# Born-Kothari condensation in an ideal Fermi gas

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"Condensation" in Fermi-Dirac statistics [D. S. Kothari and B. Nath, Nature **151**, 420 (1943)], which appears as a natural consequence of Born's reciprocity principle [M. Born, Proc. R. Soc. London A **165**, 291 (1938); Nature **141**, 328 (1938)], is examined from a theoretical perspective. Since fermions obey the Pauli exclusion principle, it is conceptually different from Bose-Einstein condensation, which permits macroscopic occupation of bosons at the single-particle level below a critical temperature. Yet, in accordance with the Cahill and Glauber [Phys. Rev. A **59**, 1538 (1999)] formulation for fermionic fields, and in close kinship to bosonic fields, we have shown that in analogy to Bose-Einstein condensation, it is possible to associate an intrinsic notion of symmetry breaking and the thermodynamic "order parameter" to characterize the foregoing hitherto unexplored phenomenon in an ideal Fermi-Dirac gas as *condensation-like coherence within fermions*.

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## I. INTRODUCTION

Trapped dilute, atomic gases have proven to be remarkable model systems for the realization of quantum statistical effects at the fundamental level, such as the direct observation of Bose-Einstein condensation (BEC) [1–3] at ultracold temperatures. Perhaps not as dramatic as the phase transition of bosons, the behavior of trapped Fermi gases also merits attention in its own right, both as a degenerate quantum system and as a possible precursor to a paired Fermi condensate at very low temperatures [4–6]. The use of mean-field theory, the pseudopotential, and consideration of fluctuations around the mean field has already led to interesting advancements in the understanding of basic physics of ultracold Fermi gases [7–9].

Here, instead, we invoke a seldom used but reasonable basis for the possible existence of a "condensed phase" for an ideal Fermi-Dirac (FD) gas [10] that is based on the notion of Born's reciprocity theory [11], which is considered one of the cornerstones for the development of the theory of elementary particles [12,13] and other related fields [14,15]. The attempt was made many decades ago by Kothari and Nath [10] in the course of examining the relationship between Born's reciprocity principle [11] with FD statistics. In the present study we put forward a convenient description of such states following Cahill and Glauber [16], on the basis of close parallelism between the expressions found for fermionic fields and the more familiar ones for bosonic fields that resulted in a unification between the two seemingly distinct seminal works. Our approach, however, is not in contradiction to the standard pairing theory for fermions [17].

A key element of our formulation is the fermionic coherent state, defined as a unitary displaced state of all singly occupied filled up modes. The transformation with this displacement operator displaces the fermionic field operators over anticommuting (Grassmann) numbers [18–20]. This resembles the harmonic oscillator coherent state, for which the displacement operator displaces the bosonic field operators over classical commuting variables [21–23]. The basic question behind the present approach is whether a state of macroscopic coherence for an FD gas can be described as a fermionic coherent state. This approach gives an equivalent result to the problem of a fixed number of particles $N$ in the limit $N \longrightarrow \infty$ for the BEC case [24–26]. In the same spirit, here, we have extended the coherent-state approach of Cahill and Glauber [16] to its fermionic counterpart. It forms an essential ingredient for demonstration of the thermodynamic limit, fermionic order parameter, and spontaneous symmetry breaking of the state comprising FD statistics.

The paper is organized as follows: In Sec. II we revisit the Born-Kothari approach to stimulate the motivation for the present work. Since the fermionic coherent state plays a crucial role in the formulation of the problem, in Sec. III we briefly review the relevant parts of the coherent state of fermions as developed by Cahill and Glauber [16]. The aspect of the thermodynamic limit, spontaneous symmetry breaking and the fermionic analog of the order parameter are then introduced to comprehend the Born-Kothari criterion for so-called "condensation" as a state of macroscopic coherence that can be depicted as a fermionic coherent state. The paper is concluded in Sec. IV.

