Coherence depletion in the Grover quantum search algorithm

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(Received 13 November 2016; published 7 March 2017)

We investigate the role of quantum coherence depletion (QCD) in the Grover search algorithm (GA) by using several typical measures of quantum coherence and quantum correlations. By using the relative entropy of coherence measure ($\mathcal{C}_r$), we show that the success probability depends on the QCD. The same phenomenon is also found by using the l_1 norm of coherence measure ($\mathcal{C}_{l_1}$). In the limit case, the cost performance is defined to characterize the behavior about QCD in enhancing the success probability of GA, which is only related to the number of searcher items and the scale of the database, regardless of using $\mathcal{C}_r$ or $\mathcal{C}_{l_1}$. In the generalized Grover search algorithm (GGA), the QCD for a class of states increases with the required optimal measurement time. In comparison, the quantification of other quantum correlations in GA, such as pairwise entanglement, multipartite entanglement, pairwise discord, and genuine multipartite discord, cannot be directly related to the success probability or the optimal measurement time. Additionally, we do not detect pairwise nonlocality or genuine tripartite nonlocality in GA since Clauser-Horne-Shimony-Holt inequality and Svetlichny's inequality are not violated.

DOI: 10.1103/PhysRevA.95.032307

I. INTRODUCTION

Quantum mechanics provides some distinctive computational resources that can be utilized to make quantum algorithms superior to some classical algorithms [1]. The origin of this speed-up in quantum computational processes has attracted many research attentions. For instance, Jozsa and Linden demonstrated that, for pure states, entanglement is needed for some certain quantum computations if the calculated results cannot be simulated classically [2]. In addition, Vidal showed that, under arbitrary bipartite cut and at all times, if the state of the quantum computer has Schmidt rank polynomial in n, then the quantum computation can be simulated classically [3]. However, a quantum computation using only separable states still surpasses classical computations [3]. The celebrated Knill-Gottesman theorem tells us that some quantum algorithms using highly entangled states can also be efficiently simulated classically [4]. Thus, the existence of entanglement is not sufficient for exponential quantum speed-up [5]. Besides entanglement, quantum discord, as another type of quantum correlations, is equally vital in quantum algorithms. For example, in the some settings of one-way algorithm for remote state preparation, discord does not vanish while entanglement vanishes, when the noise is maximal and fidelity drops to its minimum value [6]. Moreover, the effects of quantum resources, such as entanglement, discord, and nonlocality, on the process of quantum key distribution (QKD) have received widespread attention and scrutiny [7–9].

Coherence, as a quantum property from the quantum states superposition principle [10], has been widely studied in quantum information processing [11–13]. A rigorous framework for quantifying the coherence was proposed by Baumgratz *et al.* in Ref. [14]. Recently, it has been proved that coherence can be converted to other valued quantum resources, such as entanglement and discord, by suitable operations [15–17]. To some extent, coherence is as important as entanglement or discord. Moreover, coherence also exists in a single system without any correlations. What role does coherence play in quantum algorithms?

Recently, this topic has generated a great deal of interest. Hillery declared that coherence can be viewed as a resource in the Deutsch-Jozsa algorithm in the sense that a bigger amount of coherence decreases the failure of this algorithm [18]. For deterministic quantum computation with one qubit (DQC1), Matera *et al.* showed that the precision of this algorithm is directly related to the recoverable coherence [19,20]. At the heart of quantum algorithms, there lies another fundamental algorithm, Grover search algorithm (GA) [21,22]. GA was introduced for accelerating the search process [23]. It is believed that multipartite entanglement is necessary for GA to achieve the speed-up [2]. To investigated properties of entanglement, different measures of entanglement, such as concurrence and geometric measure of entanglement, have been attempted in GA [24–29]. However, the role of entanglement is not yet fully demonstrated; in particular, the quantity of entanglement is not directly related with the success probability in GA [30]. On the other hand, the behavior of quantum discord, as a nonclassical correlation beyond entanglement, has been proved to be similar to the entanglement in GA [31]. It is worth noting that coherence is potentially a more fundamental quantum resource than entanglement and discord [32]. Much attention has been paid in this direction [33–38]. Will coherence display unique characteristics, which are different from entanglement or discord in GA? To clarify the role of coherence, we investigate coherence depletion in GA and in the generalized Grover search algorithm (GGA). Other quantum correlations are also discussed for comparison.

This paper is organized as follows. In Sec. II, we briefly review GA and study its coherence dynamics of the whole

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n-qubit system in the cases of any solutions to the search problem by using two different coherence measures, namely, the relative entropy and the l_1 norm. In additional, the relationship between quantum coherence depletion (QCD) and success probability in GA is also discussed. In Sec. III, we introduce GGA and investigate the relationship between QCD of a class of states and optimal measurement time in GGA. In Sec. IV, we consider dynamics of entanglement, discord, and nonlocality for any two qubits in the simplest situation of single solution to Grover search. Moreover, multipartite entanglement, genuine quantum correlation, and genuine tripartite nonlocality are also discussed. Finally, the main results are summarized in Sec. V.

