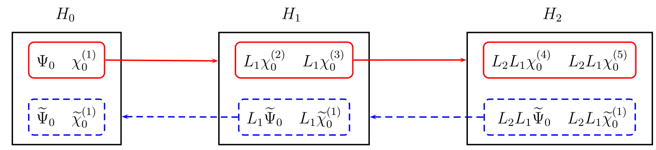

FIG. 1. Schematically illustrated action of L_1, L_2 (red arrows) and $L_1^\sharp$, $L_2^\sharp$ (dashed blue arrows) on the eigenvectors of the associated Jordan states of H_0 (left box), H_1 (center), and H_2 (right).

To prove this relation, notice first that $L_1^\sharp L_1 = (H_0 - \lambda_*)^2$ as the operators have the same kernel and the same coefficient at the highest derivative. Then, we can rewrite $[L_1^\sharp L_1, H_0] = 0$ as $(L_1^\sharp H_1 - H_0 L_1^\sharp)L_1 = 0$. Hence the relation (20) is valid on the space of all eigenstates and associated Jordan states of H_1. The operator $L_1^\sharp$ acts like the "inverse" operator to L_1; see Fig. 1 for illustration.

III. INTERTWINING OPERATORS OF HIGHER ORDER

One can construct a chain of Darboux transformations of arbitrary length n and identify it with a higher-order intertwining operator $\mathcal{L}_n$ such that

$$\mathcal{L}_n H_0 = H_n \mathcal{L}_n, \quad \mathcal{L}_n \Psi_0 = \mathcal{L}_n \chi_0^{(j)} = 0, \quad j = 1,2,\ldots,n-1. \tag{21}$$

We are going to write down the explicit form of H_n and $\mathcal{L}_n$. Let us fix the initial Hamiltonian H_0,

$$H_0 = \begin{pmatrix} -i\partial_x & q_0 \\ r_0 & i\partial_x \end{pmatrix}, \tag{22}$$

where $q_0(x)$ and $r_0(x)$ are arbitrary complex functions, and a set of $2n$ spinors,

$$S_{2n} = \{\Xi_1, \Xi_2, \ldots, \Xi_{2n-1}, \Xi_{2n}\}, \quad \Xi_i = \begin{pmatrix} f_i \\ g_i \end{pmatrix}. \tag{23}$$

Let us notice that the function in S_{2n} are sometimes called seed solutions in the literature. For the purpose of the analysis of the confluent Crum-Darboux transformation, we fix (23) as a sequence of an eigenstate Ψ_0 of H_0 corresponding to the eigenvalue λ_* and its associated Jordan states, $S_{2n} = \{\Psi_0, \chi_0^{(1)}, \chi_0^{(2)}, \ldots, \chi_0^{(2n-1)}\}$, $(H_0 - \lambda_*)\Psi_0 = 0$.

The operator $\mathcal{L}_n$ for $n = 1$ was discussed in the previous section. It can be written explicitly in terms of the seed solutions Ξ_1 and Ξ_2. It is given by

$$\mathcal{L}_1 = \begin{pmatrix} \partial_x + \frac{f_2' g_1 - f_1' g_2}{f_1 g_2 - f_2 g_1} & \frac{f_1' f_2 - f_1 f_2'}{f_1 g_2 - f_2 g_1} \\ \frac{g_1 g_2' - g_1' g_2}{f_1 g_2 - f_2 g_1} & \partial_x + \frac{f_2 g_1' - f_1 g_2'}{f_1 g_2 - f_2 g_1} \end{pmatrix}, \tag{24}$$

where the term $f_1 g_2 - f_2 g_1$ in the denominators corresponds to the Wronskian determinant for the Hamiltonian (22). It is not difficult to check that this operator annihilates Ξ_1 and Ξ_2,

$$\mathcal{L}_1 \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \mathcal{L}_1 \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = 0. \tag{25}$$

It intertwines H_0 with H_1 where the latter operator has the following explicit form:

$$H_1=\begin{pmatrix}-i\partial_x & q_0+2i\frac{f_1'f_2-f_1f_2'}{f_1g_2-f_2g_1}\\ r_0+2i\frac{g_1'g_2-g_1g_2'}{f_1g_2-f_2g_1} & i\partial_x\end{pmatrix}. \tag{26}$$

