Resonances in open quantum systems

Hichem Eleuch[1,*] and Ingrid Rotter[2,†]
[1]*Institute for Quantum Science and Engineering, Texas A&M University, College Station, Texas 77843, USA*
[2]*Max Planck Institute for the Physics of Complex Systems, D-01187 Dresden, Germany*
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The Hamilton operator of an open quantum system is non-Hermitian. Its eigenvalues are generally complex and provide not only the energies but also the lifetimes of the states of the system. The states may couple via the common environment of scattering wave functions into which the system is embedded. This causes an external mixing (EM) of the states. Mathematically, EM is related to the existence of singular (the so-called exceptional) points. The eigenfunctions of a non-Hermitian operator are biorthogonal, in contrast to the orthogonal eigenfunctions of a Hermitian operator. A quantitative measure for the ratio between biorthogonality and orthogonality is the phase rigidity of the wave functions. At and near an exceptional point (EP), the phase rigidity takes its minimum value. The lifetimes of two nearby eigenstates of a quantum system bifurcate under the influence of an EP. At the parameter value of maximum width bifurcation, the phase rigidity approaches the value one, meaning that the two eigenfunctions become orthogonal. However, the eigenfunctions are externally mixed at this parameter value. The S matrix and therewith the cross section do contain, in the one-channel case, almost no information on the EM of the states. The situation is completely different in the case with two (or more) channels where the resonance structure is strongly influenced by the EM of the states and interesting features of non-Hermitian quantum physics are revealed. We provide numerical results for two and three nearby eigenstates of a non-Hermitian Hamilton operator that are embedded in one common continuum and are influenced by two adjoining EPs. The results are discussed. They are of interest for an experimental test of the non-Hermitian quantum physics as well as for applications.

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I. INTRODUCTION

In experiments [1–3] on Aharonov-Bohm rings containing a quantum dot in one arm, both the phase and magnitude of the transmission amplitude $T = |T|e^{i\beta}$ of the dot could be extracted. The results obtained did not fit into the general understanding of the transmission process. As a function of the plunger gate voltage V_g, a series of well-separated transmission peaks of rather similar width and height has been observed and, according to expectations, the transmission phases $\beta(V_g)$ increase continuously by π across every resonance. In contrast to expectations, however, β always jumps sharply downward by π in each valley between any two successive peaks. These jumps, called phase lapses in multilevel systems, were observed in a large succession of valleys for every many-electron dot studied. They have been discussed in many theoretical papers, including Ref. [4].

In spite of much effort, the experimental results could not be explained in Hermitian quantum physics. Using the non-Hermitian formalism of the quantum physics, it was however possible to explain [5] convincingly the experimentally observed phase lapses (see also the discussion of this problem in Sec. 4.3.2 of the recent review in [6]). This example shows the meaning that non-Hermitian quantum physics *can* have for the description of a concrete physical system that is open, in contrast to a closed (or almost closed) system that is well described in the framework of Hermitian quantum physics.

Another example that shows the meaning of a non-Hermitian Hamilton operator for a concrete quantum system is the description of laser-induced continuum structures in atoms [7,8]. In these papers the motion of the complex eigenvalues of the non-Hermitian Hamiltonian is traced as a function of the field strength for different field frequencies and atomic parameters. Level repulsion in the complex plane is shown to occur at a critical field intensity. With further increasing intensity, the complex energies move differently. This effect is called resonance trapping according to similar results obtained earlier in nuclear physics [9].

Recently, non-Hermitian Hamilton operators have been used for the description or prediction of different phenomena in quantum physics also in other papers. We mention here only a few of many examples [10–13].

A non-Hermitian Hamilton operator describing an open quantum system may play an important role also in explaining well-known puzzles of quantum physics. The natural environment of a localized quantum mechanical system is the extended continuum of scattering wave functions in which the system is embedded. This environment can be changed by means of external forces, however it can never be deleted. It exists at all times and is completely independent of any observer. For this reason, radioactive dating can be used in geologic studies.

According to this statement, the properties of an open quantum system can be described by means of two projection operators, each of which is related to one of the two parts of the function space. The localized part of the quantum system is basic for spectroscopic studies. Mathematically, the localized part of the open quantum system is a subsystem that is related to another subsystem. The Hamiltonian of the (localized) system is therefore non-Hermitian, while the Hamiltonian of the total system consisting of the two subsystems is Hermitian [14].