II. COHERENCE DEPLETION IN STANDARD GROVER SEARCH ALGORITHM

The first step of GA is to initialize the n-qubit database to an equally weighted superposition of all computational basis states $|\psi_0\rangle = 1/\sqrt{2^n}\sum_{x=0}^{2^n-1}|x\rangle$, which can be realized by projecting a prepared pure state $|0,\ldots,0\rangle$ to local Hadamard gates $H^{\otimes n}$ where $H = (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)/\sqrt{2}$. It should be pointed that the initialized n-qubit database is a maximally coherent state with $N = 2^n$ equiprobable items $|x\rangle$ and our goal is to obtain desired items (in the following we call them "solutions" to the GA) from it with maximum probability after the GA. The initialized database can be written in a more convenient form,

$$|\psi_0\rangle = \sqrt{\frac{j}{N}}|X\rangle + \sqrt{\frac{N-j}{N}}|X^{\perp}\rangle, \tag{1}$$

where j represents the number of solutions and $|X\rangle = 1/\sqrt{j}\sum_{x_s}|x_s\rangle$ $[|X^{\perp}\rangle = 1/\sqrt{N-j}\sum_{x_n}|x_n\rangle]$ is constructed by states $|x_s\rangle$ $[|x_n\rangle]$ that are solutions [nonsolutions] to the GA. It is easy to confirm that both $|X\rangle$ and $|X^{\perp}\rangle$ are orthonormal. The next step is to apply Grover operation G repeatedly (called iteration) to improve proportion of solutions gradually. The Grover operation, $G = DO$, is composed of oracle $O = I - 2|X\rangle\langle X|$ and an inversion about average operation $D = 2|\psi_0\rangle\langle\psi_0| - I$ [21]. After r iterations of the Grover operation G, the global state has the following form [1,31]

$$|\psi_r\rangle \equiv G^r|\psi_0\rangle = \sin\alpha_r|X\rangle + \cos\alpha_r|X^{\perp}\rangle, \tag{2}$$

with $\alpha_r = (r + 1/2)\alpha$ and $\alpha = 2\arctan\sqrt{j/(N-j)}$. Note that $|\psi_r\rangle$ is also a pure state since G is unitary and initial state $|\psi_0\rangle$ is a pure state. The above processes are summarized in Fig. 1: (1) Initialize the n-qubit database to $|\psi_0\rangle$. (2) Oracle O reflects the vector $|\psi_0\rangle$ according to $|X^{\perp}\rangle$ and then operation D reflects the vector $O|\psi_0\rangle$ according to $|\psi_0\rangle$. Therefore, the role of Grover operation G is to rotate the vector before iteration anticlockwise by an angle α.

The final step (3) is that measure $|\psi_r\rangle$ to get $|X\rangle$ with maximum probability. The success probability is expressed as

$$P(r) = \sin^2\alpha_r. \tag{3}$$

Therefore, the optimal time to stop iteration is $r_{\rm opt} = CI[(\pi - \alpha)/(2\alpha)]$, where $CI[x]$ denotes the closest integer to x. In the following, we will confine our discussion to $0 \leqslant r \leqslant r_{\rm opt}$.

Quantum coherence describes the capability of a quantum state to exhibit quantum interference phenomena. The first rigorous framework to quantify the coherence was built by Baumgratz *et al.* in Ref. [14]. Based on this work, a number of coherence measures, such as the relative entropy of coherence, the l_1 norm of coherence, the Tsallis relative α entropy of coherence, and the coherence of formation [14,39,40], have been proposed. Recently, an interesting phenomenon has been founded in Ref. [41]: All measures of coherence are frozen for an initial state in a strictly incoherent channel if and only if the relative entropy of coherence is frozen for the state. It means that the relative entropy of coherence is an excellent coherence measure. Hence we choose it to investigate the GA and also calculate the l_1 norm of coherence for comparison. In this section, we consider coherence dynamics under the general case of any j solutions. According to Eq. (2), the density matrix of state generated by GA can be written as

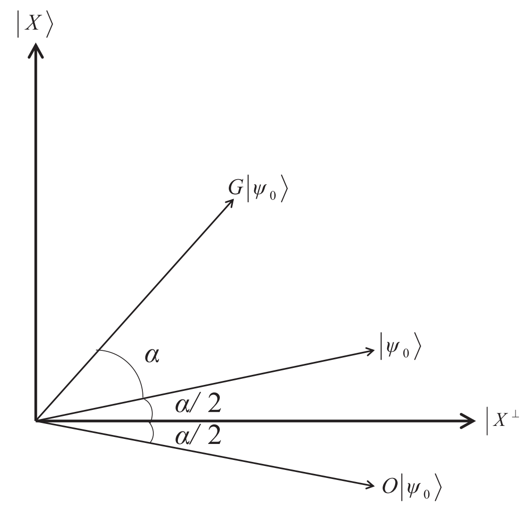

FIG. 1. An illustration to show the first two steps of GA. First, initialize the n-qubit database to $|\psi_0\rangle$; second, O reflects $|\psi_0\rangle$ according to $|X^{\perp}\rangle$ and D reflects $O|\psi_0\rangle$ according to $|\psi\rangle$. Consequently, one whole iteration G turns the vector before iteration anticlockwise by an angle α.

$$\rho(r) = \frac{a^2}{j}\sum_{x_s,y_s}|x_s\rangle\langle y_s| + b^2\sum_{x_n,y_n}|x_n\rangle\langle y_n| + \frac{ab}{\sqrt{j}}\left[\sum_{x_s}\sum_{y_n}(|x_s\rangle\langle y_n| + |y_n\rangle\langle x_s|)\right], \tag{4}$$

where subscripts s and n denote that they are solutions and nonsolutions, respectively. Here $a = \sin\alpha_r$ and $b = 1/\sqrt{N-j}\cos\alpha_r$ are brought in for convenience.