The second-order transformation $\mathcal{L}_2$ is fixed such that it annihilates the seed solutions Ξ_1, Ξ_2, Ξ_3, and Ξ_4. It can be factorized as $\mathcal{L}_2=L_2L_1=L_2\mathcal{L}_1$, where the operator L_2 satisfies $L_2(\mathcal{L}_1\Xi_3)=0$ and $L_2(\mathcal{L}_1\Xi_4)=0$. Its explicit form can be inferred from (24) by inserting the corresponding components of $\mathcal{L}_1\Xi_3$ and $\mathcal{L}_1\Xi_4$. Following the same steps [the factorization and the interative use of (24)], one could find $\mathcal{L}_n$ for any n. However, this, rather tedious, approach can be bypassed by writing $\mathcal{L}_n$ directly as

$$\mathcal{L}_n=\begin{pmatrix}\partial_x^n+\sum_{\ell=1}^{n-1}\frac{\det A[f^{(\ell)}]}{\det W}\partial_x^\ell & \sum_{\ell=1}^{n-1}\frac{\det A[g^{(\ell)}]}{\det W}\partial_x^\ell\\ \sum_{\ell=1}^{n-1}\frac{\det B[f^{(\ell)}]}{\det W}\partial_x^\ell & \partial_x^n+\sum_{\ell=1}^{n-1}\frac{\det B[g^{(\ell)}]}{\det W}\partial_x^\ell\end{pmatrix},\quad \mathcal{L}_n\Xi_j=0,\quad j=1,2,\ldots,2n. \tag{27}$$

In the definition (27), we have denoted

$$A=\begin{pmatrix}f^{(n)} & f^{(n-1)} & \ldots & f & g^{(n-1)} & \ldots & g' & g\\ f_1^{(n)} & f_1^{(n-1)} & \ldots & f_1 & g_1^{(n-1)} & \ldots & g_1' & g_1\\ f_2^{(n)} & f_2^{(n-1)} & \ldots & f_2 & g_2^{(n-1)} & \ldots & g_2' & g_2\\ \vdots & \vdots & & & & & & \\ f_{2n}^{(n)} & f_{2n}^{(n-1)} & \ldots & f_{2n} & g_{2n}^{(n-1)} & \ldots & g_{2n}' & g_{2n}\end{pmatrix}, \tag{28}$$

$$B=\begin{pmatrix}g^{(n)} & f^{(n-1)} & \ldots & f & g^{(n-1)} & \ldots & g' & g\\ g_1^{(n)} & f_1^{(n-1)} & \ldots & f_1 & g_1^{(n-1)} & \ldots & g_1' & g_1\\ g_2^{(n)} & f_2^{(n-1)} & \ldots & f_2 & g_2^{(n)} & \ldots & g_2' & g_2\\ \vdots & \vdots & & & & & & \\ g_{2n}^{(n)} & f_{2n}^{(n-1)} & \ldots & f_{2n} & g_{2n}^{(n)} & \ldots & g_{2n}' & g_{2n}\end{pmatrix}. \tag{29}$$

The matrix $A[h^{(i)}]$ ($B[h^{(i)}]$) in (27) is obtained from A (B) by substituting all the entries in the first row by zeros *except* the position $h^{(i)}$ where $h^{(i)}\to 1$. Here, h is either f or g. To get $A[f^{(1)}]$ in the case $n=2$, we have to substitute the first row $[f'',f',f,g',g]$ in the matrix A by $[0,1,0,0,0]$. The matrix W corresponds to the generalized Wronskian of the set (23),

$$W=\begin{pmatrix}f_1^{(n-1)} & f_1^{(n)} & \ldots & f_1 & g_1^{(n-1)} & \ldots & g_1' & g_1\\ f_2^{(n-1)} & f_2^{(n)} & \ldots & f_2 & g_2^{(n-1)} & \ldots & g_2' & g_2\\ \vdots & \vdots & & & & & & \\ f_{2n}^{(n-1)} & f_{2n}^{(n)} & \ldots & f_{2n} & g_{2n}^{(n-1)} & \ldots & g_{2n}' & g_{2n}\end{pmatrix}. \tag{30}$$

The Hamiltonian H_n, defined in Eq. (21), acquire the following form:

$$H_n=\begin{pmatrix}-i\partial_x & q_n\\ r_n & i\partial_x\end{pmatrix}, \tag{31}$$

where

$$q_n=q+2i\frac{\det D_q}{\det W},\quad r_n=r+2i\frac{\det D_r}{\det W}, \tag{32}$$

and

$$D_q=\begin{pmatrix}g_1^{(n-2)} & g_1^{(n-3)} & \ldots & g_1 & f_1^{(n)} & \ldots & f_1' & f_1\\ g_2^{(n-2)} & g_2^{(n-3)} & \ldots & g_2 & f_2^{(n)} & \ldots & f_2' & f_2\\ \vdots & \vdots & & & & & & \\ g_{2n}^{(n-2)} & g_{2n}^{(n-3)} & \ldots & g_{2n} & f_{2n}^{(n)} & \ldots & f_{2n}' & f_{2n}\end{pmatrix}, \tag{33}$$

