Number of superclasses of four-qubit entangled states under the inductive entanglement classification

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L. Lamata *et al.* use an inductive approach to classify the entangled pure states of four qubits under stochastic local operations and classical communication (SLOCC) [Phys. Rev. A **75**, 022318 (2007)]. The inductive method yields *a priori* ten entanglement superclasses, of which they discard three as empty. One of the remaining superclasses is split into two, resulting in eight superclasses of genuine four-qubit entanglement. Here, we show that two of the three discarded superclasses are in fact nonempty and should have been retained. We give explicit expressions for the canonical states of those superclasses, up to SLOCC and qubit permutations. Furthermore, we confirm that the third discarded superclass is indeed empty, yielding a total of ten superclasses of genuine four-qubit entanglement under the inductive classification scheme.

I. INTRODUCTION

Entangled states are an important resource in quantum information and computation. The classification of entanglement under stochastic local operations and classical communication (SLOCC) aims to group quantum states according to which quantum information tasks they can accomplish [1]. This classification is an important question in quantum theory, and it has not been resolved for more than four qubits.

This is in part because there are infinitely many distinct entanglement classes under SLOCC for four or more qubits [2]. Approaches to the classification of four-qubit entanglement group these infinitely many classes into "superclasses", which contain entanglement classes with similar structures that are nevertheless distinct under SLOCC [3,4]. In the following, we sometimes use the word "class" to refer to a superclass; it should be clear from the context which is meant.

One scheme for entanglement classification of multiqubit states is the inductive approach of Lamata *et al.* [5]. In a subsequent paper in 2007, the same authors apply this approach to the classification of four-qubit states [3], reporting eight distinct entanglement superclasses of genuine four-partite entanglement (up to qubit permutations). In the process, they consider three other potential superclasses, discarding them with the claim that they are empty. We point out that in two cases this is erroneous: the classes called $\mathfrak{W}_{0_k\Psi,W}$ and $\mathfrak{W}_{W,W}$ in the naming scheme from [3] should not have been discarded because they do contain states that do not fall into any other entanglement superclass. For these two entanglement superclasses, we not only show that they are nonempty but also derive their respective canonical states, into which any state in the given class can be transformed by SLOCC. This is analogous to the approach taken for the other entanglement classes in [3]. We, furthermore, confirm that in the third case—that of the class called $\mathfrak{W}_{\mathrm{GHZ,GHZ}}$—Lamata *et al.* were correct in discarding it.

Throughout this paper, we use the equational rules for the classification of three-qubit entanglement derived by Li *et al.* in 2006 [6].

In the following, we first recap the inductive entanglement classification in Sec. II and the equational rules in Sec. III. The new results are reported in Sec. IV.

II. THE INDUCTIVE ENTANGLEMENT CLASSIFICATION

In the inductive entanglement classification of Lamata *et al.*, introduced in [5] and applied to four-qubit states in [3], entanglement classes are distinguished as follows: Consider an n-qubit state $|\psi\rangle$. Let $|v_0\rangle$ and $|v_1\rangle$ be two linearly independent states of the first qubit. Then the full state can be written as

$$|\psi\rangle = |v_0\rangle|\phi_0\rangle + |v_1\rangle|\phi_1\rangle \tag{1}$$

for some $(n-1)$-qubit states $|\phi_0\rangle$, $|\phi_1\rangle$. Note that $|v_0\rangle$, $|v_1\rangle$ do not need to be normalized, and even if they are chosen to be normalized, $|\phi_0\rangle$ and $|\phi_1\rangle$ need not be. As SLOCC operations can change the norm of states, in general we work with unnormalized states anyway.

The entanglement class—or rather superclass, as there are infinitely many SLOCC classes on four or more qubits—is determined by the types of $(n-1)$-qubit entangled vectors found in different spanning sets for $\mathrm{span}\{|\phi_0\rangle,|\phi_1\rangle\}$. Those in turn are given by the entanglement classes of $(n-1)$-qubit states: hence the inductiveness of the classification process.

By convention, states are classified according to the spanning sets for $\mathrm{span}\{|\phi_0\rangle,|\phi_1\rangle\}$ containing vectors with the "least amount of entanglement". In the case of the classification of four-qubit entanglement, the chosen order of the entanglement classes of three-qubit states is [3]

$$000 < 0\Psi < \mathrm{GHZ}, W, \tag{2}$$

where, furthermore, GHZ is usually considered before W. Here, 000 denotes a fully separable state, and 0Ψ denotes a state which is the product of a single-qubit state and an entangled two-qubit state. This type of state is sometimes referred to as a "bipartite separable state". GHZ and W are the usual classes of fully entangled three-qubit states [2], the (unnormalized) standard representatives of which we write as $|\mathrm{GHZ}\rangle = |000\rangle + |111\rangle$ and $|W\rangle = |001\rangle + |010\rangle + |100\rangle$, respectively.

Lamata *et al.* label the entanglement classes according to the types of entangled vectors in the spanning set $\mathfrak{W}_{X,Y}$ where

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X and Y take the values 000, 0Ψ, GHZ, or W. In the case of 0Ψ, the bipartition may be specified by a subscript, i.e., $0_k\Psi$, where $k \in \{1,2,3\}$, denotes a state where the kth qubit is in a product with an entangled state of the remaining two qubits.

We give a few examples of the classification conditions. Consider a four-qubit state $|\psi\rangle$ and decompose it as in (1). Let $\mathfrak{W} = \text{span}\{|\phi_0\rangle,|\phi_1\rangle\}$. Then:

(i) If there is a spanning set for $\mathfrak{W}$ that contains two fully separable states, $|\psi\rangle$ is in the class $\mathfrak{W}_{000,000}$.

(ii) If there is a spanning set containing one fully separable state and one bipartite separable state and there are no spanning sets containing two fully separable states, $|\psi\rangle$ is in the class $\mathfrak{W}_{000,0\Psi}$.

(iii) If there is a spanning set containing one fully separable state and one GHZ state and there are no spanning sets containing two separable states of which at least one is fully separable, $|\psi\rangle$ is in the class $\mathfrak{W}_{000,\text{GHZ}}$.