A. The relative entropy of coherence

The definition of relative entropy of coherence is [14]

$$\mathcal{C}_r(\rho) = \min_{\delta\in\mathcal{I}} S(\rho\|\delta), \tag{5}$$

where $S(\rho\|\delta) = \mathrm{Tr}(\rho\log_2\rho - \rho\log_2\delta)$ is the quantum relative entropy and $\mathcal{I}$ denotes a set of incoherent quantum states whose density matrices are diagonal in the calculational basis. This formula can be rewritten as a closed form [14], avoiding the minimization

$$\mathcal{C}_r(\rho) = S(\rho_{\rm diag}) - S(\rho), \tag{6}$$

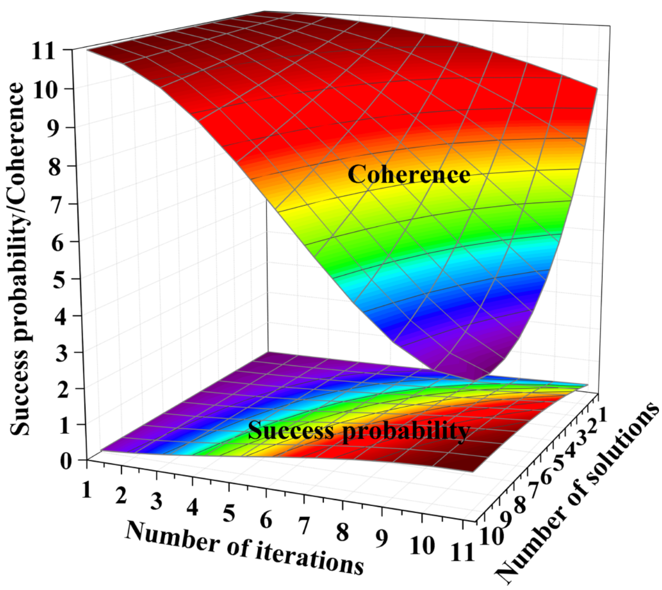

FIG. 2. Evolution of coherence in GA for the whole 11-qubit system with j (from 1 to 10) solutions.

where $\rho_{\text{diag}} = \sum_i \rho_{ii}|i\rangle\langle i|$ and $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy.

Substituting Eq. (4) into Eq. (6), we obtain the coherence dynamics of n-qubit

$$\mathcal{C}_r(\rho) = H(a^2) + \log_2(N - j) + a^2 \log_2 \frac{j}{N - j}, \tag{7}$$

where $H(x) = -x \log_2 x - (1 - x)\log_2(1 - x)$ is the binary Shannon entropy function. Note that the relative entropy of coherence is independent of the choices of solutions. In other words, it only depends on the number of solutions j since $S(\rho) = 0$ and $S(\rho_{\text{diag}})$ is only connected with the diagonal elements of $\rho(r)$. From Eq. (7), we have

$$\frac{d\mathcal{C}_r(\rho)}{dr} = \log_2 \frac{j(1 - a^2)}{(N - j)a^2} \sin(2\alpha_r)\alpha \leqslant 0 \tag{8}$$

for $0 \leqslant r \leqslant r_{\text{opt}}$ due to $a(r) = \sin \alpha_r \geqslant a(0) = \sqrt{j/N}$, which means that $\mathcal{C}_r(\rho)$ is a decreasing function of r. On the contrary, the success probability $P(r)$ is a increasing function for $0 \leqslant r \leqslant r_{\text{opt}}$. Moreover, the coherence achieves the minimal value while the probability of success reaches the maximal value 1. That is to say, the improvement of success probability depends on the QCD; see Fig. 2.

It is possible to express the coherence $\mathcal{C}_r(\rho)$ as a function of the success probability P. Due to the fact that $P = a^2$, the coherence becomes

$$\mathcal{C}_r(\rho) = H(P) + \log_2(N - j) + P \log_2 \frac{j}{N - j}. \tag{9}$$

Actually, the GA is usually applied in the situation of a few solutions in a huge database. Under this condition ($j \ll N$ and $N \gg 1$), $H(P)$ can be omitted compared with $\log_2(N - j)$ and then Eq. (9) takes the following form

$$\mathcal{C}_r(\rho) \simeq -P \log_2 \frac{N}{j} + \log_2 N, \tag{10}$$

which is a linear function of P. The ability of coherence in enhancing the success probability can be quantified as cost performance w,

$$w = -\frac{dP}{d\mathcal{C}_r} = \frac{1}{\log_2 \frac{N}{j}}. \tag{11}$$

Clearly, the cost performance is related to a constant j/N, which represents the ratio of number of solutions to the scale of database.

B. The l_1 norm of coherence

The l_1 norm of coherence is a very intuitive quantification which comes from a simple fact that coherence is linked with the off-diagonal elements of considered quantum states. The expression of the l_1 norm of coherence is defined as [14]

$$\mathcal{C}_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|. \tag{12}$$

By employing this equation, we have the coherence dynamics in GA

$$\mathcal{C}_{l_1}(\rho) = (\sqrt{j} \sin \alpha_r + \sqrt{N - j} \cos \alpha_r)^2 - 1, \tag{13}$$

when $0 \leqslant r \leqslant r_{\text{opt}}$. Using Eq. (3), the l_1 norm of coherence can be rewritten as a function of P

$$\mathcal{C}_{l_1}(\rho) = [\sqrt{jP} + \sqrt{(N - j)(1 - P)}]^2 - 1. \tag{14}$$

In the asymptotic limits $j \ll N$ and $N \gg 1$, the l_1 norm of coherence takes the simple form

$$\mathcal{C}_{l_1}(\rho) \simeq -NP + N. \tag{15}$$

The same phenomenon that the success probability depends on the QCD is also existed under the l_1 norm of coherence measure. From this perspective, we say that the QCD is of great significance in GA, and the cost performance w equals to $1/N$.

III. COHERENCE DEPLETION IN GENERALIZED GROVER SEARCH ALGORITHM

In Ref. [42], the Grover search algorithm was generalized to deal with arbitrary initial complex amplitude distributions. The only difference between GA and generalized Grover search algorithm (GGA) is that there is no initialization step in GGA. Thus the GGA includes the following steps: (1) Use any initial amplitude distribution of a system which does not need to be initialized to the uniform distribution. (2) Repeat the following two steps r times: (i) Rotate the solutions by a phase of π radians. (ii) Rotate all states according to the average amplitude of all states by π. (3) Measure the resulting state in the optimal time r_{opt}.