$$D_r = \begin{pmatrix} g_1^{(n)} & g_1^{(n-1)} & \dots & g_1 & f_1^{(n-2)} & \dots & f_1' & f_1 \\ g_2^{(n)} & g_2^{(n-1)} & \dots & g_2 & f_2^{(n-2)} & \dots & f_2' & f_2 \\ \vdots & \vdots & & & & & & \\ g_{2n}^{(n)} & g_{2n}^{(n-1)} & \dots & g_{2n} & f_{2n}^{(n-2)} & \dots & f_{2n}' & f_{2n} \end{pmatrix}. \tag{34}$$

Then the operators H_0, H_n, and $\mathcal{L}_n$ as defined (22) and (27)–(34) satisfy the intertwining relation

$$\mathcal{L}_n H_0 = H_n \mathcal{L}_n. \tag{35}$$

The formulas (27)–(34) coincide with those that appear in the literature for the usual Crum-Darboux transformations, e.g. [46].

In the end of the section, let us finally comment on the construction of the inverse intertwining operator for the operator (22) and (31). The Hamiltonians (22) and (31) satisfy the following symmetry relation:

$$H_0 = \sigma_1 H_0^t \sigma_1, \quad H_n = \sigma_1 H_n^t \sigma_1\,, \tag{36}$$

where ^{t} stands for transposition. Transposing and multiplying (35) by σ_1 from both sides, we get $\sigma_1(H_n\mathcal{L}_n)^t\sigma_1 = \sigma_1(\mathcal{L}_n H)^t\sigma_1$ that can be written as

$$\left(\sigma_1\mathcal{L}_n^t\sigma_1\right)H_n = H_0\left(\sigma_1\mathcal{L}_n^t\sigma_1\right), \tag{37}$$

which is the inverse intertwining relation for H_0 and H_n.

IV. PT-SYMMETRIC DIRAC OPERATORS

In order to simulate a balanced gain and loss in the system, we require the Hamiltonian H_n to be PT-symmetric, where P is the spatial inversion ($PxP = -x$) and T is the time reversal operator ($TxT = x$, $TiT = -i$). Let us suppose that the Hamiltonian H_0 is PT-symmetric, i.e.,

$$[H_0, PT] = 0,$$

that implies $[PT,q_0] = [PT,r_0] = 0$. Both eigenstates Ψ, $\widetilde{\Psi}$ and the associated Jordan states $\chi_0^{(n)}$, $\tilde{\chi}_0^{(n)}$ of H_0 can be fixed in such a way they have definite parity ϵ with respect to the operator PT,

$$PT\Psi_0 = \epsilon\Psi_0, \quad PT\widetilde{\Psi}_0 = -\epsilon\widetilde{\Psi}_0, \tag{38}$$

$$PT\chi_0^{(n)} = \epsilon\,\chi_0^{(n)}, \quad PT\widetilde{\chi}_0^{(n)} = -\epsilon\,\widetilde{\chi}_0^{(n)}, \quad \epsilon^2 = 1. \tag{39}$$

To satisfy these relations, the Jordan states (12) and (13) are fixed as

$$\chi_0^{(n)} = \frac{\partial^n\Psi_0}{\partial\lambda^n} + \sum_{k=0}^{n-1}\left(c_k^{(n)}\chi_0^{(k)} + i\,d_k^{(n)}\widetilde{\chi}_0^{(k)}\right),$$
$$\widetilde{\chi}_0^{(n)} = \frac{\partial^n\widetilde{\Psi}_0}{\partial\lambda^n} + \sum_{k=0}^{n-1}\left(i\,\tilde{c}_k^{(n)}\chi_0^{(k)} + \tilde{d}_k^{(n)}\widetilde{\chi}_0^{(k)}\right), \tag{40}$$

where the constants $c_k^{(n)}$, $\tilde{c}_k^{(n)}$, $d_k^{(n)}$, and $\tilde{d}_k^{(n)}$ must acquire real values.