(iv) If there is a spanning set containing one fully separable state and one W state and there are no spanning sets containing a fully separable state together with a non-W type state, then $|\psi\rangle$ is in the class $\mathfrak{W}_{000,W}$.

(v) If there is a spanning set containing two bipartite separable states and there are no fully separable states in $\mathfrak{W}$, then $|\psi\rangle$ is in a $\mathfrak{W}_{0\Psi,0\Psi}$ class.

(vi) And so on.

Always picking the spanning set with the "lowest" entanglement makes the classification unique: if, e.g., some subspace $\mathfrak{W}$ has a spanning set containing two linearly independent fully separable states, then there are certainly also spanning sets containing entangled states. Hence without the "lowest entanglement" criterion, many states would fall into multiple entanglement classes.

Determining the entanglement class of arbitrary three-qubit states is thus a crucial part of the inductive entanglement classification of four-qubit states. To do this, Lamata *et al.* use a method based on the ranks of *coefficient matrices* and some of their other properties [3,5]. Coefficient matrices arise from a state vector as partial transposes. For example, a three-qubit state expressed in the computational basis as

$$\sum_{i,j,k\in\{0,1\}} a_{ijk}|ijk\rangle, \tag{3}$$

where $a_{000},a_{001},\ldots,a_{111} \in \mathbb{C}$, has a 2×4 coefficient matrix:

$$C_{1|23} = \begin{pmatrix} a_{000} & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{101} & a_{110} & a_{111} \end{pmatrix}. \tag{4}$$

There are also two other 2×4 coefficient matrices for the same state, labeled $C_{2|13}$ and $C_{3|12}$, which arise by permuting the role of the qubits. A three-qubit state is fully separable if and only if all three coefficient matrices have rank 1. Further rules, some of them involving additional properties of the coefficient matrices or their submatrices, can be used to identify exactly to which of the three-qubit entanglement classes a given state belongs (see Theorem 1 in [3]).

Yet, in the context of the inductive entanglement classification, it is not just single three-qubit states whose entanglement class needs to be identified. Instead, it is generally necessary to determine which types of entanglement arise in a subspace spanned by two three-qubit states, whose definitions may

themselves involve free parameters. The process of determining the rank and other properties of a 2×4 matrix depending on multiple parameters for all values of these parameters is tedious and error-prone: it is easy to overlook special cases arising for specific values of some of the parameters. This seems to be the origin of the errors in [3]: the generic coefficient matrices given there for the entanglement superclasses $\mathfrak{W}_{0_k\Psi,W}$ and $\mathfrak{W}_{W,W}$ are correct, but the analysis of their properties is flawed.

III. EQUATIONS FOR THE CLASSIFICATION OF THREE-QUBIT ENTANGLEMENT

For four or more qubits, the classification (or, rather, the grouping of entanglement classes into superclasses) is not unique. Different classification schemes thus cannot easily be combined or interchanged. This problem does not arise for three-qubit states: there is only a small finite number of entanglement classes—fully separable states, three types of bipartite separable states, and GHZ and W states—whose definitions are generally accepted. We are thus free to choose any method for identifying the entanglement class of a three-qubit state. Hence, we use the following equational method derived by Li *et al.* [6], which we find much more straightforward than the classification method from [3] and [5], described above.

This equational method works as follows. Consider a nonvanishing three-qubit state expressed in the computational basis as

$$a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle$$
$$+ a_5|101\rangle + a_6|110\rangle + a_7|111\rangle, \tag{5}$$

where $a_0,a_1,\ldots,a_7 \in \mathbb{C}$ are not all 0. This state is in the GHZ SLOCC class if and only if the following expression is nonzero:

$$(a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4)^2 - 4(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7). \tag{6}$$

The state is in the W SLOCC class if and only if (6) is 0, and furthermore,

$$(a_0a_3 \neq a_1a_2 \vee a_5a_6 \neq a_4a_7)$$
$$\wedge (a_1a_4 \neq a_0a_5 \vee a_3a_6 \neq a_2a_7)$$
$$\wedge (a_3a_5 \neq a_1a_7 \vee a_2a_4 \neq a_0a_6). \tag{7}$$

We sometimes refer to formula (7) as the "W conditions".

In all other cases, the state is not genuinely three partite entangled: all three clauses are false for a fully product state; for bipartite states, one of the clauses is satisfied and two are not [6]. This classification is basis independent to some degree: as SLOCC transformations do not change the entanglement class of a state, the same equations hold when the state is expressed in any basis that arises from the computational basis via SLOCC, i.e., via applying an invertible 2×2 matrix to each qubit.

IV. THE POTENTIALLY EMPTY FOUR-QUBIT ENTANGLEMENT CLASSES

In [3], the inductive entanglement classification is applied to four-qubit states. *A priori*, and up to qubit permutations, there are 10 potential entanglement classes, given that there are four entanglement classes of three-qubit states: $\mathfrak{W}_{000,000}$, $\mathfrak{W}_{000,0\Psi}$, $\mathfrak{W}_{000,\text{GHZ}}$, $\mathfrak{W}_{000,W}$, $\mathfrak{W}_{0\Psi,0\Psi}$, $\mathfrak{W}_{0\Psi,\text{GHZ}}$, $\mathfrak{W}_{0\Psi,W}$, $\mathfrak{W}_{\text{GHZ},\text{GHZ}}$, $\mathfrak{W}_{\text{GHZ},W}$, and $\mathfrak{W}_{W,W}$.

Lamata *et al.* consider these classes one by one and discard three of them as being empty: $\mathfrak{W}_{0\Psi,W}$, $\mathfrak{W}_{\text{GHZ},\text{GHZ}}$ [7], and $\mathfrak{W}_{W,W}$. On the other hand, they split the class of states where the spanning set consists of two bipartite entangled states according to whether the two bipartitions are the same or different, resulting in $\mathfrak{W}_{0_k\Psi,0_k\Psi}$ and $\mathfrak{W}_{0_i\Psi,0_j\Psi}$. This gives a total of eight classes of genuine four-qubit entanglement. Additionally, there are also classes of four-qubit states that do not exhibit any four-partite entanglement; these are ignored here.