We denote the amplitudes of solutions by $k_i(r), i = 1, \ldots, j$, and nonsolutions by $l_i(r), i = j + 1, \ldots, N$. Let the average amplitudes over solutions and over nonsolutions be represented respectively by

$$\bar{k}(r) = \frac{1}{j} \sum_{i=1}^{j} k_i, \tag{16}$$

$$\bar{l}(r) = \frac{1}{N - j} \sum_{i=j+1}^{N} l_i. \tag{17}$$

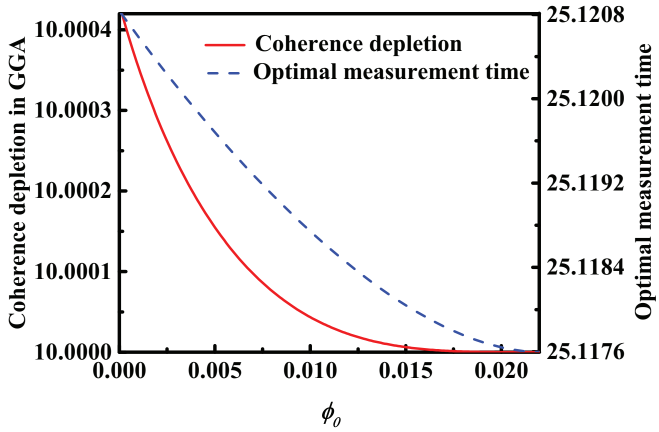

FIG. 3. Coherence depletion $\Delta\mathcal{C}_r^{\mathrm{GGA}}$ vs optimal measurement time $r_{\mathrm{opt}}^{\mathrm{GGA}}$ for the initial states of $|\phi_0\rangle$ in GGA.

The success probability in optimal measurement time was founded by Biham *et al.* [42],

$$P_{\max}^{\mathrm{GGA}} = 1 - (N-j)\sigma_l^2. \tag{18}$$

where $\sigma_l^2 = 1/(N-j)\sum_{i=j+1}^{N} |l_i(r_{\mathrm{opt}}^{\mathrm{GGA}}) - \bar{l}(r_{\mathrm{opt}}^{\mathrm{GGA}})|^2$, and the optimal measurement time is given by

$$r_{\mathrm{opt}}^{\mathrm{GGA}} = \left(\frac{\pi}{2} - \beta\right)\Big/\omega \tag{19}$$

with $\cos\omega = (1 - 2j/N)$ and $\tan\beta = \sqrt{j/(N-j)}\,\bar{k}(0)/\bar{l}(0)$. The dynamics of the amplitudes are described by [42]

$$k_i(r) = \bar{k}(r) + k_i(0) - \bar{k}(0), \tag{20}$$

$$l_i(r) = \bar{l}(r) + (-1)^r[l_i(0) - \bar{l}(0)]. \tag{21}$$

Now let us consider the initial state

$$|\phi_0\rangle = \phi_0|0\rangle + \phi_1|1\rangle + \frac{1}{\sqrt{N}}\sum_{x=2}^{N-1}|x\rangle, \tag{22}$$

where $\phi_0^2 + \phi_1^2 = 2/N$, $\phi_0, \phi_1 \in \mathbb{R}$ and $|0\rangle, |1\rangle$ are solutions. Without loss of generality, we assume that $\phi_0 \leqslant \phi_1$. From Eqs. (18) and (22), it follows immediately that the success probability of these kind of states can reach the maximum value 1, and corresponding states can be written as

$$|\phi_{\mathrm{opt}}\rangle = k_1|0\rangle + k_2|1\rangle, \tag{23}$$

with $k_1 = \sqrt{(N-2)/(2N) + 1/4(\phi_0+\phi_1)^2} + 1/2(\phi_0 - \phi_1)$ and $k_2 = \sqrt{(N-2)/(2N) + 1/4(\phi_0+\phi_1)^2} - 1/2(\phi_0 - \phi_1)$. By using Eqs. (6), (22), and (23), the QCD of these kind of states in GGA is

$$\begin{aligned}\Delta\mathcal{C}_r^{\mathrm{GGA}} &\equiv \mathcal{C}_r(|\phi_0\rangle\langle\phi_0|) - \mathcal{C}_r(|\phi_{\mathrm{opt}}\rangle\langle\phi_{\mathrm{opt}}|)\\ &= -\phi_0^2\log_2\phi_0^2 - \phi_1^2\log_2\phi_1^2\\ &\quad + \frac{N-2}{N}\log_2 N - H(k_1^2),\end{aligned} \tag{24}$$

where H is the binary Shannon entropy function. Both $\Delta\mathcal{C}_r^{\mathrm{GGA}}$ and $r_{\mathrm{opt}}^{\mathrm{GGA}}$ are increased with the decrease of ϕ_0; see Fig. 3.

It means that the optimal measurement time depends on the QCD for this kind of states in GGA. In other words, as the optimal measurement time is smaller, the QCD is smaller also.

IV. OTHER QUANTUM CORRELATIONS IN GROVER SEARCH ALGORITHM

In this section, we only consider the simplest situation of single solution ($j = 1$) for convenience, which has the benefit of capturing the essence of other quantum resource dynamics in GA. Without loss of generality, we assume that the solution is located at $|0\rangle$ and the density matrix of states generated by GA [Eq. (4)] has the following form:

$$\rho(r) = \begin{pmatrix} a^2 & ab & ab & ab & \cdots\\ ab & b^2 & b^2 & b^2 & \cdots\\ ab & b^2 & b^2 & b^2 & \cdots\\ ab & b^2 & b^2 & b^2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}_{N\times N}. \tag{25}$$

A. Entanglement in Grover search

Entanglement is widely considered as the main undertaker for quantum computational speed-up, though the role of entanglement is not clear. Here we use concurrence, a widely accepted entanglement measure, to investigate the behavior of entanglement during the GA.