We fix S_{2n} in (23) such that it contains PT-symmetric functions Ψ_0 and $\chi_0^{(1)},\chi_0^{(2)},\dots,\chi_0^{(2n-1)}$; see (38) and (39). Using directly the formulas (30), (33), and (34) we can see that

$$[PT,\det W] = 0, \quad \{PT,\det D_q\} = 0, \quad \{PT,\det D_q\} = 0, \tag{41}$$

which imply that $[PT,q_n] = [PT,r_n] = 0$. As there also holds $PT\det A[\psi^{(i)}]PT = (-1)^{n-i}\det A[\psi^{(i)}]$, the Hamiltonian H_n is therefore PT-symmetric,

$$[H_n, PT] = 0. \tag{42}$$

The intertwining operator $\mathcal{L}_n$ either commutes or anticommutes with PT, dependent on the value of n. For any Ψ, $PT\Psi = \epsilon\Psi$, we have $PT\mathcal{L}_n\Psi = \epsilon(-1)^n\mathcal{L}_n\Psi$.

It is desirable that the potential terms of H_n, more specifically q_n and r_n, do not display any new singularities. It is the Wronskian determinant, $\det W$, which encodes singularities of both q_n and r_n. Thus, when we require the Hamiltonian H_n to be regular, the Wronskian determinant should be nodeless. Let us take $n = 1$ and fix $\chi_0^{(1)} = \partial_\lambda\Psi_0 + i d_0^{(1)}\widetilde{\Psi}_0$, where $d_0^{(1)} \in \mathbb{R}$; see (40). Then we have

$$\det W\left[\Psi_0,\chi_0^{(1)}\right] = \det W[\Psi_0,\partial_\lambda\Psi_0] + i d_0^{(1)}\det W[\Psi_0,\widetilde{\Psi}_0]. \tag{43}$$

When the first term is a PT-symmetric function with a nonvanishing imaginary part, we can fix d_1 such that (43) is nodeless. Indeed, the second term is a nonzero number (Ψ_0 and $\tilde{\Psi}_0$ are the two independent solutions of the stationary equation) which is purely real as there holds $PT\det W[\Psi_0,\tilde{\Psi}_0]PT = -\det W[\Psi_0,\tilde{\Psi}_0]$. It implies that the imaginary part of (43) is independent of $d_0^{(1)}$, whereas the real part of (43) contains $d_0^{(1)}$ as an additive constant. Hence one can always fix $d_0^{(1)}$ such that the zeros of the real and imaginary parts of (43) are mismatched, giving rise to a singularity-free determinant $\det W$.

V. INTENSITY OF THE ELECTRIC FIELD IN PT-SYMMETRIC GRATINGS: EXAMPLES

Let us apply here the confluent Crum-Darboux transformation on an explicit Hamiltonian H_0, giving as a result almost isospectral ones.

We identify the following initial Dirac Hamiltonian that, in quantum mechanics, would correspond to a free particle with possibly nonvanishing mass,

$$H_0 = -i\sigma_3\partial_x + \sigma_1\delta = \begin{pmatrix} -i\partial_x & \delta \\ \delta & i\partial_x \end{pmatrix}, \quad \delta \geqslant 0. \tag{44}$$

Its spectrum[5] consists of two bands of negative and positive energies $\varepsilon \in (-\infty,-\delta] \cup [\delta,\infty)$, divided by a energy gap of

[5]Here, we borrow terminology of quantum mechanics. The eigenfunctions associated with the allowed eigenvalues are either square integrable or correspond to the quantum mechanical scattering states.

magnitude 2δ. When comparing H_1 with the Hamiltonian in (4), we find that $\rho = \frac{q_1-r_1}{2}$, $\kappa = -\frac{q_1+r_1}{2}$, i.e.,

$$H_1 = -i\sigma_3\partial_x + \begin{pmatrix} 0 & q_1 \\ r_1 & 0 \end{pmatrix} = -i\sigma_3\partial_x + \frac{q_1-r_1}{2}\sigma_2 + \frac{q_1+r_1}{2}\sigma_1. \quad (45)$$

The eigenstates of H_1 can be obtained via (17). However, we are concerned only with the zero modes of H_1 as their components represent the intensities of the electric field of the counterrunning light in the grating.

The zero modes of H_1 differ qualitatively in dependence on the spectral gap of H_1 and the utilized transformation, based on the chosen seed states. When there is a gap, the two solutions of (4) either expand exponentially towards infinity or one of them can be square integrable. If there is no gap, the zero modes correspond to the threshold of the continuous spectrum. One of them can be a bounded function. Let us elaborate these two cases below.