## II. REVISITING "CONDENSATION" IN FD STATISTICS: THE BORN-KOTHARI APPROACH

In the spirit of condensation phenomena for a Bose-Einstein gas (where a condensed phase is formed by the particles in the lowest energy state), Kothari and Nath have shown condensation in FD statistics [10] as a direct consequence of Born's reciprocity principle [11–15], where the condensed phase is formed by particles in the highest energy state. According to Born's reciprocity theory [11], the number of wave functions of a particle of weight factor $g$ (due to its internal degrees of freedom) within the momentum range $p$ to $p + dp$ is given by

$$a(p)dp = \frac{4\pi Vg}{(2\pi\hbar)^3}\frac{p^2 dp}{(1 - p^2/\mathtt{b}^2)^{1/2}}. \tag{1}$$

It is indeed essential to limit the momentum $p$ in the above equation by an absolute constant $\mathtt{b}$ (whose existence is ensured

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by the fundamental laws of quantum mechanics, which are symmetric in space-time $x^k$ and momentum energy $p_k$), since the reciprocity theory "is based on a demand for symmetry" as mentioned by Born (see, for example, letters to Einstein [15]). Therefore the total number of wave functions ($a_0$) possible in a volume $V$ is (according to Born [11])

$$a_0 = \int_0^{\mathtt{b}} a(p)dp = g\frac{\pi^2 V \mathtt{b}^3}{(2\pi\hbar)^3}. \quad (2)$$

As is evident for an FD gas, any independent wave function cannot accommodate more than one particle due to the *Pauli exclusion principle*, and Eq. (2) imposes an upper limit ($N \equiv N_0 = a_0$) on the number of particles that can be contained in volume $V$. So, the natural question arises, What will happen to an FD gas when the number of particles $N$ in the given volume exceeds $N_0$?

The answer was given by Kothari and Nath [10] that, upon replacing the sum of all possible eigenstates extending from ($p = 0$) to ($p = \mathtt{b}$) with the integral in Eq. (2), the total number of eigenstates has become limited, though it is in reality infinite. The total number of states $\mathtt{a}$ thus becomes

$$\mathtt{a} = a_0 + a_1, \quad (3)$$

where $a_1$ takes care of the infinitely large number of eigenstates corresponding to $p \to \mathtt{b}$. It follows, therefore, that unlike in an ideal Bose gas, when the number of particles $N$ in an FD gas exceeds $N_0$, irrespective of any critical temperature, a *condensed phase* (a band of dense eigenstates) containing $(N - N_0)$ particles will be formed, all possessing momentum $\mathtt{b}$ [10]. The existence of such a condensed phase, consisting of particles in the highest energy state, $p = \mathtt{b}$, depends on Born's reciprocity theory [11].

One point should be noted here. The above inference respects the Pauli exclusion principle and does not contradict our basic knowledge, which rules out the possibility of macroscopic occupation for fermions at the single-particle level [27]. The question thus remains whether it is possible to attribute the condensation in Ref. [10] with a close connection to BEC. In the following, we have shown that it is, however, possible to reach such a consensus in the framework of coherent states using the statistical properties of fermionic fields as a close equivalent to bosonic fields, pioneered by Cahill and Glauber [16]. The decisive advantage of this coherent-state formalism is that it is no longer restricted by the stringent Yang criterion [27] of a large occupation number for fermionic fields that must abide by the single-particle picture. This accounts for a consistent theory of so-called Born-Kothari condensation in terms of fermionic coherent states.