The reduced matrix of any two qubits takes the following form:

$$\rho_2 = \begin{pmatrix} \Omega_0 & \Omega_1 & \Omega_1 & \Omega_1\\ \Omega_1 & \Omega_2 & \Omega_2 & \Omega_2\\ \Omega_1 & \Omega_2 & \Omega_2 & \Omega_2\\ \Omega_1 & \Omega_2 & \Omega_2 & \Omega_2\end{pmatrix}, \tag{26}$$

where $\Omega_0 = a^2 + (\frac{N}{4} - 1)b^2$, $\Omega_1 = ab + (\frac{N}{4} - 1)b^2$, and $\Omega_2 = \frac{N}{4}b^2$. The concurrence of arbitrary two-qubit states is defined in Ref. [43] and is calculated as follows:

$$E_2(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{27}$$

where $\lambda_i s$ are square roots of the eignvalues of matrix $\rho\tilde{\rho}$ in decreasing order, $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$. Here $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where σ_y is Pauli matrix $(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix})$, and ρ^* is the conjugation of ρ. According to Eqs. (26) and (27), the expression of concurrence between any two qubits ρ_2 in GA can be obtained [24] as

$$E_2(\rho_2) = 2|\Omega_1 - \Omega_2| = 2|ab - b^2|. \tag{28}$$

The behavior of pairwise entanglement in the case of $n = 11$ is displayed in Fig. 4. The pairwise entanglement first increases to the maximal value and then decreases to almost zero when the optimal number of iterations is reached.

Now let us consider the multipartite entanglement of the n-qubit system, which may better depict the behavior of $P(r)$. The concurrence of n-qubit states is introduced in Ref [44]

$$E_n(\psi) = \frac{2}{\sqrt{N}}\sqrt{(N-2)\langle\psi|\psi\rangle^2 - \sum_\beta \mathrm{Tr}\rho_\beta^2}, \tag{29}$$

where $N = 2^n$ and β labels $(N-2)$ different reduced density matrices; i.e., there are C_N^k different terms when tracing over k different subsystems from the n-qubit system. Note that the concurrence for n-qubit states used is upper bound. From Eq. (25), we have the reduced matrix for any k-qubit:

$$\rho_k = \begin{pmatrix} a^2+(2^{n-k}-1)b^2 & ab+(2^{n-k}-1)b^2 & ab+(2^{n-k}-1)b^2 & ab+(2^{n-k}-1)b^2 & \cdots \\ ab+(2^{n-k}-1)b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & \cdots \\ ab+(2^{n-k}-1)b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & \cdots \\ ab+(2^{n-k}-1)b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & 2^{n-k}b^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{2^k\times 2^k}. \quad (30)$$

Thereby, concurrence of the whole n-qubit system can be expressed as

$$E_n = \frac{2}{\sqrt{N}}\sqrt{(N-2)-\sum_{k=1}^{n-1} C_n^k \mathrm{Tr}\rho_k^2}. \quad (31)$$

By substituting Eq. (30) into the above equation, we have

$$\begin{aligned} E_n = \frac{2}{\sqrt{2^n}}[&2^n-2-(4\times 3^n-2^{n+3}+4)a^2b^2 \\ &-(8^n+4\times 3^n-3\times 2^{2n+1}+3\times 2^n-2)b^4 \\ &-(2^n-2)a^4-4(4^n-2\times 3^n+2^n)ab^3]^{\frac{1}{2}}. \end{aligned} \quad (32)$$

By virtue of this equation, we present the behavior of multipartite entanglement of the n-qubit system in the case that $n = 11$, which is similar with the pairwise entanglement (see Fig. 4).

B. Discord in Grover search

Discord was introduced in Ref. [45] to quantify quantum correlation, which is viewed as the difference between total correlation and classical correlation,

$$\mathcal{D}(\rho) = \mathcal{I}(\rho) - \mathcal{C}(\rho), \quad (33)$$

where $\mathcal{I}$ and $\mathcal{C}$ represent the total correlation and classical correlation, respectively. In Ref. [46], the total correlation between two systems A and B is defined by the minimal amount of noise, which is wanted to destroy all the correlation between them. The total correlation is equal to the quantum mutual information,

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (34)$$

where $\rho_{A(B)} = \mathrm{Tr}_{B(A)}\rho_{AB}$. The classical correlation was proposed in Ref. [47] as the maximum information we can obtain from A by measuring B. Under projective measurements $\{\prod_i\}$, the classical correlation can be written as

$$\mathcal{C}(\rho) = \max_{\{\prod_i\}}\{S(\rho_A) - \sum_i p_i S(\rho_{A|i})\}, \quad (35)$$

where $p_i = \mathrm{Tr}_{AB}(I\otimes\prod_i)\rho_{AB}(I\otimes\prod_i)$ and $\rho_{A|i} = 1/p_i \mathrm{Tr}_B(I\otimes\prod_i)\rho_{AB}(I\otimes\prod_i)$. Put Eqs. (34) and (35) into Eq. (33), and then

$$\mathcal{D}(\rho) = \min_{\{\prod_i\}}\sum_i[p_i S(\rho_{A|i}) + S(\rho_B) - S(\rho_{AB})]. \quad (36)$$

We choose the bipartite discord to analyze discord dynamics in GA. The projective measurement can be parameterized via $0\leqslant\theta\leqslant\pi$ and $0\leqslant\phi\leqslant 2\pi$ in the form of $\{\cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle, e^{-i\phi}\sin\theta|0\rangle - \cos\theta|1\rangle\}$. Using the exact diagonalization method, we calculate pairwise discord in the case of 11-qubit system. Figure 5 shows that the behavior of pairwise discord is similar to the entanglement.