A. Exponentially decaying electric field

For $\delta > 0$, it is convenient to parametrize the eigenfunctions and eigenvalues of H_0 in the following manner:

$$\Psi_\theta = \begin{pmatrix} \cosh(\delta\,\sin\theta\,x + i\mu) \\ \cosh(\delta\,\sin\theta\,x + i\theta + i\mu) \end{pmatrix},$$

$$\widetilde{\Psi}_\theta = \begin{pmatrix} \sinh(\delta\,\sin\theta\,x + i\mu) \\ \sinh(\delta\,\sin\theta\,x + i\theta + i\mu) \end{pmatrix}, \quad (46)$$

$$H_0\Psi_\theta = \delta\,\cos\theta\,\Psi_\theta, \quad H_0\widetilde{\Psi}_\theta = \delta\,\cos\theta\,\Psi_\theta, \quad (47)$$

where θ controls the magnitude of the eigenvalues while μ reflects the translational invariance of H_0. The states (46) have a definite parity with respect to PT as long as both θ and μ are simultaneously real or imaginary. In the latter case, they represent scattering states (linear combination of plane waves) with the eigenvalues $\lambda = \delta\,\cosh|\theta|$. The scattering states corresponding to $-\lambda$ can be obtained by multiplication of the former ones by σ_2 as there holds $\{H_0,\sigma_2\} = 0$.

Let us fix the following eigenstate and the associated Jordan state of H_0,

$$\Psi = \alpha\Psi_\theta + i\beta\widetilde{\Psi}_\theta, \quad \chi^{(1)} = \frac{\partial\Psi}{\partial\theta} + \nu\Psi_\theta + i\eta\widetilde{\Psi}_\theta. \quad (48)$$

The coefficients α, β, ν, and η have to be fixed such that definite PT-parity of the states is guaranteed. For real θ and μ, all α, β, ν, and η have to be real as well. When θ and μ are purely imaginary, we have to take α and ν real and β and η purely imaginary.

Substituting (46) and (48) into either (8) or (32) with $n = 1$, the components of the potential term of the new Hamiltonian H_1 can be written as

$$q_1 = \delta\left(-1 + 4\,\sin\theta\frac{[\alpha\,\cosh(x\delta\,\sin\theta + i\mu) + i\beta\,\sinh(x\delta\,\sin\theta + i\mu)]^2}{D}\right), \quad (49)$$

$$r_1 = \delta\left(-1 - 4\,\sin\theta\frac{[\alpha\,\cosh(x\delta\,\sin\theta + i\theta + i\mu) + i\beta\,\sinh(x\delta\,\sin\theta + i\theta + i\mu)]^2}{D}\right), \quad (50)$$

where

$$D = 2\alpha\beta\,\cosh(2x\delta\,\sin\theta + i\theta + 2i\mu) - (\alpha^2-\beta^2)i\,\sinh(2x\,\sin\theta + i\theta + 2i\mu) \quad (51)$$

$$+ (\alpha^2+\beta^2+2\alpha\eta-2\beta\nu)\sin\theta - ix(\alpha^2+\beta^2)\delta\,\sin 2\theta. \quad (52)$$

The Hamiltonian H_1 possesses a *bounded* zero mode provided that $\theta = \pi/2 + k\pi$, where k is an integer. For other values of θ, the zero modes are exponentially expanding. Fixing $\theta = \pi/2$, the zero modes of H_1 are $\Psi^{(1)} = L_1\chi^{(2)}_{\pi/2}$ and $\widetilde{\Psi}^{(1)} = L_1\widetilde{\Psi}_{\pi/2}$. The state $\Psi^{(1)}$ expands exponentially for $|x| \to \infty$, whereas $\widetilde{\Psi}^{(1)}$ is exponentially decaying. Its explicit form is

$$\widetilde{\Psi}^{(1)} = \begin{pmatrix} \frac{2\alpha\delta[\alpha\,\cosh(x\delta+i\mu)+i\beta\,\sinh(x\delta+i\mu)]}{\alpha^2+\beta^2+2\alpha\eta-2\beta\nu+(\alpha^2-\beta^2)\cosh(2x\delta+2i\mu)+2i\alpha\beta\,\sinh(2x\delta+2i\mu)} \\ \frac{2\alpha\delta[\beta\,\cosh(x\delta+i\mu)-i\alpha\,\sinh(x\delta+i\mu)]}{\alpha^2+\beta^2+2\alpha\eta-2\beta\nu+(\alpha^2-\beta^2)\cosh(2x\delta+2i\mu)+2i\alpha\beta\,\sinh(2x\delta+2i\mu)} \end{pmatrix}. \quad (53)$$

Substituting (49) and (50) into (45) and utilizing the definition of $n(x)$ below (4), we can find the explicit form of the refractive index which is presented in Fig. 2 together with (53).