To proceed further toward a systematic description, we consider a noninteracting FD gas consisting of $N$ particles. The Hamiltonian can be expressed in terms of the field operator $\hat{\psi}(r)$ as

$$\hat{\mathcal{H}} = \int \left(\frac{\hbar^2}{2m}\nabla\hat{\psi}^\dagger(r)\nabla\hat{\psi}(r)\right)dr. \quad (4)$$

The field operator $\hat{\psi}(r)$ annihilates a particle at a position $r$ and can be expressed as $\hat{\psi}(r) = \sum_k \phi_k(r)\hat{a}_k$. $\hat{a}_k$ ($\hat{a}_k^\dagger$) is the annihilation (creation) operator of a particle in the single-particle state $\phi_k(r)$ that obeys anticommutation relations:

$$\{\hat{a}_k,\hat{a}_l^\dagger\} = \delta_{kl}, \quad \{\hat{a}_k,\hat{a}_l\} = \{\hat{a}_k^\dagger,\hat{a}_l^\dagger\} = 0. \quad (5)$$

The wave functions $\phi_k(r)$ satisfy the orthonormal condition

$$\int \phi_k^*(r)\phi_l(r) = \delta_{kl}. \quad (6)$$

The field operator then follows the anticommutation relation $\{\hat{\psi}(r),\hat{\psi}^\dagger(r')\} = \sum_k \phi_k(r)\phi_k^*(r') = \delta(r - r')$.

We first separate the field operator $\hat{\psi}(r)$ into a noncommuting "classical" field $\psi(r)$ and the quantum fluctuation $\delta\hat{\psi}(r)$ about its classical component as

$$\hat{\psi}(r) = \psi(r) + \delta\hat{\psi}(r). \quad (7)$$

We do this separation by taking a hint from the Bogoliubov approximation [28] for bosons and the close similarity of the expressions evaluated for bosonic and fermionic fields [16]. When the quantum fluctuation is neglected, as happens for a dilute Bose gas at a very low temperature, the field operator coincides with the classical field $\psi(r)$ and the system behaves as a classical object [24–26]. Here lies an important difference between the bosonic fields and their fermionic counterparts since the latter always remain "classical". This sense of "classicality" for fermionic fields is revisited in the next section, where we consider the fermionic "order parameter" from the physical point of view.

The ansatz for the field operator in Eq. (7) can be interpreted as an expectation value $\langle\hat{\psi}(r)\rangle$ different from 0. Such states are coherent states, which are well known for bosonic fields and form the basis for understanding BEC [24–26]. However, a straightforward extension of the scheme to their fermionic counterparts is nontrivial. The primary reason may be traced to the basics; as pointed out by Schwinger [29], since fermions anticommute, their eigenvalues must be anticommuting numbers. While such numbers are well studied in mathematics [18] and quantum-field theory [19,20], other applications of such anticommuting (Grassmann) numbers are much less popular. However, in recent times, a number of investigations of fermionic systems by adopting Grassmann variables have appeared in the literature [30–37]. Among them are the phase-space methods for degenerate Fermi gases [30], superfluidity [33] and Cooper-like pairing in trapped fermions [34], quantum Monte Carlo methods and counting statistics of strongly correlated fermions [32], the fermionic analog of a parametric amplifier [36], and treatment of dissipation and the stochastic Schrodinger equation in fermionic baths [35], to name just a few. In the next section we digress slightly to discuss this algebra of anticommuting numbers centering around the fermionic coherent state [16], which forms an integral part of our analysis in the remainder of the paper.

## III. FERMIONIC COHERENT STATES: CONNECTION TO MACROSCOPIC COHERENCE FOR FERMIONS

For a system of fermions described by the creation $a_k^\dagger$ and annihilation $a_k$ operators obeying Eq. (5), the set of Grassmann variables $\boldsymbol{\gamma} = \{\gamma_k\}$ satisfies the following anticommutation relations:

$$\gamma_k\gamma_l + \gamma_l\gamma_k \equiv \{\gamma_k,\gamma_l\} = 0. \quad (8)$$

This immediately suggests that $\gamma_k^2 = 0$ for any given $k$. For the general properties of such anticommuting numbers and a description of fermionic coherent states, we refer the reader to the classic book of Berezin [18] and other related works [30–38]. Here instead we follow the approach developed by Cahill and Glauber [16] for the statistical properties of fermionic fields and cite only the parts necessary to make our presentation self-contained.

