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Activation of monogamy in nonlocality using local contextuality

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A unified view of the phenomenon of monogamy exhibited by Bell inequalities and noncontextuality inequalities arising from the no-signaling and no-disturbance principles is presented using the graph-theoretic method introduced by Ramanathan *et al.* [Phys. Rev. Lett. **109**, 050404 (2012)]. We propose a hitherto unexplored tradeoff relation, namely, Bell inequalities that do not exhibit monogamy features of their own can be activated to be monogamous by the addition of a local contextuality term. This is illustrated by means of the well-known $\mathcal{I}_{3322}$ inequality and reveals a resource trade-off between bipartite correlations and the local purity of a single system. In the derivation of no-signaling monogamies, we uncover a unique feature, namely, that two-party Bell expressions that are trivially classically saturated can become nontrivial upon the addition of an expression involving a third party with a single measurement input.

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I. INTRODUCTION

The Bell theorem [1] and the Kochen-Specker theorem [2] are cornerstone results in the foundation of quantum mechanics. While the Bell theorem demonstrates that local hidden variable theories are incompatible with the statistical predictions of quantum mechanics, the Kochen-Specker (KS) theorem demonstrates the incompatibility of quantum theory with the assumption of outcome noncontextuality when describing systems with more than two distinguishable states. In other words, the KS theorem shows that there are quantum measurements whose outcomes cannot be predefined in a noncontextual manner, i.e., independent of the measurement's context (the choice of jointly measurable tests that may be performed together). However, Bell's theorem can be interpreted as a specific case of this feature where the measurement's context is remote.

The nonlocal correlations between spatially separated systems that lead to the violation of the well-known Clauser-Horne-Shimony-Holt (CHSH) inequality [3] exhibit the phenomenon of monogamy [4]. In particular, it was proved by Toner in [4] that in a Bell experiment with three spatially separated parties Alice, Bob, and Charlie, when Alice and Bob observe a violation of the CHSH inequality between their systems, the no-signaling principle imposes that Alice and Charlie cannot at the same time observe a violation of the inequality. This feature of nonlocal correlations reflects the monogamy of entanglement in the underlying quantum state used in the Bell experiment and is significant in cryptographic scenarios [5,6] where Alice and Bob can verify a sufficient Bell violation to guarantee that their systems are not much correlated with any eavesdropper's system. In fact, such a trade-off in the nonlocal correlations can be seen for every Bell inequality [7]. Other (stricter) no-signaling monogamy relations for specific classes of Bell inequalities have also been discovered [8–11].

A general constraint analogous to no-signaling is also imposed in a single-system-contextuality scenario. This is the no-disturbance principle [12], which is a consistency constraint that the probability distribution observed for the outcome of the measurement of any observable is independent of which set of other (co-measurable) observables it is measured alongside. It was shown in [12] that this constraint also imposes a trade-off in the violations of noncontextuality inequalities on a single system, the simplest of these inequalities being the Klyachko-Can-Binicoglu-Shumovsky (KCBS) inequality [13]. Recently, the phenomenon of monogamy between the nonlocal correlations and the single-system contextuality has been pointed out by Kurzynski *et al.* [14] and has been experimentally verified [15]. In particular, *any* violation of the CHSH inequality between Alice's and Bob's systems implies that locally Alice's system alone does not exhibit a violation of the KCBS inequality and vice versa.

In this Rapid Communication we first outline a sufficient condition to derive no-signaling and no-disturbance monogamies using a graph-theoretic method, while showing that this condition is not also necessary by means of a counterexample. This method allows us to derive hitherto unexplored monogamy relations. For instance, we show any two cycle inequalities for any length of the cycle, whether studied as noncontextuality [16,17] or Bell inequalities [18], exhibit a monogamy relation in both contextual and nonlocal scenarios under certain conditions. Then we proceed to our main result that a nonmonogamous Bell inequality can be activated to be monogamous by the addition of a local (state-dependent) noncontextuality inequality. Such a monogamy relation, which we illustrate via the well-known $\mathcal{I}_{3322}$ Bell inequality, reveals a resource trade-off between bipartite correlations and the local purity of a single system. This is a theory-independent nonlocality analog of the well-known Coffman-Kundu-Wootters trade-off [19] for entanglement. Finally, in investigating methods for the derivation of no-signaling monogamies, we uncover the feature that Bell expressions that are trivially satisfied by a classical Alice and Bob system can give rise to no-signaling violations upon the addition of a third party with a single measurement input. We explore this counterintuitive feature by means of an explicit example.