In the standard Hermitian description of a localized quantum system, the system is considered to be closed, the

*hichemeleuch@tamu.edu
†Corresponding author: rotter@pks.mpg.de

Hamiltonian is Hermitian, and the eigenstates are discrete. The eigenstates' decay is described by tunneling of particles into the function space of scattering states into which the system is embedded. The tunneling time can be calculated. It is small and could be measured experimentally only recently [15]. The experimental results have shown that the tunneling time is extremely short, which does not correspond to the expectations of standard Hermitian quantum physics. They agree, however, with the conclusions obtained when the system is considered to be an open system described by a non-Hermitian Hamilton operator. In this case, the eigenvalues $\mathcal{E}_i$ of the system are complex and the lifetime of the states is given by $\mathrm{Im}(\mathcal{E}_i)$. There is no need to consider any tunneling time.

In a similar manner, the problem of the Schrödinger cat does not exist when the system is considered to be an open quantum system. The consequence for this is the necessity to describe the system by a non-Hermitian Hamilton operator and to solve the involved mathematical problems.

Usually, the calculations with a non-Hermitian Hamiltonian give results for observable values of the quantum system that differ only a little from those obtained with a Hermitian Hamiltonian, especially in relation to the uncertainties involved in the comparison with experimental data. There are however exceptions to this rule. These exceptions arise from the mathematical existence of singular points. One example is the so-called exceptional points (EPs), which have been known in mathematics for many years [16]. Consider a family of operators of the form

$$T(\kappa) = T(0) + \kappa T', \tag{1}$$

where κ is a scalar parameter, $T(0)$ is an unperturbed operator, and $\kappa T'$ is a perturbation. Kato [16] has shown that the number of eigenvalues of $T(\kappa)$ is independent of κ, as expected, however with the exception of some special values of κ. The corresponding points in the parameter space are called EPs. Here, (at least) two eigenvalues coalesce. An example is

$$T(\kappa) = \begin{pmatrix} 1 & \kappa \\ \kappa & -1 \end{pmatrix}, \tag{2}$$

in which the two values $\kappa = +i$ and $\kappa = -i$ result in the same eigenvalue 0.

Now the following questions arise: What is the behavior of the eigenfunctions of the non-Hermitian Hamilton operator under the influence of an EP? Can EPs be observed directly in experimental results? These questions have been answered only partly in the literature, although their influence on the dynamics of open quantum systems is well known (for references see the review in [17]).

The meaning of EPs has been studied in the literature over the past 20 years, in classical as well as in quantum physics. We will not discuss here the problems of classical physics. Instead we refer to Ref. [18] and to the collection of articles on spectral analysis, stability, and bifurcation in modern nonlinear physical systems [19]. Related problems have been studied also in molecular physics (see the recent paper in [20] where references to previous papers can be found). Unfortunately, in quantum physics the problems are studied in a confusing and often contradictory manner. We will not enumerate here the references to the different papers. They will rather be cited in those sections of the present paper in which they can be discussed consistently [21].

The non-Hermiticity of the Hamiltonian is introduced mostly by adding a non-Hermitian part to the Hermitian Hamiltonian that is known to describe the system quite well (for references see the review in [17]). It appears therefore more or less as a perturbation that is able to describe the changes of the system properties under special conditions, i.e., under the influence of an EP. Although this treatment gives mostly reliable results, the question remains of what the properties of a Hamiltonian are that is really non-Hermitian, i.e., when the non-Hermiticity does *not* appear as some perturbation.

The aim of the present paper is to find an answer to this question in a mathematically exact manner, however by keeping in mind that points in the continuum are of measure zero and cannot be observed directly. It is important therefore to point to observable signatures of the EPs [27] occurring in physical values, by means of which their existence can be proven. This is, e.g., an avoided level crossing and the formation of different time scales in the two-level case. Similar signatures exist in the three- and higher-level cases. Most interesting is the so-called external mixing (EM) of the states via the common continuum into which the system is embedded. By definition, an EM of the states can occur *only* when the system is open. It is therefore one of the characteristic values of the non-Hermitian physics of open quantum systems [14].

Many years ago, EM was shown to play an important role in the open quantum mechanical nuclear system (the continuum shell model in contrast to the standard shell model) [9]. Today we know that it characterizes the main features of the influence of EPs on the dynamics of an open quantum system (see the recent review [6] on experimental and theoretical results). Experimentally, an example of EM was provided a few years ago in a mesoscopic system. It was shown in [28] that two distinct quantum states are coupled through a common continuum. In a further experiment, the authors were able to show that EM survives even under conditions of strongly nonequilibrium transport in the system [29].