In Ref. [48], a quantifier for genuine multipartite quantum correlation was proposed based on relative entropy. For tripartite pure states ρ_{ABC}, the genuine quantum correlation $\mathcal{D}^{(3)}$ is equal to half of genuine total correlation $T^{(3)}$, namely

$$\mathcal{D}^{(3)}(\rho_{\mathrm{ABC}}) = \frac{T^{(3)}(\rho_{\mathrm{ABC}})}{2}. \quad (37)$$

Here $T^{(3)}$ is defined as the difference between total correlation T and the maximum among the bipartite correlation $T^{(2)}$

$$T^{(3)}(\rho_{\mathrm{ABC}}) = T(\rho_{\mathrm{ABC}}) - T^{(2)}(\rho_{\mathrm{ABC}}), \quad (38)$$

where $T(\rho_{\mathrm{ABC}}) = S(\rho_A)+S(\rho_B)+S(\rho_C)-S(\rho_{\mathrm{ABC}})$ and $T^{(2)}(\rho_{\mathrm{ABC}}) = \max\{\mathcal{I}(\rho_{AB}),\mathcal{I}(\rho_{AC}),\mathcal{I}(\rho_{BC})\}$. Defined in this way, $T^{(3)}$ is the shortest distance to a state without tripartite correlations based on relative entropy. For pure states of n

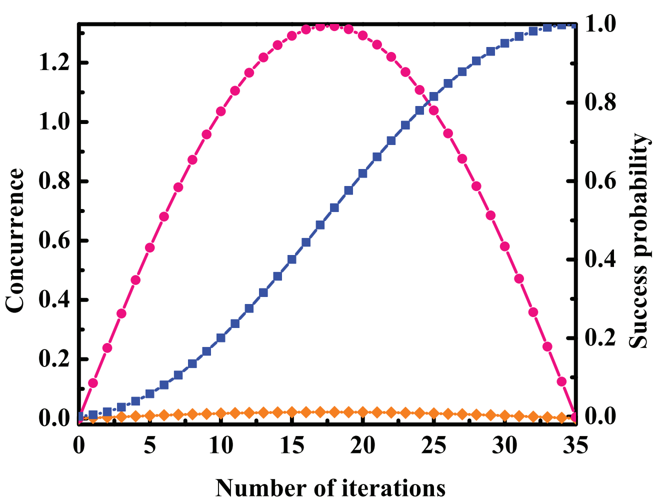

FIG. 4. The evolutions for the entanglement in the case of 11-qubit system. The pairwise entanglement is depicted by orange diamonds while the entanglement of the whole 11-qubit system is shown by pink points. The blue squares represent the success probability.

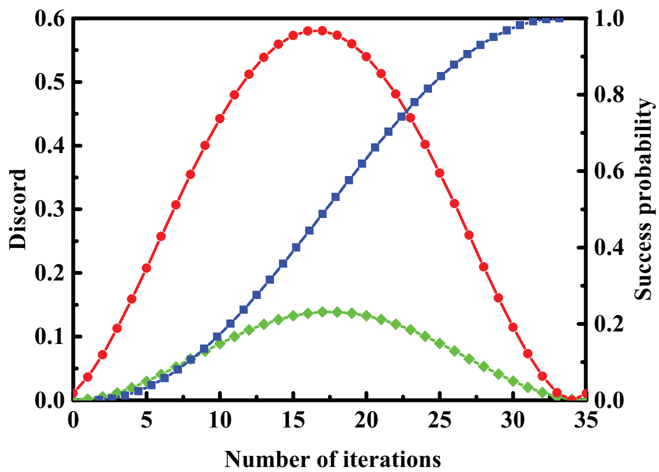

FIG. 5. The evolutions for the discord in the case of 11-qubit system. The pairwise discord is depicted by green diamonds while the genuine quantum correlation of the whole 11-qubit system is shown by red points. The blue squares represent the success probability.

qubits, genuine n-partite quantum correlation can also be expressed as [48]

$$\mathcal{D}^{(n)}(\rho) = \frac{T^{(n)}(\rho)}{2}, \tag{39}$$

where $T^{(n)}(\rho) = S(\rho||\sigma)$ and σ is a product state making $S(\rho||\sigma)$ minimum. Besides, Modi *et al.* found that σ is the reduced states of ρ in the product form. According to Eq. (30), we obtain

$$\begin{aligned} T^{(n)}(\rho) &= \min_{\sum_i k_i = n} S\left(\rho || \bigotimes_i \rho_{k_i}\right) \\ &= \min_{\sum_i k_i = n} \left[-S(\rho) - \mathrm{Tr}\left(\rho \log_2 \bigotimes_i \rho_{k_i}\right)\right] \\ &= \min_{\sum_i k_i = n} \left[-\sum_i \mathrm{Tr}\left(\rho_{k_i} \log_2 \rho_{k_i}\right)\right] \\ &= \min_{\sum_i k_i = n} \sum_i S\left(\rho_{k_i}\right) \end{aligned} \tag{40}$$

since ρ in GA is a pure state. By using the Lagrangian multiplier method, the above equation is simplify into

$$T^{(n)}(\rho) = S(\rho_1) + S(\rho_{n-1}) = 2S(\rho_1), \tag{41}$$

where ρ_1 is a reduced state of any single qubit in GA:

$$\rho_1 = \begin{pmatrix} a^2 + (2^{n-1} - 1)b^2 & ab + (2^{n-1} - 1)b^2 \\ ab + (2^{n-1} - 1)b^2 & 2^{n-1}b^2 \end{pmatrix}. \tag{42}$$

Thus, the dynamics of genuine quantum correlation in GA becomes

$$D^{(n)} = S(\rho_1) = H\left(\frac{1 + \sqrt{\Delta}}{2}\right), \tag{43}$$

where $\Delta = 1 - 4(2^{n-1} - 1)(ab - b^2)^2$. Figure 5 depicts the behavior of genuine quantum correlation of the whole 11-qubit system in GA.

