Breaking the weak Heisenberg limit

Alfredo Luis*
Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain
(Received 4 September 2016; published 13 March 2017)

We provide a very simple case showing that the weak form of the Heisenberg limit can be beaten while the prior information is improved without bias.

DOI: 10.1103/PhysRevA.95.032113

I. INTRODUCTION

The question of the ultimate quantum limits to the precision in signal detection has received a great deal of attention. One of the reasons for this interest relies on the growing importance of novel quantum technologies where quantum physics is applied to an increasing number of practical tasks.

In this work we focus on the most common scheme in quantum metrology, where the signal to be detected is encoded as a shift ϕ of the phase of an harmonic oscillator. In the most practical terms, this means an electromagnetic field mode illuminating an interferometer.

A key issue in quantum metrology is the trade-off between resolution and resources employed, usually counted as the number of photons in the probe state. The common belief points to an ultimate minimum uncertainty $\Delta\phi$ scaling as the inverse of the total number of photons N; this is $\Delta\phi_s \propto 1/N$ [1–11]. We will refer to this as the *strong form of the Heisenberg limit* or simply *strong Heisenberg limit*. In order to reach this limit all the photons must be employed in a single realization of the measurement and a highly nonclassical probe state with a very large number of photons is required. Such states are very difficult to generate and extremely fragile against practical imperfections [12–14]. These considerations may spoil the actual practical meaning of the strong Heisenberg limit.

In a more practical scenario, we should consider instead the repetition m times of the measurement with identical probes prepared in a nonclassical state with small mean number $\bar{n}$, such that $N = m\bar{n}$ can be still very large. Considering that the m repetitions are statistically independent, the minimum uncertainty would scale as [15]

$$\text{Weak form:} \quad \Delta\phi_w \propto \frac{1}{\sqrt{m\bar{n}}}, \tag{1.1}$$

which is rather different from the strong form,

$$\text{Strong form:} \quad \Delta\phi_s \propto \frac{1}{m\bar{n}} = \frac{1}{N}, \tag{1.2}$$

especially for meaningful situations where m will be far larger than $\bar{n}$. We will refer to $\Delta\phi_w$ as the *weak form of the Heisenberg limit*, or simply *weak Heisenberg limit*.

The actual meaning of the strong Heisenberg limit has been much debated [16–30]. However, the weak Heisenberg limit has not so extensively examined [5,23,30], although it has a more deep practical meaning, as discussed above.

*alluis@fis.ucm.es; http://www.ucm.es/info/gioq

In this work, we show by means of an extremely simple example the meaningful beating of the weak form of the Heisenberg limit. Moreover, this example suggests that the strong limit may be as well approached in this same scenario of large m and small $\bar{n}$.

We can benefit from many conclusions of the strong-limit scenario. To begin with, one must be careful concerning the performance estimators we can trust [16–20,23]. It is crucial to check the presence of bias and whether meaningful improvement over the prior information is achieved [3,8,10,25,30]. Because of this, we mainly focus on Bayesian-like approaches involving averages over the prior knowledge. Nevertheless, we will contrast the results also with pointwise approaches such as the Cramér-Rao bounds.

II. DETECTION SCHEME

The physical system for the detection is a single-mode electromagnetic field. The probe is prepared in the state $|\psi\rangle$ expressed in the photon-number basis as

$$|\psi\rangle = \sqrt{1-\nu^2}|0\rangle + \nu|\bar{n}/\nu^2\rangle, \tag{2.1}$$

where ν is a parameter that will be considered small enough $\nu \ll 1$, while $\bar{n}$ is the mean number of photons of each probe-state realization, assuming always that $\bar{n}/\nu^2$ is an integer. These kind of states have been considered before [5,23] and known, for example, as unbalanced cat states [31].

In this scheme, the signal to be detected induces a phase shift ϕ transforming the probe state into

$$|\psi(\phi)\rangle = \sqrt{1-\nu^2}|0\rangle + e^{i\phi\bar{n}/\nu^2}\nu|\bar{n}/\nu^2\rangle. \tag{2.2}$$

We will consider that all which is known about the signal is that ϕ is included in the interval $[0,W]$. This knowledge is often summarized in a *prior distribution* $P(\phi)$, in our case as $P(\phi) = 1/W$ for $\phi \in [0,W]$ and $P(\phi) = 0$ otherwise.