II. GRAPH-THEORETIC METHOD TO DERIVE MONOGAMY RELATIONS

Let us first state and explain the graph-theoretic method to derive general monogamy relations for both noncontextuality and Bell inequalities [12]. All the observables $\{A_1, \ldots, A_n\}$ that appear in the combined Bell and contextuality experiment are represented by means of a commutation graph G that is constructed as follows. Each vertex of G represents an observable and two vertices are connected by an edge if the corresponding observables can be measured together. Notice that in the commutation graph each clique represents a context, i.e., a jointly measurable system of observables.

A graph is said to be chordal if all cycles of length 4 or more in the graph have a chord, i.e., an edge connecting two nonadjacent vertices in the cycle. As explained in the Supplemental Material, chordal graphs are known to have equivalent characterizations in terms of admitting a maximal clique tree as well as having a simplicial elimination scheme [20]. The importance of chordal commutation graphs comes from the following proposition [12] stating that a set of observables admits a joint probability distribution for its outcomes when its corresponding commutation graph is chordal. This statement can be seen as the generalization of the statement by Fine [21] showing that tree graphs admit a joint probability distribution. A similar statement in the context of relational databases was proven in [22]. For completeness, we give an explicit proof in the Supplemental Material [23] using the notion of simplicial elimination ordering for chordal graphs. It is also noteworthy that the above condition is not necessary for existence of a monogamy relation, as discussed in the Supplemental Material [23].

III. METHOD

Consider a set of Bell and noncontextuality inequalities $\{\mathcal{I}_k\}$ and let $\omega_c = \sum_k \omega_c(\mathcal{I}_k)$ define the maximum classical value of the combined expression, i.e., $\sum_k \mathcal{I}_k \leqslant \omega_c$ in any classical theory. Let $G_{\{\mathcal{I}_k\}}$ denote the commutation graph representing the observables measured by all the parties in the noncontextuality or Bell scenario. A no-signaling or no-disturbance monogamy relation holds if $G_{\{\mathcal{I}_k\}}$ can be decomposed into a set of induced chordal subgraphs $\{G^{(j)}_{\text{c-sub}}\}$ such that the sum of the algebraic values of the reduced Bell expressions in each of the chordal graphs equals ω_c.

Lemma. It is sufficient to consider *induced* chordal subgraphs in the method described above.

To see this, suppose that a set of chordal graphs $G^{(j)}_{\text{c-sub}}$ with associated (classical equals no-signaling) values for the reduced Bell expressions $\omega^{(j)}_c$ exists satisfying $\sum_j \omega^{(j)}_c = \omega_c$. Consider the optimal classical deterministic strategy achieving value ω_c for the whole graph; this strategy must therefore necessarily achieve value $\omega^{(j)}_c$ on each of the chordal subgraphs. In other words, a set of optimal and compatible classical (deterministic) strategies for all of the $G^{(j)}_{\text{c-sub}}$ exists, where compatibility means that the values are assigned by the strategies to any observable A_m that is the same in each of the chordal subgraphs A_m appears in. From this observation, it follows that each $G^{(j)}_{\text{c-sub}}$ may be taken to be induced, i.e., any edges between two vertices $A_i, A_j \in V(G^{(j)}_{\text{c-sub}})$ that are present in $G_{\{\mathcal{I}_k\}}$ may also be included in $G^{(j)}_{\text{c-sub}}$.

However, we remark that when considering different Bell expressions, it may happen that the classical deterministic strategies for these expressions are not compatible with each other, in the sense that the same observable A_i is assigned different values in the optimal strategies for different expressions. In such cases one might have classical monogamies, where the classical value of the sum is strictly smaller than the sum of the individual classical values, i.e., $\omega_c < \sum_k \omega_c(\mathcal{I}_k)$ [10,12].