We reanalyze the three discarded classes using the equational classification criteria of Li *et al.* [6] (cf. Sec. III). With these methods, we show that, while $\mathfrak{W}_{\text{GHZ},\text{GHZ}}$ is indeed empty, there are in fact states that belong in $\mathfrak{W}_{0\Psi,W}$ or $\mathfrak{W}_{W,W}$, respectively, according to the inductive classification.

A. The entanglement class $\mathfrak{W}_{0_k\Psi,W}$

The entanglement class labeled $\mathfrak{W}_{0_k\Psi,W}$ in the naming scheme in [3] is the class containing states

$$|\psi\rangle = |v_0\rangle|\phi_0\rangle + |v_1\rangle|\phi_1\rangle, \tag{8}$$

where span$\{|\phi_0\rangle,|\phi_1\rangle)\}$ contains no vectors of type 000, exactly one vector of type 0Ψ, and no vectors of GHZ type. Up to permutations of the last three qubits, a generic representative of this class can be written as

$$|\phi\rangle|\varphi_1\rangle|\Psi_1\rangle + |\bar{\phi}\rangle(|\varphi_2\psi_2\bar{\theta}_2\rangle + |\varphi_2\bar{\psi}_2\theta_2\rangle + |\bar{\varphi}_2\psi_2\theta_2\rangle)). \tag{9}$$

Here, lowercase Greek letters label single-qubit states. Throughout, states do not need to be normalized (though they may not have norm 0). Overbars denote linear independence, i.e., $|\phi\rangle$ and $|\bar{\phi}\rangle$ are linearly independent, as are $|\varphi_2\rangle$ and $|\bar{\varphi}_2\rangle$, and so on. Different indices denote states that may or may not be linearly independent, e.g., $|\varphi_1\rangle$ may be linearly dependent on $|\varphi_2\rangle$, or $|\bar{\varphi}_2\rangle$, or neither. The state $|\Psi_1\rangle$ is an entangled two-qubit state.

To simplify the representative state, we can apply a SLOCC operation that maps $\{|\phi\rangle,|\bar{\phi}\rangle\}$ to the computational basis and, similarly, all pairs of linearly independent states with index 2. This yields

$$|0\rangle|\varphi\rangle|\Psi\rangle + |1\rangle(|001\rangle + |010\rangle + |100\rangle)$$
$$= |0\rangle|\varphi\rangle|\Psi\rangle + |1\rangle|W\rangle, \tag{10}$$

where $|\varphi\rangle$ is the result of applying the SLOCC operation described above to $|\varphi_1\rangle$, and similarly for $|\Psi\rangle$. Now $|\varphi\rangle$ and $|\Psi\rangle$ can be expressed in the computational basis as $|\varphi\rangle = \varphi_0|0\rangle + \varphi_1|1\rangle$ and

$$|\Psi\rangle = \Psi_{00}|00\rangle + \Psi_{01}|01\rangle + \Psi_{10}|10\rangle + \Psi_{11}|11\rangle, \tag{11}$$

where entanglement of $|\Psi\rangle$ implies that $\Psi_{00}\Psi_{11} - \Psi_{01}\Psi_{10} \neq 0$. Furthermore, φ_0 and φ_1 cannot both be 0.

Lamata *et al.* claim that span$\{|\varphi\Psi\rangle,|W\rangle\}$ always contains a GHZ-type vector and that therefore the $\mathfrak{W}_{0_k\Psi,W}$ class is empty, as any state with a spanning set of type $0\Psi,W$ also has one of type $0\Psi,\text{GHZ}$ and thus falls into the $\mathfrak{W}_{0_k\Psi,\text{GHZ}}$ class by the ordering of the classes [3]. We show by analysis of the different combinations of parameter values that this is not correct and, furthermore, identify a canonical state for the $\mathfrak{W}_{0_k\Psi,W}$ class.

An arbitrary state in span$\{|\varphi\Psi\rangle,|W\rangle\}$ can be written as $x|\varphi\Psi\rangle + y|W\rangle$ for some $x,y \in \mathbb{C}$. Expanding this yields

$$x\varphi_0\Psi_{00}|000\rangle + (x\varphi_0\Psi_{01} + y)|001\rangle + (x\varphi_0\Psi_{10} + y)|010\rangle$$
$$+ x\varphi_0\Psi_{11}|011\rangle + (x\varphi_1\Psi_{00} + y)|100\rangle + x\varphi_1\Psi_{01}|101\rangle$$
$$+ x\varphi_1\Psi_{10}|110\rangle + x\varphi_1\Psi_{11}|111\rangle. \tag{12}$$

Now, using (6), we find that this state is in the GHZ SLOCC class if and only if

$$\left(\left(\varphi_0^2\Psi_{11}^2 + 2\varphi_0\varphi_1\Psi_{01}\Psi_{11} + 2\varphi_0\varphi_1\Psi_{10}\Psi_{11} + \varphi_1^2\Psi_{01}^2 \right.\right.$$
$$\left.\left. - 2\varphi_1^2\Psi_{01}\Psi_{10} + \varphi_1^2\Psi_{10}^2 + 4\varphi_1^2\Psi_{00}\Psi_{11}\right)x + 4\varphi_1\Psi_{11}y\right)xy^2 \neq 0. \tag{13}$$

There are no GHZ-type states in the subspace if and only if this polynomial is 0 for all values of x and y, i.e., if and only if

$$0 = \varphi_0^2\Psi_{11}^2 + 2\varphi_0\phi_1\Psi_{01}\Psi_{11} + 2\varphi_0\varphi_1\Psi_{10}\Psi_{11} + \varphi_1^2\Psi_{01}^2$$
$$- 2\varphi_1^2\Psi_{01}\Psi_{10} + \varphi_1^2\Psi_{10}^2 + 4\varphi_1^2\Psi_{00}\Psi_{11} \tag{14}$$

and also

$$4\varphi_1\Psi_{11} = 0. \tag{15}$$

For each case in which the subspace contains no GHZ vectors, we, furthermore, need to check that it contains no separable vectors other than $|\varphi\Psi\rangle$. To exclude $|\varphi\Psi\rangle$, we assume that $y \neq 0$ and, for simplicity, rescale so that $y = 1$. We distinguish cases according to the solutions of (15).