C. Nonlocality in Grover search

Nonlocality is another manifestation of nonclassical correlation which tells us that reproducing the predictions of quantum theory by considering local hidden variables (LHV) is impossible. It is well known that the entanglement is necessary for the existence of nonlocality but nonlocality is not necessary for entanglement [49]. We are interested about whether nonlocality appears in GA or not. Unfortunately, there is a lack of necessary and sufficient criteria or suitable measurements for nonlocality. Violating the CHSH (Clauser, Horne, Shimony, and Holt) inequality provides a powerful tool to recognize the nonlocality of two-qubit systems. Consequently, we choose the CHSH inequality to investigate the nonlocality of any two qubits during the Grover search.

The CHSH inequality is described as [50]

$$|\langle \mathcal{B}_{\mathrm{CHSH}} \rangle| = |\mathrm{Tr}(\mathcal{B}_{\mathrm{CHSH}}\rho)| \leqslant 2, \tag{44}$$

where

$$\mathcal{B}_{\mathrm{CHSH}} = \vec{a} \cdot \vec{\sigma_1} \otimes (\vec{b} + \vec{b}') \cdot \vec{\sigma_2} + \vec{a}' \cdot \vec{\sigma_1} \otimes (\vec{b} - \vec{b}') \cdot \vec{\sigma_2} \tag{45}$$

and $\vec{a},\vec{a}',\vec{b},\vec{b}'$ are unit vectors in $\mathbb{R}^3$. In Ref. [51], a theorem that a two-qubit system violates the CHSH inequality if and only if $M(\rho) > 1$ has been given. Note that obeying the CHSH inequality does not mean that the system is local. Here, $M(\rho) = \max_{i \neq j}\{u_i + u_j\}$ with u_i being the three eigenvalues of the matrix $T^T T$, where $T = T_{ij} = \mathrm{Tr}\rho(\sigma_i \otimes \sigma_j)$ is the correlation matrix. The correlation matrix for ρ_2 is given by

$$T = \begin{pmatrix} 2\Omega_1 + 2\Omega_2 & 0 & 2\Omega_1 - 2\Omega_2 \\ 0 & 2\Omega_2 - 2\Omega_1 & 0 \\ 2\Omega_1 - 2\Omega_2 & 0 & \Omega_0 - \Omega_2 \end{pmatrix} \tag{46}$$

and the corresponding eigenvalues are $\lambda_1 = 2\Omega_2 - 2\Omega_1$, $\lambda_2 = (\Omega_0 + 2\Omega_1 + \Omega_2 - \sqrt{\Delta})/2$, and $\lambda_3 = (\Omega_0 + 2\Omega_1 + \Omega_2 + \sqrt{\Delta})/2$ with $\Delta = \Omega_0^2 + 20\Omega_1^2 + 25\Omega_2^2 - 4\Omega_0\Omega_1 - 6\Omega_0\Omega_2 - 20\Omega_1\Omega_2$. Therefore, we have

$$M(\rho_2) = \begin{cases} \lambda_2^2 + \lambda_3^2, & \lambda_1 \leqslant \lambda_2; \\ \lambda_1^2 + \lambda_3^2, & \lambda_1 > \lambda_2. \end{cases} \tag{47}$$

In the asymptotic limits $N \gg 1$, we have $\lambda_1 \leqslant \lambda_2$ and

$$\begin{aligned} M(\rho_2) &= \lim_{N\to\infty} \lambda_2^2 + \lambda_3^2 = \lim_{N\to\infty} \frac{(\Omega_0 + 2\Omega_1 + \Omega_2)^2 + \Delta}{2} \\ &= \lim_{N\to\infty} \Omega_0^2 + 12\Omega_1^2 + 13\Omega_2^2 - 2\Omega_0\Omega_2 - 8\Omega_1\Omega_2 \\ &= 1 - 2\sin^2\left(\frac{2r+1}{2}\alpha\right)\cos^2\left(\frac{2r+1}{2}\alpha\right) \\ &\leqslant 1, \end{aligned} \tag{48}$$

which means that the pairwise nonlocality does not exist in this limit case.

Next we will discuss genuine tripartite nonlocality of reduced tripartite states in the GA by using the Svetlichny's inequality. The violation of Svetlichny's inequality means that the correlations cannot be simulated by a hybrid nonlocal-local ensemble [52]; thus the correlation is genuine tripartite nonlocality. Svetlichny's inequality is in the form of [52]

$$|\langle \mathcal{B}_S \rangle| = |\mathrm{Tr}(\mathcal{B}_S \rho)| \leqslant 4. \tag{49}$$

where $\mathcal{B}_S$ is the Svetlichny's operator and defined as

$$\begin{aligned}\mathcal{B}_S = & A[(B+B')C + (B-B')C'] \\ & + A'[(B-B')C - (B+B')C'].\end{aligned} \tag{50}$$