The present paper is organized as follows. First we consider the Hamiltonian that describes the properties of an open quantum system. By definition, an open quantum system is localized in space and embedded in the continuum of scattering wave functions due to which the states of the system become resonance states and generally have a finite lifetime. This Hamiltonian is non-Hermitian. In Secs. II and III, respectively, we consider the eigenvalues and eigenfunctions of a 2×2 and of a 3×3 non-Hermitian Hamilton operator. The eigenstates are coupled via one common continuum and show the typical EM. The eigenfunctions of a non-Hermitian Hamilton operator are biorthogonal and their phases are not rigid in approaching an EP.

We show in Sec. IV that the EPs cause nonlinear effects in an open quantum system. They can be traced best in the resonance structure of the scattering cross section under the condition that it is influenced by two adjoining EPs. The further results given in Sec. V show that, in the one-channel case, the resonance structure of the cross section is almost independent of the EM. It is therefore impossible to reason, in this case, the existence of EPs from a study of the resonance structure

of the cross section. The situation is another one in the case with two (or more) channels as discussed in Sec. VI. Here EPs and EM cause interesting observable effects. (A channel is nothing but an environment in which the system is embedded. The different channels or environments are orthogonal to one another.)

In Sec. VII we summarize and discuss the results obtained in the present paper. We conclude the paper in Sec. VIII with some general remarks on EPs and, above all, on the eigenfunctions of a non-Hermitian Hamilton operator. By doing this, we hope to stimulate experimental studies in order to prove, on the one hand, the theoretical results and to use, on the other hand, the rich possibilities they provide for applications.

II. EIGENVALUES AND EIGENFUNCTIONS OF A 2 × 2 NON-HERMITIAN HAMILTONIAN

Let us consider the 2×2 non-Hermitian matrix

$$\mathcal{H}^{(2)} = \begin{pmatrix} \varepsilon_1 \equiv e_1 + \frac{i}{2}\gamma_1 & \omega \\ \omega & \varepsilon_2 \equiv e_2 + \frac{i}{2}\gamma_2 \end{pmatrix}. \tag{3}$$

Here the ε_i are the complex eigenvalues of the basic non-Hermitian operator [30]. The ω stand for the coupling matrix elements of the two states via the common environment [14]. Their mathematical expression is derived in Sec. 3 of [17]. They are complex where $\mathrm{Re}(\omega)$ is the principal value integral and $\mathrm{Im}(\omega)$ is the residuum [17]. The imaginary part is responsible for coherent processes occurring in the system, while the real part contains decoherences. The non-Hermitian matrix

$$\mathcal{H}_0^{(2)} = \begin{pmatrix} \varepsilon_1 \equiv e_1 + \frac{i}{2}\gamma_1 & 0 \\ 0 & \varepsilon_2 \equiv e_2 + \frac{i}{2}\gamma_2 \end{pmatrix} \tag{4}$$

describes the system without any mixing of its states via the environment. In other words, $\omega = 0$ corresponds to vanishing EM of the eigenstates.

In this paper, our main interest is in the effects caused by ω. Most visible are the changes in the widths of the states: The original widths γ_i of the states turn into the widths Γ_i of the eigenstates of $\mathcal{H}^{(2)}$ due to $\omega \neq 0$.

A. Eigenvalues of $\mathcal{H}^{(2)}$

The eigenvalues $\mathcal{E}_i \equiv E_i + \frac{1}{2}\Gamma_i$ of $\mathcal{H}^{(2)}$ are generally complex:

$$\mathcal{E}_{1,2} \equiv E_{1,2} + \frac{i}{2}\Gamma_{1,2} = \frac{\varepsilon_1 + \varepsilon_2}{2} \pm Z, \tag{5}$$

with

$$Z \equiv \tfrac{1}{2}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 4\omega^2}\,. \tag{6}$$

Here E_i is the energy and Γ_i the width of the eigenstate i.

The properties of the $\mathcal{E}_i$ trajectories as a function of a certain parameter are well known. They contain level repulsion, where two states repel each other in accordance with $\mathrm{Re}(Z)$; width bifurcation, where the widths of two states bifurcate in accordance with $\mathrm{Im}(Z)$; an avoided level crossing, where two discrete (or narrow resonance) states avoid crossing [31,32] because $(\varepsilon_1 - \varepsilon_2)^2 + 4\omega^2 > 0$ and therefore always $Z \neq 0$; and the appearance of an EP, where two states cross when $Z = 0$.