III. BOUNDS

In general terms, many variable factors affect the estimation performance, such as probe state, signal codification, measurement performed, data analysis followed, experimental imperfections, and so on. Because of this, most performance analysis focus on the derivation of lower bounds on the estimation error, rather than dealing with exact precision limits. This is clearly discussed in Ref. [32], where several lower bounds to the estimation uncertainty $\Delta\tilde{\phi}$ are presented and discussed.

Among them, the most popular is the Cramér-Rao bound, which focuses on the variance of the signal estimator $\tilde{\phi}$

$$\Delta^2\tilde{\phi} = \sum_k P(k|\phi)[\tilde{\phi}(k) - \phi]^2, \tag{3.1}$$

where k are the outcomes of the measurement performed, assumed discrete for simplicity, $P(k|\phi)$ is the probability of outcome k when the signal is ϕ, and $\tilde{\phi}(k)$ is the estimation of ϕ after the outcome k. Then it can be seen that

$$\Delta^2\tilde{\phi} \geqslant \Delta^2\tilde{\phi}_{CR} = \frac{1}{mF}, \tag{3.2}$$

where F is the Fisher information of a single measurement,

$$F = \sum_k \frac{1}{P(k|\phi)}\left(\frac{\partial P(k|\phi)}{\partial\phi}\right)^2. \tag{3.3}$$

The dependence of $\Delta\tilde{\phi}_{CR}$ on the measurement performed can be removed, leading to the quantum Cramér-Rao bound $\Delta\tilde{\phi}_{QCR}$, which is again of the form (3.2) but where the Fisher information (3.3) is replaced by the quantum Fisher information F_Q. For probes in pure states F_Q reads simply as the variance of the generator G of the phase shift on the probe state

$$F_Q = 4[\langle\psi(\phi)|G^2|\psi(\phi)\rangle - \langle\psi(\phi)|G|\psi(\phi)\rangle^2], \tag{3.4}$$

and in our case G is the number operator $G = \hat{n}$. Note that in general these pointwise bounds depend on the unknown signal value ϕ, ignore prior information $P(\phi)$, and do not address whether the estimation protocol is efficiently reaching the lower bound.

An alternative picture that can take into account all these points is provided by Bayesian approaches that aim to construct a posterior distribution for the signal estimator $\tilde{\phi}$, for example, in the form

$$P(\tilde{\phi}|\phi) \propto P[k(\tilde{\phi})|\phi], \tag{3.5}$$

where $k(\tilde{\phi})$ is given by inverting the relation $\tilde{\phi}(k)$. A clear advantage of this approach is that it solves many issues at once, such as unbiasedness and efficiency. Within this Bayesian scenario the uncertainty on $\tilde{\phi}$ can be estimated, including the prior information contained in $P(\phi)$, for example, via the mean square estimation error averaged over the prior distribution $P(\phi)$

$$\Delta^2\tilde{\phi} = \int d\phi \sum_k P(k|\phi)P(\phi)[\tilde{\phi}(k) - \phi]^2. \tag{3.6}$$

A suitable lower bound for $\Delta^2\tilde{\phi}$ is the Ziv-Zakai bound [3]

$$\Delta^2\tilde{\phi} \geqslant \frac{1}{2}\int_0^W d\phi\phi\left(1 - \frac{\phi}{W}\right)[1 - \sqrt{1 - |\langle\psi|\psi(\phi)\rangle|^{2m}}], \tag{3.7}$$

where

$$|\langle\psi|\psi(\phi)\rangle|^2 = (1 - \nu^2)^2 + \nu^4 + 2\nu^2(1 - \nu^2)\cos(\bar{n}\phi/\nu^2). \tag{3.8}$$

In order to deal with practicable expressions, let us assume, as a first condition, that

$$W\bar{n}/\nu^2 \ll 1 \quad \text{1st condition}, \tag{3.9}$$

so that in due course we may approximate $\cos(\bar{n}\phi/\nu^2)$ as $\cos x \simeq 1 - x^2/2$ as well as $(1 - y)^m \simeq e^{-my}$ within the range of values allowed for ϕ by the prior distribution. In such a case

$$|\langle\psi|\psi(\phi)\rangle|^{2m} \simeq (1 - \bar{n}^2\phi^2/\nu^2)^m \simeq e^{-m\bar{n}^2\phi^2/\nu^2}. \tag{3.10}$$