IV. CYCLE INEQUALITIES

As mentioned earlier, any Bell inequality can also be viewed as a noncontextuality inequality of the combined system of distant parties. By incorporating other Bobs into Alice's system monogamy relations in nonlocal-contextual and contextual-contextual scenarios can be inferred from those in the Bell scenario. However, the most interesting case is when the contextuality test is performed on a single system (for example, qutrit) that does not exhibit nonlocality. Recently, n-cycle ($n \geqslant 5$) noncontextual inequalities have been proposed and shown to be maximally violated by qutrits [16,17]. The analogous cycle Bell inequality [18] with n inputs each has also been studied. Motivated by these facts, we first consider cycle inequalities that are the simplest nontrivial case to study unified monogamy relations. It is shown that monogamy exists for two n-cycle Bell inequality of same length in the nonlocal-nonlocal scenario [24]. In the nonlocal-contextual scenario, monogamy between the CHSH and n-cycle noncontextuality inequalities and, in the contextual-contextual scenario, monogamy between any two cycle inequalities are pointed out [25]. Here we show a more general result in this direction.

Proposition. Any two cycle inequalities with possible different cycle lengths, having at least two common observables, are monogamous in any theory satisfying the no-disturbance principle. For the monogamy to hold in the nonlocal-contextual and contextual-contextual scenarios, suitable additional commutation relations are required.

The proof of this proposition with the decompositions of induced chordal subgraphs is explicitly described in the Supplemental Material [23]. This result can also be generalized for many outcome cycle inequalities [24] in all three scenarios (see the Supplemental Material [23]).

V. ACTIVATION OF THE MONOGAMY RELATION IN THE BELL INEQUALITY

The monogamy relations for entanglement establish a strict trade-off in the shareability of this resource. In [19], Coffman *et al.* established a monogamy relation for the tangle, which is a well-known measure of entanglement. The monogamy relation for the tangle reads

$$\tau^{(1)}_{A|BC} \geqslant \tau^{(2)}_{AB} + \tau^{(2)}_{AC}, \tag{1}$$

where $\tau^{(1)}_{A|BC}$ is the tangle between qubit A and the pair BC. This relation says that the amount of entanglement that qubit A has with BC cannot be less than the sum of the individual entanglements with qubits B and C separately.

We now propose an analogous relationship between nonlocality and contextuality. Consider the nonlocal-contextual scenario with three observers Alice, Bob, and Charlie. Alice performs six possible ± 1-valued measurements, say, A_i, with $i \in \{1,\ldots,6\}$. In each run of the experiment, Alice randomly measures two compatible observables from the set $\{A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_1\}$ or a single observable A_6 on her subsystem while Bob and Charlie randomly perform one of their three ± 1-valued observables, say, B_1, B_2, B_3 and C_1, C_2, C_3, on their respective subsystems. The $\mathcal{I}_{3322}$ inequality [26] involving Alice's observables A_1, A_4, A_6 and Bob's observable B_1, B_2, B_3 is given by

$$\begin{aligned}\mathcal{I}^{A_1A_4A_6}_{B_1B_2B_3} = {}& \langle A_1\rangle + \langle A_4\rangle + \langle B_1\rangle + \langle B_2\rangle - \langle A_1B_1\rangle \\ & - \langle A_1B_2\rangle - \langle A_1B_3\rangle - \langle A_4B_1\rangle - \langle A_4B_2\rangle \\ & + \langle A_4B_3\rangle - \langle A_6B_1\rangle + \langle A_6B_2\rangle \leqslant 4. \qquad (2)\end{aligned}$$

A no-signaling box exists that achieves the value 8 for this inequality. This is a tight Bell inequality in the scenario of two parties, with three dichotomic inputs each, which exhibits several remarkable properties [26–28]. With regard to its monogamy properties, it was shown in [26] that the nonlocality that it reveals can be shared, i.e., there exist three qubit states $|\psi\rangle_{ABC}$ such that both $\rho_{AB} = \mathrm{tr}_C(|\psi\rangle\langle\psi|)$ and $\rho_{AC} = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ violate the inequality at the same time. Here we show another remarkable property of the $\mathcal{I}_{3322}$ inequality, namely, that the addition of a minimal local state-dependent contextuality term can make the $\mathcal{I}_{3322}$ inequality monogamous. More precisely,

$$\mathcal{I}^{A_1A_4A_6}_{B_1B_2B_3} + \mathcal{I}^{A_1A_4A_6}_{C_1C_2C_3} + 2\mathcal{I}(5) \overset{ND}{\leqslant} 14, \qquad (3)$$

where $\mathcal{I}(5) = \sum_{i=1}^{4}\langle A_iA_{i+1}\rangle - \langle A_5A_1\rangle \leqslant 3$ is the noncontextual bound.