1. Case $\varphi_1 = 0$

If $\varphi_1 = 0$, then we must also have $\varphi_0 \neq 0$. In this case, (14) reduces to $\varphi_0^2\Psi_{11}^2 = 0$. As φ_0 cannot vanish, we must have $\Psi_{11} = 0$ and hence $\Psi_{01}\Psi_{10} \neq 0$ by entanglement of $|\Psi\rangle$. The W conditions, (7), become

$$(0 \neq (\varphi_0\Psi_{01}x + 1)(\varphi_0\Psi_{10}x + 1))$$
$$\wedge (\varphi_0\Psi_{01}x + 1 \neq 0) \wedge (\varphi_0\Psi_{10}x + 1 \neq 0). \tag{16}$$

All parameters appearing in those inequalities are nonzero, thus there are always separable states in the subspace: set $x = -1/(\varphi_0\Psi_{01})$ or $x = -1/(\varphi_0\Psi_{10})$.

2. Case $\Psi_{11} = 0$

If $\Psi_{11} = 0$, then we must also have $\Psi_{01}\Psi_{10} \neq 0$. In this case, (14) reduces to $\varphi_1^2(\Psi_{01} - \Psi_{10})^2 = 0$. Hence there are two subcases.

(i) $\varphi_1 = 0$, which implies $\varphi_0 \neq 0$. This brings us back to the case considered in Sec. IV A 1, above.

(ii) $\Psi_{01} = \Psi_{10}$: under this assumption, the state is in the W SLOCC class if

$$\left((0 \neq (\varphi_0 \Psi_{01} x + 1)^2)\right) \vee \left(\varphi_1^2 \Psi_{01}^2 x^2 \neq 0\right)$$
$$\wedge \left((\varphi_0 \Psi_{01} + \varphi_1 \Psi_{00})x + 1 \neq 0\right). \tag{17}$$

The last inequality represents two clauses in the original set of three, which have become identical under the current choice of parameter values. Now, the following cases occur:

(a) If $\varphi_1 = 0 \neq \varphi_0$, there exists a 000-type state in the subspace: setting $x = -1/(\varphi_0 \Psi_{01})$ makes all the inequalities false.

(b) If $\varphi_1 \neq 0$ and $\varphi_0 \Psi_{01} + \varphi_1 \Psi_{00} \neq 0$, there is a separable state in the space, which can be constructed by setting $x = -1/(\varphi_0 \Psi_{01} + \varphi_1 \Psi_{00})$. (Recall that the last inequality represents two clauses of the original set.)

(c) If $\varphi_1 \neq 0$ and $\varphi_0 \Psi_{01} + \varphi_1 \Psi_{00} = 0$, any state in the subspace (other than $|\varphi\Psi\rangle$) is in the W class. To see this, note that $\varphi_0 \Psi_{01} + \varphi_1 \Psi_{00} = 0$ implies that the last inequality in (17) is always satisfied. The first two inequalities would both need to be false simultaneously for the state not to be in the W class. But the second inequality is false only for $x = 0$, for which the first inequality is satisfied. Hence states of this form are classified into $\mathfrak{W}_{0_k \Psi, W}$ by the inductive scheme.

This concludes the analysis of all cases in which span$\{|\varphi\Psi\rangle, |W\rangle\}$ contains no GHZ-type states.

3. The canonical state for $\mathfrak{W}_{0_k\Psi,W}$

From the above, the canonical state for $\mathfrak{W}_{0_k\Psi,W}$ satisfies $\Psi_{11} = 0$, $\Psi_{01} = \Psi_{10}$, $\varphi_1 \neq 0$, and $\varphi_0 \Psi_{01} + \varphi_1 \Psi_{00} = 0$. Since $\varphi_1 \neq 0$, we can write $\Psi_{00} = -\varphi_0 \Psi_{01}/\varphi_1$. Then the canonical generator is

$$(\varphi_0|0\rangle + \varphi_1|1\rangle)\left(-\frac{\varphi_0 \Psi_{01}}{\varphi_1}|00\rangle + \Psi_{01}|\Psi^+\rangle\right), \tag{18}$$

where $|\Psi^+\rangle = |01\rangle + |10\rangle$. Let $\lambda = \varphi_0/\varphi_1$, and the generator becomes

$$\varphi_1 \Psi_{01}(\lambda|0\rangle + |1\rangle)(-\lambda|00\rangle + |\Psi^+\rangle). \tag{19}$$

From the previous conditions, we must have $\lambda, \varphi_1, \Psi_{01} \in \mathbb{C}$ and $\varphi_1 \Psi_{01} \neq 0$. The canonical state is

$$\varphi_1 \Psi_{01}|0\rangle(\lambda|0\rangle + |1\rangle)(-\lambda|00\rangle + |\Psi^+\rangle) + |1\rangle|W\rangle. \tag{20}$$

Note that the nonzero factor $\varphi_1 \Psi_{01}$ can be removed by a SLOCC operation on the first qubit. Then the canonical state becomes

$$|0\rangle(\lambda|0\rangle + |1\rangle)(-\lambda|00\rangle + |\Psi^+\rangle) + |1\rangle|W\rangle, \tag{21}$$

where λ is arbitrary.

B. The entanglement class $\mathfrak{W}_{\text{GHZ,GHZ}}$

Lamata *et al.* consider the class $\mathfrak{W}_{\text{GHZ},W}$ before $\mathfrak{W}_{\text{GHZ,GHZ}}$, i.e., if a state qualifies for both, it is considered to be in $\mathfrak{W}_{\text{GHZ},W}$. This means that a state is in $\mathfrak{W}_{\text{GHZ,GHZ}}$ if and only if span$\{|\phi_0\rangle, |\phi_1\rangle\}$ contains only GHZ-type vectors, where the state has been decomposed as in (1). Now, Lamata *et al.* argue that the span of two GHZ-type vectors always contains a W state or a separable state and that therefore $\mathfrak{W}_{\text{GHZ,GHZ}}$ is empty [3]. To ensure that our revisions of the classification are complete, we confirm this result using the methods from [6].