Moreover, in order to proceed with user-friendly expressions we further assume as a second condition that

$$\sqrt{m}\bar{n}W/\nu \gg 1 \quad \text{2nd condition}. \tag{3.11}$$

This is natural since we expect that a very large number of repetitions m is needed to reach a meaningful resolution. Thus we may approximate in Eq. (3.7) $1 - \sqrt{1 - z} \simeq z/2$ with $z = \exp(-m\bar{n}^2\phi^2/\nu^2)$. Finally, after the ϕ integration we have

$$\Delta^2\tilde{\phi} \geqslant \Delta^2\tilde{\phi}_{ZZ} = \frac{\nu^2}{8m\bar{n}^2} = \frac{m\nu^2}{8N^2}. \tag{3.12}$$

This is the final form for the Ziv-Zakai bound. The first conclusion is that this scheme clearly includes the possibility of beating the weak Hesienberg limit thanks to the small factor ν in the numerator of $\Delta\tilde{\phi}_{ZZ}$. Moreover, now we get that the condition (3.11) means that the bound is clearly smaller than the prior information, $\Delta\tilde{\phi}_{ZZ} \ll W$.

The conjunction of the two above conditions (3.9) and (3.11) implies that $m\nu^2 \gg 1$, which agrees with the results in Ref. [3] regarding the strong Heisenberg limit, since this would imply that $\Delta\tilde{\phi}_{ZZ} \gg \Delta\tilde{\phi}_s$. Nevertheless, we recall that conditions (3.9) and (3.11) were imposed in order to get manageable expressions. But the scheme can work equally well without them, as demonstrated by the following example.

After Eq. (3.12) it is not excluded that this strategy may approach the strong Heisenberg limit $\Delta\tilde{\phi}_{ZZ} \simeq \Delta\tilde{\phi}_s$ provided that $m\nu^2 \simeq 1$. This suggests that the attainability of the strong limit scaling refers essentially to the $\bar{n}$ scaling. The other variables m,ν are customarily fixed as function of $\bar{n}$ to obtain the desired result for $\Delta\tilde{\phi}$ up to constant factors.

Finally, as shown in Ref. [32], there is the Bayesian Cramér-Rao bound suitably mixing both strategies in the form

$$\Delta^2\tilde{\phi}_{BCR} = \frac{1}{m\int d\phi P(\phi)F + \mathcal{I}}, \tag{3.13}$$

where F is the Fisher information associated to the measurement, while $\mathcal{I}$ represents Fisher information corresponding to the prior information

$$\mathcal{I} = \int d\phi \frac{1}{P(\phi)}\left[\frac{\partial P(\phi)}{\partial\phi}\right]^2. \tag{3.14}$$

IV. POSTERIOR DISTRIBUTION

The above result (3.12) for the Ziv-Zakai bound does not prove that the weak Heisenberg limit can be actually beaten in a specific scheme since this is a bound that may or may not be reached. In this section we proceed by showing a specific detection scheme, providing a proof of principle that the bound can be actually approached. We also compare the resolution reached with the other bounds in Sec. III as a consistence check. To this end we begin with by deriving explicitly the *posterior distribution* $P(\tilde{\phi}|\phi)$ for the estimated phase $\tilde{\phi}$ conditioned to the true unknown phase shift ϕ as presented in Eq. (3.5).