Proof. To show the validity of the above inequality, let us decompose the commutation graph of the whole quantity in the following way:

$$\begin{aligned}& G\big[\mathcal{I}^{A_1A_4A_6}_{B_1B_2B_3}\big] + G\big[\mathcal{I}^{A_1A_4A_6}_{C_1C_2C_3}\big] + 2\times G[\mathcal{I}(5)] \\ & \to G\big[\mathcal{I}^{A_1A_4A_6}_{B_1,C_2,A_5}\big] + G\big[\mathcal{I}^{A_1A_4A_6}_{B_2,C_1,A_5}\big] + G\big[\mathcal{I}(5)^{A_1A_2A_3A_4}_{B_3}\big] \\ & \quad + G\big[\mathcal{I}(5)^{A_1A_2A_3A_4}_{C_3}\big], \qquad (4)\end{aligned}$$

where $G[\mathcal{I}]$ denotes the commutation graph corresponding to the set of observables appearing in the inequality $\mathcal{I}$. It can be checked (as shown in Fig. 1) that all the decomposed induced graphs are chordal and hence possess joint probability distributions.

To see the correspondence between (1) and (3), one can reinterpret the monogamy relation (1) in terms of the purity of the subsystem A (denoted by $\mathfrak{p}_A$) as $\tau_{AB} + \tau_{AC} + \mathfrak{p}_A \leqslant 1$. Similarly, the purity of a system can be related to the resource of state-dependent contextuality [29]. This generalizes the view presented in [14], where a single CHSH inequality was shown to have a trade-off in violation with a local contextuality term. In the above scenario, on the other hand, it is worth noting that a single $\mathcal{I}_{3322}$ inequality does not exhibit monogamy with the local noncontextuality inequality $\mathcal{I}(5)$. In fact, a no-signaling box that violates both inequalities can be found [23].

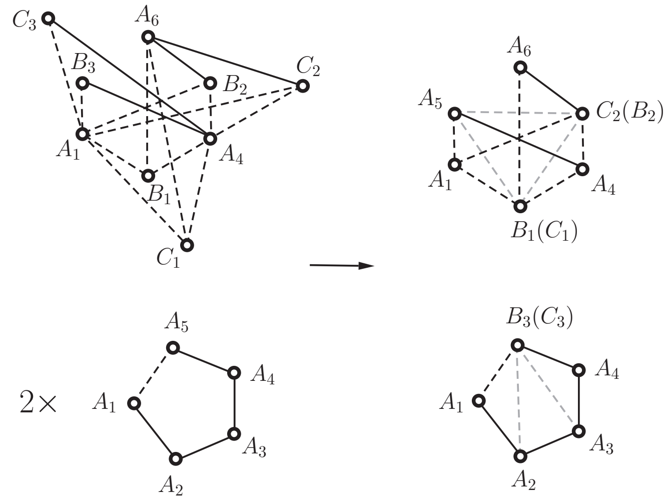

FIG. 1. Chordal decomposition given in Eq. (4) to show the monogamy relation between the bipartite $\mathcal{I}_{3322}$ correlations of Alice and Bob, Alice and Charlie, and between commuting measurements on Alice's subsystem. Here the dashed line represents the contradiction edge and gray lines denote the additional commutation relations.

VI. LACK OF MONOGAMIES DUE TO NONTRIVIAL BELL INEQUALITIES WITH SINGLE INPUTS FOR SOME PARTIES

In this section we study a unique feature of Bell inequalities that appears in the derivation of no-signaling monogamy relations. There exists a two-party Bell expression, which has the same classical and no-signaling value, that can be turned into a nontrivial inequality upon the addition of an expression involving a third party with a single measurement input. We present such an example in the scenario of three parties with one, two, and three inputs, respectively, and four outputs per setting. While this appears counterintuitive at first sight, we explain that this arises due to an incompatibility between the optimal classical strategies for the subexpressions into which the commutation graph of the whole Bell expression is decomposed.