B. Power-law decay of the electric field

When H_0 lacks the mass term, $\delta = 0$, the parametrization (46) is not suitable. Instead, we fix the eigenfunctions in the following manner:

$$\Psi_\lambda = (1,0)^t e^{i\lambda x}, \quad \widetilde{\Psi}_\lambda = (0,1)^t e^{-i\lambda x}, \quad (54)$$

$$H_0\Psi_\lambda = \lambda\,\Psi_\lambda, \quad H_0\widetilde{\Psi}_\lambda = \lambda\,\widetilde{\Psi}_\lambda. \quad (55)$$

We fix the kernel of the intertwining operator L_1 as

$$\Psi = \alpha\Psi_0 + \beta\widetilde{\Psi}_0, \quad \chi^{(1)} = \alpha\chi_0^{(1)} + \beta\widetilde{\chi}_0^{(1)} + \nu\Psi_0 + \eta\widetilde{\Psi}_0, \quad (56)$$

i.e., $L_1\Psi = L_1\chi^{(1)} = 0$. Ψ_0 and $\widetilde{\Psi}_0$ are zero modes of H_0. The associated Jordan states are $\chi_0^{(1)} = (ix,0)^t e^{i\lambda x}$ and $\widetilde{\chi}_0^{(1)} = (0,-ix)^t e^{-i\lambda x}$ and satisfy $H_0\chi_0^{(1)} = \Psi_0$, $H_0\widetilde{\chi}_0^{(1)} = \widetilde{\Psi}_0$. Here, the states have definite parity provided that the coefficients α, β, ν, and η are all either purely real or purely imaginary. Then the potential term of the Hamiltonian H_1 reads

$$q = \frac{2\alpha^2 e^{2i\lambda x}}{\alpha\eta - \beta\nu - 2ix\alpha\beta}, \quad r = -\frac{2\beta^2 e^{-2i\lambda x}}{\alpha\eta - \beta\nu - 2ix\alpha\beta}. \quad (57)$$

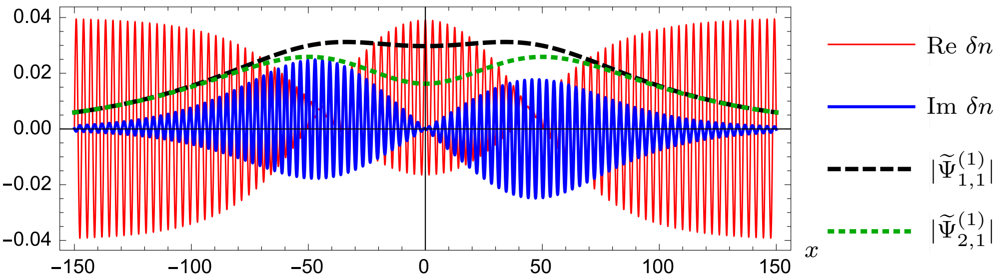

FIG. 2. Real (solid red) and imaginary (solid blue) part of the fluctuations δn of the (dimensionless) refractive index $n(x)$, $\delta n(x) = n_0 - n(x)$ (notice that the real part is even while the imaginary part is odd with respect to PT). Absolute values of the upper (dashed black) and lower (dotted green) components of $\widetilde{\Psi}^{(1)}$ (measured in volts). We fixed $\rho(x) = \sigma(x)$, $\theta = \pi/2$, $\delta = 0.02$, $\mu = 0.5$, $\eta = 1$, $\alpha = 1$, $\beta = 0$, $\nu = 0$, and $n_0 = 1$.

From the two zero modes of H_1 only one is bounded and reads

$$\widetilde{\Psi}^{(1)} = \frac{i\beta}{\alpha\eta - \beta\nu - 2ix\alpha\beta}\begin{pmatrix}\alpha \\ -\beta\end{pmatrix}, \tag{58}$$

and is obtained from $\widetilde{\Psi}^{(1)} \sim L_1\Psi_0 \sim L_1\widetilde{\Psi}_0$. Let us notice that the upper and lower components of $\widetilde{\Psi}^{(1)}$ coincide for $\beta = -\alpha$. The plot of the modulus of the components of $\widetilde{\Psi}^{(1)}$ together with the corresponding fluctuations of the refractive index are in Fig. 3.

VI. DISCUSSION AND OUTLOOK

In the current article, we focused on the analysis of the optical systems described in (4) with a PT-symmetric potential that simulates a balanced gain and loss of signal in the optical setting. In order to construct exactly solvable models with these properties, we presented the extension of the confluent Crum-Darboux transformation mechanism for the one-dimensional Dirac equation. The construction of the intertwining operator between Dirac Hamitonians as well as chains of Crum-Darboux transformations were discussed in detail. We presented the explicit form of the higher-order intertwining operator and the associated new Hamiltonian, see (27) and (31), which can be used not only for the confluent Crum-Darboux transformation but also for usual and combined ones. The machinery was illustrated in the preceding section, where the transformation was utilized for construction of exactly solvable Dirac Hamiltonians that corresponded to mild, complex fluctuations of the refractive index. We presented regimes where the intensity of the electric field is localized and decays either exponentially or as $\sim 1/x$.