The measurement in this example has just two outputs labeled $\pm$ whose probabilities $P(\pm|\phi)$ are given by projecting the signal-transformed state $|\psi(\phi)\rangle$ in Eq. (2.2) on the orthogonal vectors $|\pm\rangle$, expressed in the number basis as

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|\bar{n}/\nu^2\rangle). \tag{4.1}$$

This is

$$P(\pm|\phi) = \tfrac{1}{2} \pm \nu\sqrt{1-\nu^2}\sin(\phi\bar{n}/\nu^2). \tag{4.2}$$

When repeating the measurement m times, the probability that we get k positive outcomes and $m-k$ negatives is the binomial

$$P(k|\phi) = \binom{m}{k} P^k(+|\phi)P^{m-k}(-|\phi). \tag{4.3}$$

In the limit of large m, which is our case here, the binomial (4.3) can be well approximated by the Gaussian

$$P(k|\phi) \simeq \frac{\exp\left\{-\frac{[k-mP(+|\phi)]^2}{2mP(+|\phi)P(-|\phi)}\right\}}{\sqrt{2\pi m P(+|\phi)P(-|\phi)}}. \tag{4.4}$$

Following a maximum likelihood strategy, to each result with k positive outcomes we can assign the estimator $\tilde{\phi}$ given by relation

$$P(+|\tilde{\phi}) = k/m, \tag{4.5}$$

as the phase shift that maximizes the probability $P(k|\phi)$ of obtaining the result actually obtained. Using again that the first condition (3.9) holds,

$$P(+|\tilde{\phi}) \simeq \tfrac{1}{2} \pm \phi\bar{n}/\nu = k/m, \tag{4.6}$$

so that

$$\tilde{\phi} = \frac{k-m/2}{m\bar{n}/\nu}, \quad k(\tilde{\phi}) = m/2 + m\bar{n}\tilde{\phi}/\nu. \tag{4.7}$$

With this, the final posterior distribution becomes after Eqs. (3.5), (4.4), and (4.7)

$$P(\tilde{\phi}|\phi) \simeq \sqrt{\frac{2m\bar{n}^2}{\pi\nu^2}}\exp\left[-2\frac{m\bar{n}^2}{\nu^2}(\tilde{\phi}-\phi)^2\right], \tag{4.8}$$

where we have approximated $P(+|\phi)P(-|\phi) \simeq 1/4$. Note that $P(\tilde{\phi}|\phi)$ readily provides the uncertainty of the estimator as

$$\Delta^2\tilde{\phi} \simeq \frac{\nu^2}{4m\bar{n}^2}, \tag{4.9}$$

which is just twice the Ziv-Zakai bound (3.12). We can check also in Eq. (4.8) that in this limit there is no bias, since the mean value of the estimator $\tilde{\phi}$ coincides with the true value ϕ.

Therefore this scheme is able to beat the weak Heisenberg limit for a suitable choice of large m and small ν. At difference with the strong Heisenberg form, in this case the violation is not jeopardized by any relation between m and ν. The only requirement is that $m \gg 1$ and $\nu \ll 1$. In the next section we present a specific numerical example.

Regarding the Cramér-Rao bounds we have that the Fisher information in Eq. (3.3) after condition (3.9) and using $P(k = \pm|\phi)$ in Eq. (4.2) becomes $F \simeq 4\bar{n}^2/\nu^2$ so that the Cramér-Rao bound (3.2) is

$$\Delta^2\tilde{\phi}_{CR} \simeq \frac{\nu^2}{4m\bar{n}^2}. \tag{4.10}$$

This is just equal to the actual Bayesian uncertainty achieved in this scheme $\Delta^2\tilde{\phi}$ in Eq. (4.9) so our scheme saturates the Cramér-Rao bound. Moreover, it saturates also the quantum Cramér-Rao bound since the quantum Fisher information in Eq. (3.4) for $\nu \ll 1$ matches the Fisher information $F_Q \simeq F \simeq 4\bar{n}^2/\nu^2$, so that

$$\Delta\tilde{\phi} \simeq \Delta\tilde{\phi}_{CR} \simeq \Delta\tilde{\phi}_{QCR}. \tag{4.11}$$

Regarding the Bayesian Cramér-Rao bound in Eq. (3.13) and considering in our case $\mathcal{I} \simeq 1/W^2$ we get

$$m\int d\phi P(\phi)F + \mathcal{I} \simeq 4m\bar{n}^2/\nu^2 + 1/W^2 \simeq 4m\bar{n}^2/\nu^2, \tag{4.12}$$

where we have used condition (3.11). Therefore we get that our scheme also saturates this Bayesian Cramér-Rao bound, so that

$$\Delta\tilde{\phi} \simeq \Delta\tilde{\phi}_{CR} \simeq \Delta\tilde{\phi}_{QCR} \simeq \Delta\tilde{\phi}_{BCR}. \tag{4.13}$$