In [10] it was shown that in the paradigmatic example of correlation inequalities for binary outcomes, the parameter known as the contradiction number gives a sufficient characterization of the monogamy. Namely, if the removal of a certain number m of observables of any one party (m is called the contradiction number of the inequality) results in the residual expression having a local hidden variable description, then monogamy manifests itself when Alice performs the correlation Bell experiment with $m+1$ Bobs. The existence of nontrivial Bell inequalities with single inputs for some of the parties as stated above then implies that this result does not readily extend to general inequalities for many outcomes.

We study a specific example of an inequality $\mathcal{I}^{A_1A_2A_3}_{B_1B_2B_3}$ that has a contradiction number of one, i.e., upon the removal of the input B_3 for Bob, the residual expression $\mathcal{I}^{A_1A_2A_3}_{B_1B_2}$ admits a local hidden variable description [a joint probability distribution $P(A_1,A_2,A_3,B_1,B_2)$]. The residual expression $\mathcal{I}^{A_1A_2A_3}_{C_1}$ also evidently admits a classical description $P(A_1,A_2,A_3,C_1)$ simply because Charlie only measures a single input in this expression. Intuitively, one might expect that the expression

TABLE I. Description of the no-signaling probability distribution for which the tripartite inequality (6) attains its algebraic value 9. Each column corresponds to six different inputs A_k, B_l, C_1, and for a particular input only the events $P(a_0^{(k)}a_1^{(k)}, b_0^{(l)}b_1^{(l)}, c_0^{(1)}c_1^{(1)}|A_k, B_l, C_1)$ with nonzero probability are listed where the probability of each of these events is $\frac{1}{8}$.

$A_1B_1C_1$	$A_1B_2C_1$	$A_2B_1C_1$	$A_2B_2C_1$	$A_3B_1C_1$	$A_3B_2C_1$
(00, 00, 00)	(00, 00, 00)	(00, 00, 00)	(00, 00, 00)	(00, 00, 01)	(00, 00, 01)
(00, 00, 01)	(00, 00, 01)	(00, 00, 10)	(00, 10, 10)	(00, 00, 10)	(00, 11, 10)
(00, 01, 00)	(00, 01, 00)	(01, 00, 01)	(01, 01, 01)	(01, 00, 00)	(01, 01, 00)
(00, 01, 01)	(00, 01, 01)	(01, 00, 11)	(01, 11, 11)	(01, 00, 11)	(01, 10, 11)
(01, 00, 10)	(01, 10, 10)	(10, 01, 01)	(10, 00, 01)	(10, 01, 00)	(10, 00, 00)
(01, 00, 11)	(01, 10, 11)	(10, 01, 11)	(10, 10, 11)	(10, 01, 11)	(10, 11, 11)
(01, 01, 10)	(01, 11, 10)	(11, 01, 00)	(11, 01, 00)	(11, 01, 01)	(11, 01, 01)
(01, 01, 11)	(01, 11, 11)	(11, 01, 10)	(11, 11, 10)	(11, 01, 10)	(11, 10, 10)

$\mathcal{I}^{A_1A_2A_3}_{B_1B_2C_1}$ also admits such a description, for instance, via the Fine trick [21]

$$
\begin{aligned}
&P(A_1,A_2,A_3,B_1,B_2,C_1)\\
&\quad = \frac{P(A_1,A_2,A_3,B_1,B_2)P(A_1,A_2,A_3,C_1)}{P(A_1,A_2,A_3)}. \qquad (5)
\end{aligned}
$$

Here the subtlety arises. The optimal strategy that achieves the classical value for $\mathcal{I}^{A_1A_2A_3}_{B_1B_2}$ need not give rise to the same marginal distribution $P(A_1,A_2,A_3)$ as the optimal strategy for $\mathcal{I}^{A_1A_2A_3}_{C_1}$. This implies that the above construction does not automatically work and therefore one might encounter violation of Bell expressions $\mathcal{I}^{A_1A_2A_3}_{B_1B_2C_1}$ for which $\mathcal{I}^{A_1A_2A_3}_{B_1B_2}$ is trivially saturated by a classical strategy and yet the contradiction arises from a third party measuring a single setting. We give below an example of a Bell expression having such a property.