The Grassmann number $\gamma_k$ and its complex conjugate $\gamma_k^*$ are independent numbers and obey $\{\gamma_k,\gamma_k^*\} = 0$. They also anticommute with their fermionic operators $\{\gamma_k,\hat{a}_l\} = 0$; $\{\gamma_k,\hat{a}_l^\dagger\} = 0$. Hermitian conjugation reverses the order of all fermionic quantities, both the operators and the anticommuting variables.

### A. Fermionic coherence theory

In analogy to the harmonic oscillator coherent state $|\boldsymbol{\alpha}\rangle$ [21], which is defined as a displaced state where the displacement operator $\hat{\mathbf{D}}(\boldsymbol{\alpha}) = \exp(\sum_i(\alpha_i\hat{a}_i^\dagger - \alpha_i^*\hat{a}_i))$ acts on the vacuum $|0\rangle$ as $|\boldsymbol{\alpha}\rangle = \hat{\mathbf{D}}(\boldsymbol{\alpha})|0\rangle$, $\{\alpha_i\}$ being a set of complex numbers [21–23], it is possible to construct the displacement operator for fermions [16] as

$$\hat{\mathcal{D}}(\boldsymbol{\gamma}) = \exp\left(\sum_i(\hat{a}_i^\dagger\gamma_i - \gamma_i^*\hat{a}_i)\right) \quad (9)$$

for a set of $\boldsymbol{\gamma} = \{\gamma_i\}$ Grassmann variables.

The *normalized* fermionic coherent state can then be constructed by the action of this displacement operator [Eq. (9)] on the vacuum state as $|\boldsymbol{\gamma}\rangle = \hat{\mathcal{D}}(\boldsymbol{\gamma})|0\rangle$. Using the displacement operator [Eq. (9)] we may show that the coherent state is an eigenstate of every annihilation operator $\hat{a}_k$, $\hat{a}_k|\boldsymbol{\gamma}\rangle = \gamma_k|\boldsymbol{\gamma}\rangle$, with eigenvalue $\gamma_k$. The adjoint of the coherent state $|\boldsymbol{\gamma}\rangle$ can be similarly defined as $\langle\boldsymbol{\gamma}|\hat{a}_k^\dagger = \langle\boldsymbol{\gamma}|\gamma_k^*$. The inner product of two coherent states is given by [36]

$$\langle\boldsymbol{\gamma}|\boldsymbol{\beta}\rangle = \exp\left(\sum_i \gamma_i^*\beta_i - \frac{1}{2}(\gamma_i^*\gamma_i + \beta_i^*\beta_i)\right), \quad (10)$$

and using the completeness relation of the coherent states, any arbitrary coherent state $|\boldsymbol{\beta}\rangle$ can be expanded as $|\boldsymbol{\beta}\rangle = \int d^2\boldsymbol{\gamma}\langle\boldsymbol{\gamma}|\boldsymbol{\beta}\rangle|\boldsymbol{\gamma}\rangle$, which immediately follows from the resolution of identity $\int d^2\boldsymbol{\gamma}|\boldsymbol{\gamma}\rangle\langle\boldsymbol{\gamma}| = \mathbf{I}$. It is worth pointing out that for fermionic fields the integrations are taken over anticommuting numbers and for such pairs we confine ourselves to typical notation, $\int d^2\boldsymbol{\gamma} \equiv \int\prod_i d^2\gamma_i$, where $\int d^2\gamma_i = \int d\gamma_i^* d\gamma_i$, and we should keep in mind that $d\gamma_i^* d\gamma_i = -d\gamma_i d\gamma_i^*$.