A generic representative of $\mathfrak{W}_{\text{GHZ,GHZ}}$ can be written as

$$|\phi\rangle(|\varphi_1\psi_1\theta_1\rangle + |\bar{\varphi}_1\bar{\psi}_1\bar{\theta}_1\rangle) + |\bar{\phi}\rangle(|\varphi_2\psi_2\theta_2\rangle + |\bar{\varphi}_2\bar{\psi}_2\bar{\theta}_2\rangle), \tag{22}$$

with the same conventions as before. Via a SLOCC operation, this state can be transformed to

$$|0\rangle(|\varphi\psi\theta\rangle + |\bar{\varphi}\bar{\psi}\bar{\theta}\rangle) + |1\rangle(|000\rangle + |111\rangle)$$
$$= |0\rangle(|\varphi\psi\theta\rangle + |\bar{\varphi}\bar{\psi}\bar{\theta}\rangle) + |1\rangle|\text{GHZ}\rangle, \tag{23}$$

where $|\varphi\rangle$ is the result of applying the SLOCC operation to $|\varphi_1\rangle$, and similarly for the other variable states.

Rather than expanding each single-qubit state in the computational basis (leading to 12 parameters and three linear-independence conditions), it will be easier to expand the full state $|\varphi\psi\theta\rangle + |\bar{\varphi}\bar{\psi}\bar{\theta}\rangle$ instead, which only requires 8 parameters and the GHZ condition. Thus let

$$|\varphi\psi\theta\rangle + |\bar{\varphi}\bar{\psi}\bar{\theta}\rangle = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle$$
$$+ a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle, \tag{24}$$

where $a_0, \ldots, a_7 \in \mathbb{C}$ satisfy

$$0 \neq (a_0 a_7 - a_2 a_5 + a_1 a_6 - a_3 a_4)^2$$
$$- 4(a_2 a_4 - a_0 a_6)(a_3 a_5 - a_1 a_7). \tag{25}$$

A general element of span$\{|\varphi\psi\theta\rangle + |\bar{\varphi}\bar{\psi}\bar{\theta}\rangle, |\text{GHZ}\rangle\}$ then has the form

$$x(|\varphi\psi\theta\rangle + |\bar{\varphi}\bar{\psi}\bar{\theta}\rangle) + y|\text{GHZ}\rangle \tag{26}$$

or, equivalently,

$$(a_0 x + y)|000\rangle + a_1 x|001\rangle + a_2 x|010\rangle + a_3 x|011\rangle$$
$$+ a_4 x|100\rangle + a_5 x|101\rangle + a_6 x|110\rangle + (a_7 x + y)|111\rangle, \tag{27}$$

where $x, y \in \mathbb{C}$. For a state to be in the class $\mathfrak{W}_{\text{GHZ,GHZ}}$, all nonvanishing elements of the subspace must be of the GHZ type, i.e., we need the polynomial,

$$\begin{aligned}&\left(a_3^2 a_4^2 - 2a_2 a_3 a_4 a_5 + a_2^2 a_5^2 - 2a_1 a_3 a_4 a_6 - 2a_1 a_2 a_5 a_6 + 4a_0 a_3 a_5 a_6 + a_1^2 a_6^2 + 4a_1 a_2 a_4 a_7 - 2a_0 a_3 a_4 a_7 - 2a_0 a_2 a_5 a_7\right.\\&\left. - 2a_0 a_1 a_6 a_7 + a_0^2 a_7^2\right)x^4 + \left(4a_1 a_2 a_4 - 2a_0 a_3 a_4 - 2a_0 a_2 a_5 - 2a_0 a_1 a_6 + 4a_3 a_5 a_6 + 2a_0^2 a_7 - 2a_3 a_4 a_7 - 2a_2 a_5 a_7\right.\\&\left. - 2a_1 a_6 a_7 + 2a_0 a_7^2\right)x^3 y + \left(a_0^2 - 2a_3 a_4 - 2a_2 a_5 - 2a_1 a_6 + 4a_0 a_7 + a_7^2\right)x^2 y^2 + 2(a_0 + a_7)xy^3 + y^4, \end{aligned} \tag{28}$$

to be nonzero whenever at least one of x and y is nonzero. This means that, given any fixed nonzero value for x (or y), the

remaining polynomial in the other variable must have no roots. A polynomial has no roots over $\mathbb{C}$ only if it is equal to some

nonzero constant. Hence for any nonzero x, the polynomial must be independent of y. That cannot be achieved here for any choice of $a_0, \ldots, a_7$, as the coefficient of y^4 is independent of the parameters and of x. Hence the subspace spanned by two GHZ states always contains non-GHZ states and the class $\mathfrak{W}_{\text{GHZ,GHZ}}$ is indeed empty.

C. The entanglement class $\mathfrak{W}_{W,W}$

Finally, consider the class $\mathfrak{W}_{W,W}$, i.e., the class containing states

$$|\psi\rangle = |v_0\rangle|\phi_0\rangle + |v_1\rangle|\phi_1\rangle, \qquad (29)$$

where $\text{span}\{|\phi_0\rangle, |\phi_1\rangle\}$ contains only W-type vectors. A generic representative of this class can be written as

$$|\phi\rangle(|\varphi_1\psi_1\bar{\theta}_1\rangle + |\varphi_1\bar{\psi}_1\theta_1\rangle + |\bar{\varphi}_1\psi_1\theta_1\rangle)$$
$$+ |\bar{\phi}\rangle(|\varphi_2\psi_2\bar{\theta}_2\rangle + |\varphi_2\bar{\psi}_2\theta_2\rangle + |\bar{\varphi}_2\psi_2\theta_2\rangle). \qquad (30)$$

As before, overbars denote linear independence. States with different indices may or may not be linearly dependent, as long as the two W-type states are linearly independent of each other.