It is worth noticing that, in [41], the confluent Crum-Darboux transformation coincided with the intertwining operator between two Schrödinger-type operators, obtained as the squares of the first-order Dirac Hamiltonians. Hence the construction presented there was limited to Dirac Hamiltonians with pseudoscalar potentials. In our case, the intertwining operator is constructed such that it intertwines directly the Dirac Hamiltonians and the potential term of the Hamiltonian is not restricted. In [21], an optical realization of PT-symmetric relativistic systems was discussed such that a more general ansatz for the electric field in Bragg grating was used; the functions u and v were allowed to depend on time. Then (4) acquires the form of time-dependent Dirac equation; see the footnote below (4). The confluent Darboux transformation was utilized in [40] for the analysis of the optical systems with invisible defects. It might be interesting to analyze similar systems described by Hamiltonians of the Dirac type with the use of our current results.

The framework presented here is based on the time-independent, one-dimensional Dirac Hamiltonian. The scheme presented here could be extended to the analysis of the time-dependent systems that are useful in the context of solutions of nonlinear integrable equations, e.g., equations of the integrable Ablowitz-Kaup-Newell-Segur hierarchy

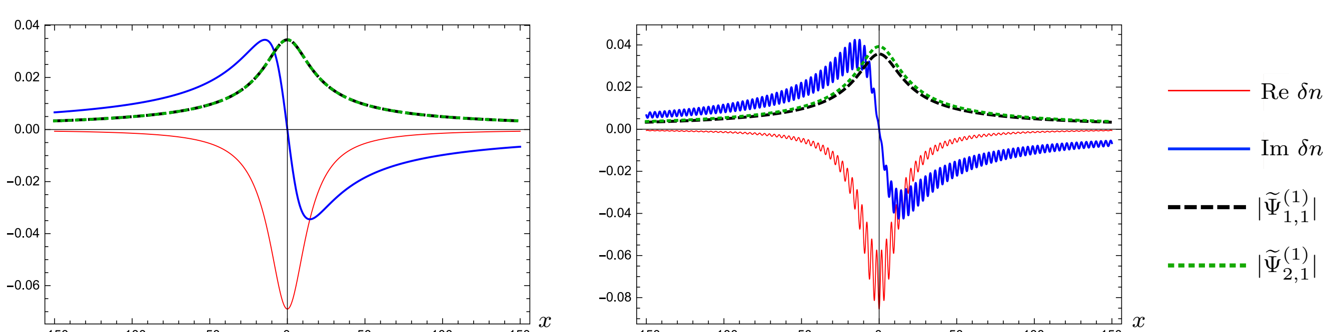

FIG. 3. Real (solid red) and imaginary (solid blue) part of the fluctuations δn of the (dimensionless) refractive index $n(x)$, $\delta n(x) = n_0 - n(x)$. Absolute values of the upper (dashed black) and lower (dotted green) components of $\widetilde{\Psi}^{(1)}$ (measured in volts). The two components of $\widetilde{\Psi}^{(1)}$, $H_1\widetilde{\Psi}^{(1)} = 0$, coincide for $\alpha = -\beta$. We fixed $\alpha = -1$, $\beta = 1$, $\nu = 18$, $\eta = 11$, and $n_0 = 1$ in the left figure while we fixed $\beta = 1.1$ in the right figure.

[47]. In this context, confluent Crum-Darboux transformations of the Schrödinger-type Hamiltonian were utilized to find PT-symmetric multisoliton solutions with nontrivial behavior [48].

The current results can be also employed for the analysis of the Schrödinger operators with matrix potentials. In [49], Crum-Darboux transformation for Dirac Hamiltonian was utilized for construction of Schrödinger operators with transparent matrix potential. It is worth noticing that these operators were analyzed recently in a different manner in [50,51]. However, such an analysis would go beyond the scope of the present article.