Unlike the harmonic oscillator, in addition to a lower bound, the fermionic oscillator possesses an upper bound [19]. Thus, for any set $\boldsymbol{\alpha} = \{\alpha_i\}$ of Grassmann numbers, the normalized coherent state $|\boldsymbol{\alpha}\rangle'$ for fermionic fields can also be defined as an eigenstate of every creation operator,

$$\hat{a}_k^\dagger|\boldsymbol{\alpha}\rangle' = \alpha_k^*|\boldsymbol{\alpha}\rangle', \quad (11)$$

as the displaced state

$$|\boldsymbol{\alpha}\rangle' = \hat{\mathcal{D}}(\boldsymbol{\alpha})|\mathbf{1}\rangle, \quad (12)$$

where $|\mathbf{1}\rangle$ denotes the state in which every mode is filled,

$$|\mathbf{1}\rangle = \prod_k \hat{a}_k^\dagger|0\rangle. \quad (13)$$

The corresponding adjoint relation and the identity operator are given by

$$'\langle\boldsymbol{\alpha}|\hat{a}_k = '\langle\boldsymbol{\alpha}|\alpha_k, \qquad \int\prod_i(-d^2\alpha_i)|\boldsymbol{\alpha}\rangle'\,'\langle\boldsymbol{\alpha}| = \mathbf{I}. \quad (14)$$

It is shown that the eigenstate of the creation operators $|\boldsymbol{\alpha}\rangle'$ plays a central role in our theoretical formulation because it stems from the characteristic upper bound displayed by a general system of fermions. It may be noted that in Eq. (9) the creation operator $\hat{a}_i^\dagger$ stands to the left of the Grassmann variable $\gamma_i$. For a fermionic field, since both operators and the Grassmann numbers anticommute with each other, special care must be taken with the ordering of all fermionic quantities. Apart from these ordering procedures, Eqs. (9)–(14) appear to be rather similar to their bosonic counterparts and they can be employed in the same analytical techniques as used for bosonic fields [30–38]. The differences are merely in their mathematical backgrounds [18–20].

We conclude this section with one pertinent remark. A principal use of the Glauber-Sudarshan $P$ representation for bosonic fields has been the evaluation of normally ordered correlation functions, which play an important role in the theory of coherence and the statistics of photon-counting experiments [22]. Analogously, *correlation functions* $G^{(n)}(x_1,\ldots,x_n,y_n,\ldots,y_1)$ *for fermionic fields* as a function of a space-time variable are defined as the central dogma [16,38] in *fermionic coherence theory*, which can be shown to play a similar role in the description of fermionic atom-counting experiments [32,36,38].

### B. Characterization of the "condensate" state

In the case of BEC, the condensed phase, which is the ground state of the many-body system, can be interpreted a displaced vacuum state $|\boldsymbol{\alpha}\rangle = \hat{\mathbf{D}}(\boldsymbol{\alpha})|0\rangle$, where $\boldsymbol{\alpha}$ is the complex numbers. For an FD gas, due to the Pauli exclusion principle each eigenstate can accommodate only a single particle. Hence a condensed phase, when formed as the ground state of the many-body system, can only occur in the form of a thick band in the highest energy state $p = \mathtt{b}$, since Born's reciprocity principle [11] allows an *infinitely large number of eigenstates corresponding to* $p \to \mathtt{b}$. Taking into account the fermionic coherent states proposed by Cahill and Glauber [16], and in close analogy to a Bose condensate in an ideal Bose gas [24–26], we may visualize that the condensate of an ideal FD gas [10], as indicated by Kothari and Nath, can best be described, in view of our Eq. (12), as a displaced state $|\boldsymbol{\alpha}\rangle' = \hat{\mathcal{D}}(\boldsymbol{\alpha})|\mathbf{1}\rangle$, where $|\mathbf{1}\rangle = \prod_{n=N_0}^{n=N} a_n^\dagger|0\rangle$, in principle, corresponds to an infinite number of single-occupancy dense modes approaching the $p \to \mathtt{b}$ limit. We therefore emphasize that our preliminary surmise is compatible with the *condensation* criterion, which emerges as a manifestation of Born's reciprocity principle and does not rely on fermion pairing. However, it is possible to account for a BCS-like condensation incorporating Grassmann variables, which calls for a separate discussion [33,34]. The basis for characterization of the so-called condensate is the thermodynamic limit, order parameter, and spontaneous symmetry breaking, as detailed below.