V. BEATING THE WEAK HEISENBERG LIMIT

Let us show explicitly that the above analysis provides a proof of principle that the weak Heisenberg limit can be beaten. To this end we present a numerical evaluation of the posterior distribution without any approximation directly by combining Eqs. (4.3), (4.5), and (3.5). This is compared with the approximation (4.8). In Fig. 1 we have plotted both posterior distributions for

$$W = 10^{-3}, \quad \bar{n} = 1, \quad m = 10^6, \quad \phi = 10^{-4}, \quad \nu = 0.1. \tag{5.1}$$

Both conditions (3.9) and (3.11) are satisfied with $W\bar{n}/\nu^2 = 0.1$, $\sqrt{m}\bar{n}W/\nu = 10$, and $m\nu^2 = 10^4$. So the exact and approximate expressions are indistinguishable and the uncertainty is readily given by Eq. (4.9). More specifically, in this case we get

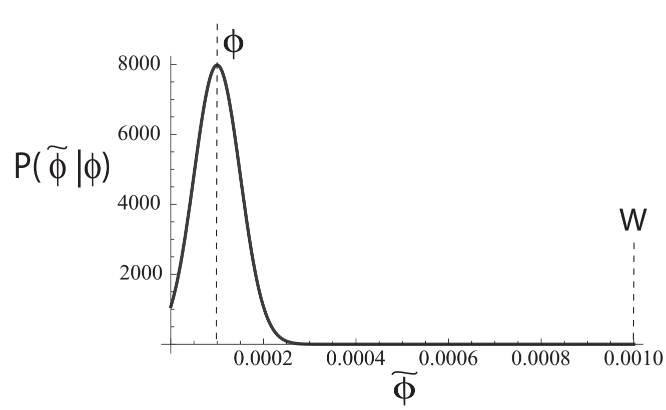

FIG. 1. Posterior distribution $P(\tilde{\phi}|\phi)$ for the estimator $\tilde{\phi}$ for $W = 10^{-3}$, $\bar{n} = 1$, $m = 10^6$, $\nu = 0.1$, and $\phi = 10^{-4}$. The exact and approximate expressions are indistinguishable. We have marked with vertical dashed lines the values of ϕ and W.

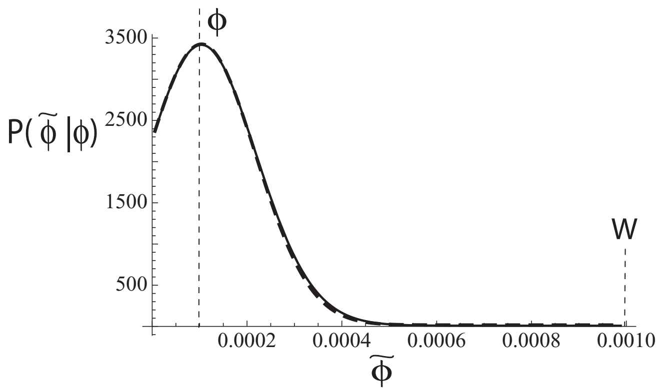

FIG. 2. Posterior distribution $P(\tilde{\phi}|\phi)$ for the estimator $\tilde{\phi}$ for $W = 10^{-3}$, $\bar{n} = 1$, $m = 1.6 \times 10^4$, $\nu = 0.03$, and $\phi = 10^{-4}$. The exact (solid line and approximate (dashed line) expressions are indistinguishable. We have marked with vertical dashed lines the values of ϕ and W.

$\Delta\tilde{\phi} = 5 \times 10^{-5}$, which is clearly below both the weak limit and the prior, being above the strong limit

$$\Delta\tilde{\phi} = 0.05\Delta\phi_w = 0.05W = 50\Delta\phi_s, \tag{5.2}$$

where for definiteness we have considered $\Delta\phi_w = 1/(\sqrt{m\bar{n}})$ and $\Delta\phi_s = 1/(m\bar{n})$.