This is a Bell expression $\mathcal{I}^{A_1A_2A_3}_{B_1B_2B_3}$ belonging to the family of tight Bell expressions in the (2,3,3,4,4) scenario found by Cabello [30] and has been experimentally tested [31]. The expression of interest to us is actually the reduced Bell expression $\mathcal{I}^{A_1A_2A_3}_{B_1B_2C_1}$ given as

$$
\begin{aligned}
\mathcal{I}^{A_1A_2A_3}_{B_1B_2C_1} &= \langle A_1^{10}B_1^{10}\rangle + \langle A_1^{01}B_2^{10}\rangle + \langle A_1^{11}C_1^{10}\rangle\\
&\quad + \langle A_2^{10}B_1^{01}\rangle + \langle A_2^{01}B_2^{01}\rangle + \langle A_2^{11}C_1^{01}\rangle\\
&\quad + \langle A_3^{10}B_1^{11}\rangle + \langle A_3^{01}B_2^{11}\rangle - \langle A_3^{11}C_1^{11}\rangle \leqslant 7, \qquad (6)
\end{aligned}
$$

where $\langle A_k^{i_k,j_k}B_l^{i_l,j_l}\rangle$ denotes the mean value of the product of the i_kth and j_kth bits (assigned ± 1 values) of the result of measuring A_k times the i_lth and j_lth bits of the result of measuring B_l (or C_l), respectively. Evidently, a classical strategy can be found for the residual expression $\mathcal{I}^{A_1A_2A_3}_{B_1B_2}$ when B_3 is removed, for instance, Alice and Bob output 00. The residual expression $\mathcal{I}^{A_1A_2A_3}_{C_1}$ also trivially admits a classical strategy, for instance, Charlie outputs 00 and Alice outputs 00 for inputs A_1 and A_2 and 10 for input A_3. On the other hand, no classical strategy exists to saturate both expressions simultaneously and it can be shown that $\mathcal{I}^{A_1A_2A_3}_{B_1B_2C_1} \leqslant 7$ in any local hidden variable theory. From the extremal box of the no-signaling polytope given in Table I, it can be verified that the algebraic value of the inequality is 9. This implies that the Bell inequalities and the corresponding classical polytopes with single inputs for some of the parties have some interesting property.

VII. CONCLUSION

In this paper we have presented a unified view on monogamy relations for the Bell and noncontextuality inequalities derived from the physical principles of no signaling and no disturbance. The unification was achieved by considering a graph-theoretic decomposition of the graph representing all the observables in the experiment in induced chordal subgraphs. We have used this method to show that any two generalized cycle inequalities (for any number of outputs) exhibit a monogamy relation. As a main result, the feature of activation of monogamy of the Bell inequality by considering local contextuality term is proposed. Finally, we uncovered an interesting characteristic of Bell inequalities that the trivial Bell expression can have no-signaling violations upon the addition of a third party with a single input. This implies that the study of Bell inequalities (and the corresponding classical polytopes) with single inputs for some of the parties becomes interesting.

Several open questions remain. An interesting question is to settle the computational complexity of the chordal decomposition method for identifying monogamies. Another important question is to show that local state-independent contextuality inequalities cannot replace the state-dependent ones in the monogamies we have formulated in the paper. If true, this would lend added strength to the resource-theoretic character of the trade-offs as suggested in [14]. It is also relevant to study the correspondence between such a monogamous relation and the feature in which nonlocality can be revealed from local contextuality [32–34]. It would also be of interest to identify the quantum boundaries of the trade-offs between nonlocality and contextuality identified here, such as done for the case of Bell inequalities in [35,36]. The extension of the above results to the multiparty scenario and network configurations such as in [36] is of special interest, particularly with regard to cryptographic applications such as secret sharing [37]. Finally, it would also be interesting to investigate the class of minimal noncontextuality inequalities whose addition activates the monogamy of any Bell inequality.

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[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.95.030104 for (a) the proof that every chordal commutation graph admits a joint probability distribution based on simplicial elimination ordering; (b) counterexample showing the non-necessity of chordal decompositions to derive monogamy relations: (c) proof of the proposition regarding the monogamy relations of two cycle inequalities; and (d) the explicit no-signalling box that violates $\mathcal{I}_{3322}$ and $\mathcal{I}(5)$.
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