*(1) Thermodynamic limit.* Since we are interested in the behavior of a gas of fermionic atoms, i.e., in a large particle number and volume, it is necessary to consider the thermodynamic limit, $N \longrightarrow \infty, V \longrightarrow \infty$, keeping the density $\rho = \frac{N}{V}$ constant. In this limit, the anticommutation relation between the fermionic operators $\hat{a}_n$ and $\hat{a}_n^\dagger$ becomes

$$\frac{\{\hat{a}_n,\hat{a}_n^\dagger\}}{V} = \frac{\hat{a}_n\hat{a}_n^\dagger + \hat{a}_n^\dagger\hat{a}_n}{V} = \frac{1}{V} \longrightarrow 0. \quad (15)$$

So, in this limit, $V \longrightarrow \infty$, we are allowed to forget the operator character of $\hat{a}_n$ and $\hat{a}_n^\dagger$ and they can be replaced by numbers. To make this point explicit, we define

$$\hat{a}_n^\dagger\hat{a}_n = \hat{N}_n. \quad (16)$$

Once the ground state of a system of $N$ fermionic atoms is realized as the coherent state $|\boldsymbol{\alpha}\rangle'$, no $\hat{a}_n$ or $\hat{a}_n^\dagger$ can annihilate the state. Because $'\langle\boldsymbol{\alpha}|\hat{N}_n|\boldsymbol{\alpha}\rangle' = \alpha_n^*\alpha_n \neq 0$ and the Grassmann variables have anticommuting properties, it follows that

$$\frac{'\langle\boldsymbol{\alpha}|\{\hat{a}_n,\hat{a}_n^\dagger\}|\boldsymbol{\alpha}\rangle'}{V} = \frac{1+\{\alpha_n,\alpha_n^*\}}{V} = \frac{1}{V} \longrightarrow 0; \quad (17)$$

i.e., we obtain the "classical" limit of the fermionic operators for which $\hat{a}_n$ and $\hat{a}_n^\dagger$ are replaced with their corresponding Grassmann variables. Here we identify $\sum_n \alpha_n^*\alpha_n$ as the average number of particles in the thermodynamic limit. In the following we interpret the Grassmann variables from a practical point of view.

*(2) Order parameter.* The field operator $\hat{\psi}(r)$ in Eq. (7) can be expanded in terms of its mode functions $\phi_k(r)$, and then its eigenvalue in the coherent state $|\boldsymbol{\alpha}\rangle'$,

$$'\langle\boldsymbol{\alpha}|\hat{\psi}(r) =' \langle\boldsymbol{\alpha}|\psi(r), \quad (18)$$

corresponds to the *amplitude*,

$$\psi(r) = \sum_n \alpha_n\phi_n(r), \quad (19)$$

of the fermionic field or Grassmann field in which the annihilation operators in $\hat{\psi}(r)$ are replaced with the Grassmann variables $\boldsymbol{\alpha} = \{\alpha_n\}$. For bosonic fields, the function $\psi(r)$ plays the role of an order parameter which is used to characterize the underlying phenomena of BEC. Similarly here for fermionic fields, one can always multiply $\psi(r)$ by a numerical phase factor as $e^{i\theta}$ (i.e., replacing $\alpha_n$ with $\alpha_n e^{i\theta}$), without changing any physical property. This beautifully reflects the gauge symmetry exhibited by all the physical equations of the fermionic fields. Making an explicit choice for the value of the order parameter (hence for the phase) actually corresponds to a formal breaking of the gauge symmetry which is guaranteed to be the *necessary and sufficient condition* [39,40] for the occurrence of BEC in bosonic fields. In due course, we reveal its significance to the fermionic counterpart, but first we would like to make a few remarks.