To simplify this representative state, apply a SLOCC transformation that maps $\{|\phi\rangle, |\bar{\phi}\rangle\}$ to the computational basis, and similarly, all pairs of linearly independent states with index 2. This yields

$$|0\rangle(|\varphi\psi\bar{\theta}\rangle + |\varphi\bar{\psi}\theta\rangle + |\bar{\varphi}\psi\theta\rangle) + |1\rangle(|001\rangle + |010\rangle + |100\rangle), \qquad (31)$$

where $|\varphi\rangle$ is the result of applying the SLOCC operation described above to $|\varphi_1\rangle$, and similarly for the other variable

states. Rather than expanding each single-qubit state in the computational basis, it is easier to expand the full state and use the W conditions from Sec. III. Thus let

$$|\varphi\psi\bar{\theta}\rangle + |\varphi\bar{\psi}\theta\rangle + |\bar{\varphi}\psi\theta\rangle$$
$$= a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle$$
$$+ a_5|101\rangle + a_6|110\rangle + a_7|111\rangle, \qquad (32)$$

where $a_0, \ldots, a_7 \in \mathbb{C}$ satisfy

$$0 = (a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4)^2$$
$$- 4(a_2a_4 - a_0a_6)(a_3a_5 - a_1a_7) \qquad (33)$$

and

$$(a_0a_3 \neq a_1a_2 \vee a_5a_6 \neq a_4a_7)$$
$$\wedge (a_1a_4 \neq a_0a_5 \vee a_3a_6 \neq a_2a_7)$$
$$\wedge (a_3a_5 \neq a_1a_7 \vee a_2a_4 \neq a_0a_6). \qquad (34)$$

A general element of

$$\text{span}\{|\varphi\psi\bar{\theta}\rangle + |\varphi\bar{\psi}\theta\rangle + |\bar{\varphi}\psi\theta\rangle, |W\rangle\} \qquad (35)$$

has the form

$$x(|\varphi\psi\bar{\theta}\rangle + |\varphi\bar{\psi}\theta\rangle + |\bar{\varphi}\psi\theta\rangle) + y|W\rangle \qquad (36)$$

or

$$a_0x|000\rangle + (a_1x + y)|001\rangle + (a_2x + y)|010\rangle + a_3x|011\rangle$$
$$+ (a_4x + y)|100\rangle + a_5x|101\rangle + a_6x|110\rangle + a_7x|111\rangle, \qquad (37)$$

where $x, y \in \mathbb{C}$. We need all nonvanishing elements of this subspace to be of the W type (the case $y = 0$ automatically includes $|\varphi\psi\bar{\theta}\rangle + |\varphi\bar{\psi}\theta\rangle + |\bar{\varphi}\psi\theta\rangle$ itself). By (6), this requires the following polynomial in x and y to be identically 0:

$$(a_3^2a_4^2 - 2a_2a_3a_4a_5 + a_2^2a_5^2 - 2a_1a_3a_4a_6 - 2a_1a_2a_5a_6 + 4a_0a_3a_5a_6 + a_1^2a_6^2 + 4a_1a_2a_4a_7 - 2a_0a_3a_4a_7$$
$$- 2a_0a_2a_5a_7 - 2a_0a_1a_6a_7 + a_0^2a_7^2)x^4 + (2a_3^2a_4 - 2a_2a_3a_5 - 2a_3a_4a_5 + 2a_2a_5^2 - 2a_1a_3a_6 - 2a_3a_4a_6$$
$$- 2a_1a_5a_6 - 2a_2a_5a_6 + 2a_1a_6^2 + 4a_1a_2a_7 - 2a_0a_3a_7 + 4a_1a_4a_7 + 4a_2a_4a_7 - 2a_0a_5a_7 - 2a_0a_6a_7)x^3y$$
$$+ (a_3^2 - 2a_3a_5 + a_5^2 - 2a_3a_6 - 2a_5a_6 + a_6^2 + 4a_1a_7 + 4a_2a_7 + 4a_4a_7)x^2y^2 + 4a_7xy^3. \qquad (38)$$

Additionally, by (7), it requires the following three statements to be true whenever x and y are not both 0:

$$(y^2 + (a_1 + a_2)xy + (a_1a_2 - a_0a_3)x^2 \neq 0)$$
$$\vee (a_7xy + (a_4a_7 - a_5a_6)x^2 \neq 0), \qquad (39)$$

$$(y^2 + (a_1 + a_4)xy + (a_1a_4 - a_0a_5)x^2 \neq 0)$$
$$\vee (a_7xy + (a_2a_7 - a_3a_6)x^2 \neq 0), \qquad (40)$$

$$(a_7xy + (a_1a_7 - a_3a_5)x^2 \neq 0)$$
$$\vee (y^2 + (a_2 + a_4)xy + (a_2a_4 - a_0a_6)x^2 \neq 0). \qquad (41)$$

The GHZ polynomial, (38), is identically 0 if and only if each coefficient is 0. For the coefficient of xy^3 this implies

$a_7 = 0$. Given that assumption, the coefficient of x^2y^2 vanishes if

$$a_3^3 - 2a_3a_5 - 2a_3a_6 + a_5^2 - 2a_5a_6 + a_6^2 = 0. \qquad (42)$$

We consider several cases.

1. Case $a_6 = 0 \wedge a_5 = a_3$

With $a_6 = 0$ and $a_5 = a_3$, the GHZ polynomial reduces to $(a_2 - a_4)^2 a_3^2 x^4$. Hence there are two subcases:

(a) $a_3 = 0$, in which case the W conditions, (39)–(41), become

$$(0 \neq (a_1x + y)(a_2x + y))$$
$$\wedge ((a_1x + y)(a_4x + y) \neq 0)$$
$$\wedge ((a_2x + y)(a_4x + y) \neq 0). \qquad (43)$$

As the $y = 0$ case must be a W-type state, this implies $a_1, a_2, a_4 \neq 0$. But then the subspace always contains separable states by setting $y = -a_1 x$ or $y = -a_2 x$ or $y = -a_4 x$ for any nonzero x.

(b) $a_3 \neq 0$ and $a_4 = a_2$, in which case the W conditions become

$$((a_1 a_2 - a_0 a_3)x^2 + (a_1 + a_2)xy + y^2 \neq 0)$$
$$\wedge ((a_3^2 x^2 \neq 0) \vee ((a_2 x + y)^2 \neq 0)), \tag{44}$$

where the first inequality represents both (39) and (40). Hence for the $y = 0$ case to be a W-type state, we need $a_0 a_3 \neq a_1 a_2$. But then the subspace always contains separable states by setting y to a root of $(a_1 a_2 - a_0 a_3)x^2 + (a_1 + a_2)xy + y^2$ (where x is considered another parameter and we are free to assume that it is nonzero).