Here the measurements are spin projections onto unit vectors: $A = \vec{a}\cdot\vec{\sigma}_1$ ($A' = \vec{a}'\cdot\vec{\sigma}_1$) on the first qubit, $B = \vec{b}\cdot\vec{\sigma}_2$ ($B' = \vec{b}'\cdot\vec{\sigma}_2$) on the second qubit, and $C = \vec{c}\cdot\vec{\sigma}_3$ ($C' = \vec{c}'\cdot\vec{\sigma}_3$) on the third qubit. By defining $\vec{b}+\vec{b}' = 2\vec{d}\cos t$ and $\vec{b}-\vec{b}' = 2\vec{d}'\sin t$ ($\vec{d}\cdot\vec{d}' = 0$), $\mathcal{B}_S$ can be further simplified as

$$\begin{aligned}|\langle \mathcal{B}_S \rangle| = & 2|(\langle AD'C'\rangle \sin t - \langle A'DC'\rangle \cos t) \\ & + (\langle A'D'C\rangle \sin t + \langle ADC\rangle \cos t)| \\ \leqslant & 2\left(\sqrt{\langle AD'C'\rangle^2 + \langle A'DC'\rangle^2}\right. \\ & \left. + \sqrt{\langle A'D'C\rangle^2 + \langle ADC\rangle^2}\right)\end{aligned} \tag{51}$$

where $D = \vec{d}\cdot\vec{\sigma}_2$ and $D' = \vec{d}'\cdot\vec{\sigma}_2$.

Any tripartite states can be expressed as

$$\begin{aligned}\rho_3 = \frac{1}{8}\Bigg(& I + \sum_i e_i \sigma_i \otimes I \otimes I + \sum_i f_i I \otimes \sigma_i \otimes I \\ & + \sum_i g_i I \otimes I \otimes \sigma_i + \sum_{ij} M_{ij}^a I \otimes \sigma_i \otimes \sigma_j \\ & + \sum_{ij} M_{ij}^b \sigma_i \otimes I \otimes \sigma_j + \sum_{ij} M_{ij}^c \sigma_i \otimes \sigma_j \otimes I \\ & + \sum_{ijk} T_{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k \Bigg)\end{aligned} \tag{52}$$

and

$$T_{ijk} = \mathrm{Tr}(\sigma_i \otimes \sigma_j \otimes \sigma_k \rho_3). \tag{53}$$

In the asymptotic limits $N \gg 1$, by using Eqs. (30) and (53), the T of reduced tripartite states generated by GA has only two nonzero elements: $T_{111} = (\cos^2\alpha_r)/4$ and $T_{333} = (\sin^2\alpha_r)/8$. Let $\tilde{T}_{ij} = \sum_k T_{ijk} c_k'$, and it gives

$$\begin{aligned}\langle AD'C'\rangle^2 + \langle A'DC'\rangle^2 & \leqslant \max\{[\vec{a}\cdot(\tilde{T}\vec{d}')]^2 + [\vec{a}'\cdot(\tilde{T}\vec{d})]^2\} \\ & = \max\{||\tilde{T}\vec{d}'||^2 + ||\tilde{T}\vec{d}||^2\} \\ & = v_1 + v_2,\end{aligned} \tag{54}$$

where v_1 and v_2 are two greater eigenvalues of $\tilde{T}^T\tilde{T}$, $v_1 = c_1'^2/16\cos^4\alpha_r$, and $v_2 = c_3'^2/64\sin^4\alpha_r$. Thus,

$$\langle AD'C'\rangle^2 + \langle A'DC'\rangle^2 \leqslant 1. \tag{55}$$

Similarly, we can also obtain

$$\langle A'D'C\rangle^2 + \langle ADC\rangle^2 \leqslant 1. \tag{56}$$

According to Eqs. (51), (55), and (56), we have

$$|\mathrm{Tr}(\mathcal{B}_S \rho_3)| \leqslant 4, \tag{57}$$

which means genuine tripartite nonlocality is not detected by using Svetlichny's inequality.

V. CONCLUSIONS

In this work, we have systematically studied the evolutions of coherence and other typical quantum correlations in the process of Grover search. Eventually, we find that both success probability in GA and optimal measurement time in GGA can be directly related to a scalar function of state, QCD. By using the relative entropy measure of coherence, we show that the improvement of success probability relies on the coherence depletion for any number of solutions in GA. Explicitly, in the limit case of a few searcher items $j \ll N$ and large database $N \gg 1$, the cost performance about coherence in enhancement the success probability is associated with the ratio of number of searched solutions to the scale of database, j/N. The same phenomenon also exists by using the l_1 norm of coherence and corresponding cost performance equals to $1/N$. In GGA, we discover a class of states where the required optimal measurement time increases with the QCD.

For pairwise entanglement, multipartite entanglement, pairwise discord, and genuine quantum correlation, they are always present during the whole process of GA. Their behaviors generally start from zero, then reach the maximum, and decrease to almost zero. But we fail to connect them with success probability. Moreover, in the limit case, the nonlocalities with respect to two-qubit and three-qubit systems have not been detected during GA by using CHSH-type Bell inequality and Svetlichny's inequality.

Our research exhibits the significance of QCD in Grover search algorithm, contributing to the resource theory of quantum coherence and providing deep insights into the role of coherence in quantum algorithms. On one hand, QCD increases the success probability in GA. On the other hand, a smaller amount of QCD decreases the required optimal measure time in GGA. Therefore, the coherence can be viewed as a potential resource in the Grover search algorithm. Our method is also worth applying to investigate QCD in other quantum information processes, such as Shor's algorithm, teleportation, and so on.

ACKNOWLEDGMENTS

We thank L. C. Zhao, J. X. Hou, and Y. H. Shi for their valuable discussions. This work was supported by the NSFC (Grants No. 11375141, No. 11425522, No. 91536108, and No. 11647057), the Special Research Funds of Shaanxi Province Department of Education (No. 203010005), Northwest University Scientific Research Funds (No. 338020004), and the Double First-Class University Construction Project of Northwest University.

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