It may appear that the fermionic order parameter does not bear any classical analogy, since it involves anticommutating numbers which do not have any classical analog. This should not lead to misunderstanding. For fermionic fields, only quantities such as the charge, energy, and current density, which are only bilinear in field operators, can be measured classically. The field operator $\hat{\psi}(r)$ is linear in $\hat{a}_n$ and hence linear in the Grassmann variables, represents the amplitude of the fermionic field, and is not an experimentally measurable quantity, while the $\psi^*(r)\psi(r) = \sum_n \alpha_n^*\alpha_n|\phi_n(r)|^2$, which is bilinear in Grassmann amplitudes, makes it experimentally relevant without any ambiguity. Since coherent states are defined in terms of bilinear forms in anticommuting variables, there is no need to adopt any ordering for the modes or extra minus sign to compute $N = \int \psi^*(r)\psi(r)dr$.

The above remarks may be corroborated by another observation. The number operator $\hat{N} = \sum_k \hat{N}_k$ and the energy operator $\hat{H} = \sum_k \epsilon_k\hat{N}_k$ have classical limits because they are bilinear in $\hat{a}_k$ and $\hat{a}_k^\dagger$ and, hence, commute with each other. Anticommutation in quantum mechanics is something special because it incorporates the Pauli exclusion principle, which does not make sense at the classical level. Extrapolating this idea a bit further, we may say that Grassmann fields themselves and fermionic-field operators are, by construction, fermionic, while c-numbers and bosonic-field operators are bosonic. The product of an even number of Grassmann variables is bosonic, which makes it experimentally relevant [34–36].

Although the anticommuting nature of Grassmann variables precludes the possibility of interpreting the fermionic order parameter in physical terms, it can be shown that, unlike the bosonic field, fermionic fields are bound to satisfy the fermionic analog of the classical Liouvilles equation [31,36] and therefore most closely resemble classical phase-space distribution functions. In this sense, the fermionic-field operator, which is linear in Grassmann variables, is always "classical" except for the fact that it incorporates the Pauli exclusion principle. This is not surprising, for the simple reason that Grassmann algebra does not allow any derivative higher than second order. This is the same reason that leads the Dirac equation to take a simple linear form [31].

*(3) Spontaneous symmetry breaking.* Finally, we consider the properly parametrized coherent states $|\boldsymbol{\alpha}\rangle' = \exp(\sqrt{V}\sum_n(a_n^\dagger\alpha_n - \alpha_n^* a_n))|\mathbf{1}\rangle$. Accordingly, the states are not invariant under the number operator $\hat{N} = \sum_k \hat{a}_k^\dagger\hat{a}_k$, while the Hamiltonian $\hat{\mathcal{H}}$ [Eq. (4)] commutes with $\hat{N}$; i.e.,

$$e^{i\theta\hat{N}}|\boldsymbol{\alpha}\rangle' = |e^{-i\theta}\boldsymbol{\alpha}\rangle', \quad e^{i\theta\hat{N}}\hat{\mathcal{H}}e^{-i\theta\hat{N}} = \hat{\mathcal{H}}. \quad (20)$$

The operator $e^{i\theta\hat{N}}$ applied to $|\boldsymbol{\alpha}\rangle'$ produces a different state, $|e^{-i\theta}\boldsymbol{\alpha}\rangle'$, which leaves the scalar product invariant:

$$'\langle e^{-i\theta}\boldsymbol{\alpha}|e^{-i\theta}\boldsymbol{\alpha}\rangle' =' \langle\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle'. \quad (21)$$

This suggests that, similarly to harmonic oscillator coherent states, one can always multiply the fermionic coherent state by an arbitrary phase factor $e^{-i\theta}$ without changing any physical property. From the symmetry point of view, the situation is quite outstanding for fermionic fields and can be further elaborated as follows.