The cases $((a_5{=}0) \wedge (a_6{=}a_3))$ and $((a_3{=}0) \wedge (a_6{=}a_5))$ are analogous.

2. Case $a_3, a_5, a_6 \neq 0 \wedge a_6 = a_3 + a_5 \pm 2\sqrt{a_3 a_5}$

First, note that $a_3 a_5 \pm a_3\sqrt{a_3 a_5} = 0$ would imply that at least one of a_3, a_5, and a_6 is 0. Hence, in the current case we must have $a_3 a_5 \pm a_3\sqrt{a_3 a_5} \neq 0$. Thus, for the coefficient of $x^3 y$ to vanish, we require

$$a_4 = \frac{2a_1 a_3 a_5 - a_2 a_3 a_5 \pm \sqrt{a_3 a_5}(a_1 a_3 + a_1 a_5 - a_2 a_5)}{a_3 a_5 \pm a_3\sqrt{a_3 a_5}}, \tag{45}$$

and this expression is well defined. Then the GHZ polynomial becomes

$$\frac{\left(a_1^2 - 2a_1 a_2 + a_2^2 + 4a_0 a_3\right)\left(a_3^2 + 6a_3 a_5 + a_5^2 \pm 4\sqrt{a_3 a_5}a_3 \pm 4\sqrt{a_3 a_5}a_5\right)a_5^2 x^4}{\left(a_5 \pm \sqrt{a_3 a_5}\right)^2}. \tag{46}$$

Again, this is well defined, as $a_5 \pm \sqrt{a_3 a_5} = 0$ implies either $a_5 = 0$ or $a_3 = a_5$ and $a_6 = 0$; so under the assumptions of the current case, $a_5 \pm \sqrt{a_3 a_5}$ must always be nonzero.

There are two subcases.

(a) $a_3^2 + 6a_3 a_5 + a_5^2 \pm 4\sqrt{a_3 a_5}a_3 \pm 4\sqrt{a_3 a_5}a_5 = 0$. This implies

$$\left(a_3^2 + 6a_3 a_5 + a_5^2\right)^2 = 16a_3 a_5(a_3 + a_5)^2, \tag{47}$$

which is equivalent to $(a_3 - a_5)^4 = 0$, so it implies $a_3 = a_5$. Then the equation $a_3^2 + 6a_3 a_5 + a_5^2 \pm 4\sqrt{a_3 a_5}a_3 \pm 4\sqrt{a_3 a_5}a_5 = 0$ becomes

$$8a_3\left(a_3 \pm \sqrt{a_3^2}\right) = 0. \tag{48}$$

We assumed $a_3 \neq 0$, so this can only be satisfied if the square root function and sign are such that $a_3 \pm \sqrt{a_3^2} = 0$. But $a_3 = a_5$ and $a_3 \pm \sqrt{a_3^2} = 0$ together imply that $a_3 + a_5 \pm 2\sqrt{a_3 a_5} = 0$, which contradicts the assumption that $a_6 \neq 0$. Hence this case cannot happen.

(b) $a_1^2 - 2a_1 a_2 + a_2^2 + 4a_0 a_3 = 0$, i.e., $a_0 = -(a_1 - a_2)^2/(4a_3)$. In this case, the subspace contains no GHZ states (by construction). The W conditions, (39)–(41), become

$$\left(-\tfrac{1}{4}((a_1 + a_2)x + 2y)^2 \neq 0\right) \vee ((a_3 + a_5 \pm 2\sqrt{a_3 a_5})a_5 x^2 \neq 0), \tag{49}$$

$$(P(x,y) \neq 0) \vee ((a_3 + a_5 \pm 2\sqrt{a_3 a_5})a_3 x^2 \neq 0), \tag{50}$$

and

$$(a_3 a_5 x^2 \neq 0) \vee (Q(x,y) \neq 0), \tag{51}$$

where

$$P(x,y) = (4a_3(a_5 \pm \sqrt{a_3 a_5}))^{-1}\big(4(3a_1 a_3 a_5 - a_2 a_3 a_5 \pm 2\sqrt{a_3 a_5}a_1 a_3 \pm \sqrt{a_3 a_5}a_1 a_5 \mp \sqrt{a_3 a_5}a_2 a_5)xy$$
$$+ 4a_3(a_5 \pm \sqrt{a_3 a_5})y^2 + \big(8a_1^2 a_3 a_5 - 4a_1 a_2 a_3 a_5 + a_1^2 a_5^2 - 2a_1 a_2 a_5^2 + a_2^2 a_5^2 \pm 4\sqrt{a_3 a_5}a_1^2 a_3$$
$$\pm 5\sqrt{a_3 a_5}a_1^2 a_5 \mp 6\sqrt{a_3 a_5}a_1 a_2 a_5 \pm \sqrt{a_3 a_5}a_2^2 a_5)x^2\big) \tag{52}$$

and

$$Q(x,y) = (4a_3(a_5 \pm \sqrt{a_3 a_5}))^{-1}\big(4(2a_1 a_3 a_5 \pm \sqrt{a_3 a_5}a_1 a_3 \pm \sqrt{a_3 a_5}a_2 a_3 \pm \sqrt{a_3 a_5}a_1 a_5 \mp \sqrt{a_3 a_5}a_2 a_5)xy$$
$$+ 4a_3(a_5 \pm \sqrt{a_3 a_5})y^2 + \big(3a_1^2 a_3 a_5 + 2a_1 a_2 a_3 a_5 - a_2^2 a_3 a_5 + a_1^2 a_5^2 - 2a_1 a_2 a_5^2 + a_2^2 a_5^2 \pm \sqrt{a_3 a_5}a_1^2 a_3$$
$$\pm 2\sqrt{a_3 a_5}a_1 a_2 a_3 \pm \sqrt{a_3 a_5}a_2^2 a_3 \pm 3\sqrt{a_3 a_5}a_1^2 a_5 \mp 2\sqrt{a_3 a_5}a_1 a_2 a_5 \mp \sqrt{a_3 a_5}a_2^2 a_5)x^2\big). \tag{53}$$

By assumption, a_3, a_5, and $a_3 + a_5 \pm 2\sqrt{a_3 a_5}$ are nonzero, so for $x \neq 0$ the W conditions are satisfied. With $x = 0 \neq y$, the W conditions can be seen to reduce to just one inequality, $y^2 \neq 0$, which is clearly satisfied (as it should be, since the $x = 0$ state

is $y|W\rangle$ and thus a W state by construction). This means that the subspace contains no separable states and states of this type belong in $\mathfrak{W}_{W,W}$ according to the classification.