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[1] P. W. Milonni, in *Coherence Phenomena in Atoms and Molecules in Laser Fields*, edited by A. D. Bandrauk and S. C. Wallace (Plenum, New York, 1992), p. 45.
[2] H. Kogelnik and C. V. Shank, J. Appl. Phys. **43**, 2327 (1972).
[3] J. Carroll, J. Whiteaway, and D. Plumb, *Distributed Feedback Semiconductor Lasers* (INSPEC, London, 1998), Chap. 5.
[4] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. **103**, 093902 (2009).
[5] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. **6**, 192 (2010).
[6] A. Regensburger, C. Bersch, M.-A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Nature (London) **488**, 167 (2012).
[7] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[8] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. **7**, 1191 (2010).
[9] A. Mostafazadeh, Phys. Scr. **82**, 038110 (2010).
[10] A. Mostafazadeh, J. Phys. A **36**, 7081 (2003).
[11] C. M. Bender, J. Brod, A. Refig, and M. Reuter, J. Phys. A **37**, 10139 (2004).
[12] M. Znojil, Rendic. Circ. Mater. Palermo, Ser. II, Suppl. **72**, 211 (2004).
[13] H. F. Jones, J. Phys. A **45**, 135306 (2012).
[14] A. Mostafazadeh, Phys. Rev. A **90**, 023833 (2014); **90**, 055803 (2014).
[15] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, Phys. Rev. Lett. **106**, 213901 (2011).
[16] A. Mostafazadeh, Phys. Rev. A **91**, 063812 (2015).
[17] S. Longhi, J. Phys. A: Math. Theor. **44**, 485302 (2011).
[18] A. Mostafazadeh, Phys. Rev. A **87**, 012103 (2013).
[19] S. Klaiman, U. Gunther, and N. Moiseyev, Phys. Rev. Lett. **101**, 080402 (2008).
[20] E. M. Graefe and H. F. Jones, Phys. Rev. A **84**, 013818 (2011).
[21] S. Longhi, Phys. Rev. Lett. **105**, 013903 (2010).
[22] J. E. Sipe, L. Poladian, and C. Martijn de Sterke, J. Opt. Soc. Am. A **11**, 1307 (1994).
[23] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251**, 267 (1995).
[24] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991).
[25] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, Opt. Lett. **32**, 2632 (2007).
[26] A. Andrianov and F. Cannata, J. Phys. A **37**, 10297 (2004).
[27] S. M. Chumakov and K. B. Wolf, Phys. Lett. A **193**, 51 (1994).
[28] J. Bai and D. S. Citrin, Opt. Express **14**, 4043 (2006).
[29] M.-A. Miri, M. Heinrich, R. El-Ganainy, and D. N. Christodoulides, Phys. Rev. Lett. **110**, 233902 (2013).
[30] B. Midya, Phys. Rev. A **89**, 032116 (2014).
[31] S. Longhi, J. Opt. **17**, 045803 (2015).
[32] J. Pappademos, U. Sukhatme, and A. Pagnamenta, Phys. Rev. A **48**, 3525 (1993).
[33] T. A. Weber and D. L. Pursey, Phys. Rev. A **50**, 4478 (1994).
[34] D. J Fernández and E. Salinas-Hernández, J. Phys. A: Math. Gen. **36**, 2537 (2003).
[35] A. Schulze-Halberg, Eur. Phys. J. Plus **128**, 68 (2013).
[36] A. Contreras-Astorga and A. Schulze-Halberg, Ann. Phys. (NY) **354**, 353 (2015).
[37] D. Bermudez, D. J. Fernandez, and C. N. Fernandez-Garcia, Phys. Lett. A **376**, 692 (2012).
[38] J. F. Cariñena and M. S. Plyushchay, Phys. Rev. D **94**, 105022 (2016).
[39] S. Longhi and G. Della Valle, Ann. Phys. (NY) **334**, 35 (2013).
[40] F. Correa, V. Jakubsky, and M. S. Plyushchay, Phys. Rev. A **92**, 023839 (2015).
[41] A. Contreras-Astorga and A. Schulze-Halberg, J. Math. Phys. **55**, 103506 (2014).
[42] L. M. Nieto, A. A. Pecheritsin, and B. Samsonov, Ann. Phys. (NY) **305**, 151 (2003).
[43] Y. Charles Li and A. Yurov, *Lie-Bäcklund-Darboux Transformations* (International Press of Boston, Inc., Boston, 2014).
[44] C. Gu, A. Hu, and Z. Zhou, *Darboux Transformations in Integrable Systems* (Springer, New York, 2015).
[45] D. Bermudez, Ann. Phys. (NY) **364**, 35 (2016).
[46] A. V. Yurov, Phys. Lett. A **225**, 51 (1997).
[47] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. **53**, 249 (1974).
[48] F. Correa and A. Fring, J. High Energy Phys. 09 (2016) 008.
[49] F. Correa and V. Jakubsky, Phys. Rev. D **90**, 125003 (2014).
[50] A. A. Andrianov and A. V. Sokolov, Theor. Math. Phys. **186**, 2 (2016).
[51] A. A. Andrianov and A. V. Sokolov, Phys. Lett. A **379**, 279 (2015).