Since the overlap of coherent states can be calculated from Eq. (10) as

$$ {}^{\backprime}\langle \boldsymbol{\alpha}'|\boldsymbol{\alpha}\rangle' \, {}^{\backprime}\langle \boldsymbol{\alpha}|\boldsymbol{\alpha}'\rangle' = \exp\left(-V\sum_n(\alpha_n^{*\prime}-\alpha_n^*)(\alpha_n'-\alpha_n)\right), \tag{22} $$

it implies that any two different states $|\boldsymbol{\alpha}\rangle'$ and $|\boldsymbol{\alpha}'\rangle'$ become orthogonal in the limit $V \to \infty$. Thus states with different phase factors, $|\boldsymbol{\alpha}\rangle'$ and $|e^{-i\theta}\boldsymbol{\alpha}\rangle'$, are macroscopically distinct. This striking observation in the present analysis hints at our assertion that, similarly to BEC, the *specific macroscopic ground condensed state* of a Fermi gas [10], realized as a coherent state, forms a degenerate manifold parametrized by a phase variable, $0 < \theta < 2\pi$. While the microscopic Hamiltonian ($\hat{\mathcal{H}}$) [Eq. (4)] has global $\mathbf{U}(1)$ symmetry, the so-called "condensed" state in Ref. [10], viewed as a fermionic coherent state, does not possess such symmetry, since adding a phase factor to state $|\boldsymbol{\alpha}\rangle'$ produces a different state altogether.

Finally, we anticipate the difficulty of achieving the foregoing Born-Kothari condensation (BKC) in an experiment. It is quite natural that the above "condensate" will obviously be significant for the astrophysical applications of the reciprocity principle [10]. The possibility of unambiguous observation of this phenomenon at very low temperatures primarily depends on the extent to which an FD gas can be prepared experimentally so that it becomes almost noninteracting and the number of particles $N$ in a given volume $V$ is macroscopically large compared to the number of particles $N_0$ allowed by Eq. (2). Second, the energy $\epsilon_{\max}$ of all $(N - N_0)$ particles must correspond to the maximum value of the momentum $\mathtt{b}$ given by the relation $\epsilon_{\max}^2 + 2\epsilon_{\max}mc^2 = \mathtt{b}^2c^2$ [15]. In view of the tremendous experimental progress in the last two decades with ultracold Fermi gases down to degeneracy temperature, we expect that one may have sufficient control over the strength of interaction between the particles and the density of atomic gases [4–9], by tuning experimental parameters such as the trap frequency and the applied magnetic fields, to achieve the desired condensate.

## IV. CONCLUSION

In this article, we have elucidated the appealing features of the so-called Born-Kothari condensation (BKC) in ideal Fermi-Dirac gases in close association with BEC. We emphasize that, unlike BEC, the most conspicuous feature of BKC is that the condensation is primarily guided by the density of states rather than only statistics itself. Our analysis is based on the mathematical methods that have been used to analyze the statistical properties of boson fields, and in particular the coherence of photons in quantum optics, have their counterparts for Fermi fields. To be specific, the coherent states, displacement operators, and $P$ representation all possess surprisingly close fermionic analogs, and upon using a practical calculus of anticommuting numbers, they can be utilized to calculate correlation functions and counting distributions for general systems of fermions. In the same spirit, we have explained that, despite their fundamental differences, the so-called condensation of an FD gas can be envisaged by fermionic coherent states just as harmonic oscillator coherent states describe BEC in an ideal Bose gas. It has been shown that, in contrast to BEC, which is defined as a displaced vacuum state, the above condensate should correspond to a unitary displaced state where the displacement operator displaces the state of infinitely dense filled-up modes. Notwithstanding their mathematical differences, the present formulation in combination with thermodynamic consideration allows us to characterize the "condensate" in terms of the fermionic order parameter and interpret fermionic field variables in "classical" terms. Most remarkably, we have pointed out, similarly to bosonic fields, the coherent and the rotated coherent state can be distinguished as macroscopically distinct ground states of the FD gas. This enables us to capture the essence of BKC in ideal Fermi gases as a close parallel to BEC as *condensation-like coherence within fermions*.

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