The canonical generator has the form

$$-\frac{(a_1 - a_2)^2}{4a_3}|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + \frac{(2a_1 - a_2)a_3a_5 \pm (a_1a_3 + a_1a_5 - a_2a_5)\sqrt{a_3a_5}}{a_3a_5 \pm \sqrt{a_3a_5}a_3}|100\rangle$$
$$+ a_5|101\rangle + (a_3 + a_5 \pm 2\sqrt{a_3a_5})|110\rangle, \qquad (54)$$

where a_3 and a_5 are nonzero, with $a_3 + a_5 \pm 2\sqrt{a_3a_5}$ also nonzero for the given choice of square root function and sign, and a_1 and a_2 are arbitrary.

This concludes the investigation of all potential members of $\mathfrak{W}_{W,W}$.

3. The canonical state for $\mathfrak{W}_{W,W}$

Given the canonical generator above, the canonical state for $\mathfrak{W}_{W,W}$ can be written (up to SLOCC) as

$$|0\rangle\left(-\frac{(a_1 - a_2)^2}{4a_3}|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + \frac{(2a_1 - a_2)a_3a_5 \pm (a_1a_3 + a_1a_5 - a_2a_5)\sqrt{a_3a_5}}{a_3a_5 \pm \sqrt{a_3a_5}a_3}|100\rangle\right.$$
$$\left. + a_5|101\rangle + (a_3 + a_5 \pm 2\sqrt{a_3a_5})|110\rangle\right) + |1\rangle|W\rangle, \qquad (55)$$

where a_3 and a_5 are nonzero, with $a_3 + a_5 \pm 2\sqrt{a_3a_5}$ also nonzero for the given choice of square root function and sign, and a_1 and a_2 are arbitrary.

In fact, we can remove one parameter via SLOCC. Note that

$$-\frac{(a_1 - a_2)^2}{4a_3}|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + \frac{(2a_1 - a_2)a_3a_5 \pm (a_1a_3 + a_1a_5 - a_2a_5)\sqrt{a_3a_5}}{a_3a_5 \pm \sqrt{a_3a_5}a_3}|100\rangle$$
$$+ a_5|101\rangle + (a_3 + a_5 \pm 2\sqrt{a_3a_5})|110\rangle \qquad (56)$$

is equal to

$$-\frac{(a_1 - a_2)^2}{4a_3}|000\rangle + (a_2 - a_1)|010\rangle + a_3|011\rangle + \frac{(a_1 - a_2)(a_3a_5 \pm a_5\sqrt{a_3a_5})}{a_3a_5 \pm a_3\sqrt{a_3a_5}}|100\rangle$$
$$+ a_5|101\rangle + (a_3 + a_5 \pm 2\sqrt{a_3a_5})|110\rangle + a_1|W\rangle, \qquad (57)$$

where we have separated out a copy of $a_1|W\rangle$. Now let $\mu = a_1 - a_2$; then the canonical state can be transformed to

$$|0\rangle\left(-\frac{\mu^2}{4a_3}|000\rangle - \mu|010\rangle + a_3|011\rangle + \mu\frac{a_5(a_3 \pm \sqrt{a_3a_5})}{a_3(a_5 \pm \sqrt{a_3a_5})}|100\rangle + a_5|101\rangle + (a_3 + a_5 \pm 2\sqrt{a_3a_5})|110\rangle\right) + |1\rangle|W\rangle \qquad (58)$$

by a SLOCC operation on the first qubit.

V. CONCLUSIONS

In their inductive classification of four-qubit states, Lamata *et al.* discard three potential superclasses as not having any members, namely, the ones labeled $\mathfrak{W}_{0_k\Psi,W}$, $\mathfrak{W}_{\text{GHZ,GHZ}}$, and $\mathfrak{W}_{W,W}$. We show that, while they are correct in stating that $\mathfrak{W}_{\text{GHZ,GHZ}}$ is empty, the other two classes are in fact nonempty.

In particular, we find that the $\mathfrak{W}_{0_k\Psi,W}$ class consists of the following one-parameter family of states, up to SLOCC and

permutations of the last three qubits:

$$|0\rangle(\lambda|0\rangle + |1\rangle)(-\lambda|00\rangle + |\Psi^+\rangle) + |1\rangle|W\rangle, \qquad (59)$$

where $\lambda \in \mathbb{C}$. Furthermore, we show that $\mathfrak{W}_{W,W}$ consists of the following three-parameter family of states, again up to SLOCC:

$$|0\rangle\left(-\frac{\mu^2}{4a_3}|000\rangle - \mu|010\rangle + a_3|011\rangle + \mu\frac{a_5(a_3 \pm \sqrt{a_3a_5})}{a_3(a_5 \pm \sqrt{a_3a_5})}|100\rangle + a_5|101\rangle + (a_3 + a_5 \pm 2\sqrt{a_3a_5})|110\rangle\right) + |1\rangle|W\rangle, \qquad (60)$$

where a_3 and a_5 are nonzero complex numbers satisfying $a_3 + a_5 \pm 2\sqrt{a_3a_5} \neq 0$ for the given choice of square root function and sign, and μ is an arbitrary complex number.

In the inductive classification scheme, there are hence ten (rather than eight) entanglement superclasses of four-qubit genuinely entangled states, with corresponding effects on the expected number of entanglement superclasses of five or more qubits.

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