Next we can examine whether this strategy can approach the strong Heisenberg limit for a proper choice of parameters. This is the case, for example, of a slight variation of the above example as

$$W = 10^{-3}, \quad \bar{n} = 1, \quad m = 1.6 \times 10^4,$$
$$\phi = 10^{-4}, \quad \nu = 0.03, \tag{5.3}$$

leading to $\Delta\tilde{\phi} = 1.1 \times 10^{-4}$ with

$$\Delta\tilde{\phi} = 0.015\Delta\phi_w = 0.11W = 2\Delta\phi_s, \tag{5.4}$$

so this is just twice the strong Heisenberg limit. In Fig. 2 we have plotted both the exact and approximate posterior distributions, showing again that they almost coincide, so Eq. (4.9) holds. In this case the first condition (3.9) is not satisfied since $W\bar{n}/\nu^2 = 1.1$ while the second one (3.11) is close to be satisfied as $\sqrt{m\bar{n}}W/\nu = 5$, leading to $m\nu^2 = 20$.

VI. DISCUSSION

We have provided analytical and numerical evidences showing that the weak form of the Heisenberg limit can be beat while the prior information is improved without bias. The probe and measurement presented may be regarded as unpractical, but the main goal was to provide a simple as possible proof of principle of this.

The key point is that in any case the weak-probe scenario is better than the bright-probe case. This is because special states of light, such as the probe (2.1) or even NOON states, are much more accessible and robust against practical imperfections for small mean numbers that for large numbers. In this regard, the use of weak coherent states is excluded since they would deprive us of the ν parameter which has been crucial in the above analysis.

One of the main questions addressed in quantum metrology is how to use resources in the most efficient way. The answer is not simple and one must trust the conclusions provided by the performance estimator chosen, in this case the mean square error averaged over prior distribution. Coherent states are much easily produced but they are not efficient enough according to these statistical tools. We may say that although each run provides a minuscule amount of information, it is of a much larger quality regarding long-run cumulative effects when compared to a single measurement with all photons gathered in a bright coherent state.

Quantum metrology protocols proceed without explicit references to the physical meaning of the parameter to be estimated. Nevertheless, in our case we may question what the phase in Eq. (2.2) is relative to. This point is settled by the measurement process that directly or indirectly embodies the reference phase. We may say that this is a somewhat sophisticated version of homodyne or heterodyne quadrature measurements, where the phase of a single-mode field can be suitably observed relative to the phase of the local oscillator that defines the actual quadratures being measured.

Saturation of bounds is a quite tricky point in every estimation procedure. Actually, this was the main reason for focusing on Bayesian approaches. We have shown that our scheme saturates all bounds including the Cramér-Rao bounds with quantum and classical Fisher informations.

Finally, a merit of these results is that they point to a largely dismissed possibility in quantum metrology, the multiple repetitions in a weak-probe scenario. To secure this point, we have shown that it avoids most of the difficulties that face more naive approaches such as prior information and bias. We show that is a quite interesting route to be followed to obtain suggestive nontrivial results.

ACKNOWLEDGMENTS

I thank Dr. Ángel Rivas and Dr. Jesús Rubio for enlightening discussions and acknowledge support from Spanish MINECO Grant no. FIS2012-35583 and CAM research consortium QUITEMAD+ Grant No. S2013/ICE-2801.

[1] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: beating the standard quantum limit, Science **306**, 1330 (2004).

[2] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photon. **5**, 222 (2011).

[3] M. Tsang, Ziv-Zakai Error Bounds for Quantum Parameter Estimation, Phys. Rev. Lett. **108**, 230401 (2012).

[4] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum Measurement Bounds Beyond the Uncertainty Relations, Phys. Rev. Lett. **108**, 260405 (2012).

[5] D. W. Berry, M. J. W. Hall, M. Zwierz, and H. M. Wiseman, Optimal Heisenberg-style bounds for the average performance of arbitrary phase estimates, Phys. Rev. A **86**, 053813 (2012).

[6] M. J. W. Hall, D. W. Berry, M. Zwierz, and H. M. Wiseman, Universality of the Heisenberg limit for estimates of random phase shifts, Phys. Rev. A **85**, 041802 (2012).

[7] R. Nair, Fundamental limits on the accuracy of optical phase estimation from rate-distortion theory, arXiv:1204.3761.

[8] V. Giovannetti and L. Maccone, Sub-Heisenberg Estimation Strategies are Ineffective, Phys. Rev. Lett. **108**, 210404 (2012).

[9] M. J. W. Hall and H. M. Wiseman, Does Nonlinear Metrology Offer Improved Resolution? Answers From Quantum Information Theory, Phys. Rev. X **2**, 041006 (2012).

[10] M. J. W. Hall and H. M. Wiseman, Heisenberg-style bounds for arbitrary estimates of shift parameters including prior information, New J. Phys. **14**, 033040 (2012).

[11] Y. Gao and H. Lee, Generalized limits for parameter sensitivity via quantum Ziv-Zakai bound, J. Phys. A **45**, 415306 (2012).

[12] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Improvement of Frequency Standards with Quantum Entanglement, Phys. Rev. Lett. **79**, 3865 (1997).

[13] R. Demkowicz-Dobrzański, K. Banaszek, and R. Schnabel, Fundamental quantum interferometry bound for the squeezed-light-enhanced gravitational wave detector GEO 600, Phys. Rev. A **88**, 041802(R) (2013).

[14] R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kołodyński, Quantum limits in optical interferometry, Prog. Opt. **60**, 345 (2015).

[15] L. Pezzè, P. Hyllus, and A. Smerzi, Phase-sensitivity bounds for two-mode interferometers, Phys. Rev. A **91**, 032103 (2015).

[16] J. H. Shapiro, S. R. Shepard, and N. W. Wong, Ultimate Quantum Limits on Phase Measurement, Phys. Rev. Lett. **62**, 2377 (1989).

[17] J. H. Shapiro and S. R. Shepard, Quantum phase measurement: A system-theory perspective, Phys. Rev. A **43**, 3795 (1991).

[18] Z. Hradil and J. H. Shapiro, Quantum phase measurements with infinite peak-likelihood and zero phase information, Quantum Opt. **4**, 31 (1992).

[19] Z. Hradil, Performance measures of quantum-phase measurement, Phys. Rev. A **46**, R2217 (1992).

[20] S. L. Braunstein, A. S. Lane, and C. M. Caves, Maximum-Likelihood Analysis of Multiple Quantum Phase Measurements, Phys. Rev. Lett. **69**, 2153 (1992).

[21] S. L. Braunstein, Quantum Limits on Precision Measurements of Phase, Phys. Rev. Lett. **69**, 3598 (1992).

[22] A. S. Lane, S. L. Braunstein, and C. M. Caves, Maximum-likelihood statistics of multiple quantum phase measurements, Phys. Rev. A **47**, 1667 (1993).

[23] S. L. Braunstein, Some limits to precision phase measurement, Phys. Rev. A **49**, 69 (1994).

[24] P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, S. D. Huver, H. Lee, and J. P. Dowling, Quantum Metrology with Two-Mode Squeezed Vacuum: Parity Detection Beats the Heisenberg Limit, Phys. Rev. Lett. **104**, 103602 (2010).

[25] A. Rivas and A. Luis, Sub-Heisenberg Estimation of non-random phase shifts, New J. Phys. **14**, 093052 (2012).

[26] A. Luis and A. Rodil, Alternative measures of uncertainty in quantum metrology: Contradictions and limits, Phys. Rev. A **87**, 034101 (2013).

[27] A. Luis, Signal detection without finite-energy limits to quantum resolution, Ann. Phys. (Amsterdam) **331**, 1 (2013).

[28] Y. R. Zhang, G. R. Jin, J. P. Cao, W. M. Liu, and H. Fan, Unbounded quantum Fisher information in two-path interferometry with finite photon number, J. Phys. A **46**, 035302 (2013).

[29] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, The elusive Heisenberg limit in quantum-enhanced metrology, Nat. Commun. **3**, 1063 (2012).

[30] L. Pezzè, Sub-Heisenberg phase uncertainties, Phys. Rev. A. **88**, 060101(R) (2013).

[31] P. A. Knott, T. J. Proctor, K. Nemoto, J. A. Dunningham, and W. J. Munro, Effect of multimode entanglement on lossy optical quantum metrology, Phys. Rev. A **90**, 033846 (2014).

[32] M. Jarzyna and R. Demkowicz-Dobrzański, True precision limits in quantum metrology, New J. Phys. **17**, 013010 (2015).