Altogether, the crossing scenario that is caused by an EP in non-Hermitian quantum physics, with generally complex eigenvalues $\mathcal{E}_i \equiv E_i + \frac{1}{2}\Gamma_i$ of the Hamiltonian, needs to be considered in terms of a combined behavior of energy E_i and width Γ_i trajectories of the two states $i = 1,2$. A level repulsion will generally appear in the $\mathrm{Re}(\mathcal{E}_i) = E_i$ trajectories together with a free crossing of the $\mathrm{Im}(\mathcal{E}_i) \propto \Gamma_i$ trajectories, while a bifurcation of the widths Γ_i is accompanied generally by a free crossing of the energy trajectories E_i. The last case is illustrated in Figs. 1–3 of Ref. [33]. Sometimes, the crossing phenomenon in non-Hermitian quantum physics is called an avoided level crossing in the complex plane (see, e.g., [34]). In the present paper we use the term avoided level crossing or level repulsion according to the standard definition for the $\mathrm{Re}(\mathcal{E}_i) = E_i$ trajectories, while the term width bifurcation is used for the corresponding phenomenon appearing in the $\mathrm{Im}(\mathcal{E}_i) \propto \Gamma_i$ trajectories. We underline once more that both phenomena are combined in non-Hermitian quantum physics.

In [35], the case with equal widths $\gamma_1 = \gamma_2$ of the two states and with imaginary coupling $\omega = i\,\omega_0$ is solved analytically. As a result, two EPs appear [see Eqs. (14)–(16) and Figs. 1(a)–1(d) in [35]]. Between the two EPs, the widths bifurcate up to a maximum value. In the present paper, we consider complex ω_i where only one EP can be seen clearly. Nevertheless, also the second EP has some influence onto the dynamical properties of the system (see the numerical results given in the present paper).

When $\omega = 0$, the energies ε_i vary smoothly as a function of any parameter. According to (5) and (6), $Z = \pm\frac{1}{2}(\varepsilon_1 - \varepsilon_2)$ and $\mathcal{E}_{1,2} \to \varepsilon_{1,2}$ in this case. This means that no EP can be related to the Hamiltonian $\mathcal{H}_0^{(2)}$.

B. Eigenfunctions of $\mathcal{H}^{(2)}$

The following properties of the eigenfunctions Φ_i of a non-Hermitian operator are less known.

Biorthogonality. The eigenfunctions and eigenvalues of every Hamilton operator have to fulfill the two conditions

$$\mathcal{H}|\Phi_i\rangle = \mathcal{E}_i|\Phi_i\rangle, \quad \langle\Psi_i|\mathcal{H} = \mathcal{E}_i\langle\Psi_i|. \tag{7}$$

A Hermitian operator has real eigenvalues such that $\langle\Psi_i| = \langle\Phi_i|$ in this case. The eigenvalues of a non-Hermitian operator are generally complex such that the left and right eigenfunctions differ from one another, $\langle\Psi_i| \neq \langle\Phi_i|$. This is valid also for the eigenvalues and eigenfunctions of the two symmetric operators $\mathcal{H}^{(2)}$ and $\mathcal{H}_0^{(2)}$. In this case, the relation between the left and right eigenfunctions is given by [33,36,37]

$$\langle\Psi_i| = \langle\Phi_i^*|. \tag{8}$$

Normalization. In the case of a Hermitian operator, $\langle\Phi_i|\Phi_j\rangle$ is real and the eigenfunctions are usually normalized to $\langle\Phi_i|\Phi_j\rangle = 1$. To smoothly describe the transition from a closed system with discrete states to a weakly open one with narrow resonance states (described by $\mathcal{H}^{(2)}$), it is meaningful to use the normalization

$$\langle\Phi_i^*|\Phi_j\rangle = \delta_{ij} \tag{9}$$

for the eigenfunctions. The value $\langle\Phi_i^*|\Phi_j\rangle \equiv (\Phi_i|\Phi_j)$ is however complex such that the phases of the two eigenfunctions $\Phi_{1,2}$ relative to one another cannot be rigid. They are rather parameter dependent since $\langle\Phi_i^*|\Phi_j\rangle$ has to be real, according to (9), for every parameter value.

It follows from (9) that the values of the standard expressions are changed [17],

$$\langle\Phi_i|\Phi_i\rangle = \mathrm{Re}(\langle\Phi_i|\Phi_i\rangle), \quad A_i \equiv \langle\Phi_i|\Phi_i\rangle \geqslant 1; \tag{10}$$

$$\begin{aligned}\langle\Phi_i|\Phi_{j\neq i}\rangle &= i\,\mathrm{Im}(\langle\Phi_i|\Phi_{j\neq i}\rangle) = -\langle\Phi_{j\neq i}|\Phi_i\rangle,\\ \left|B_i^j\right| &\equiv |\langle\Phi_i|\Phi_{j\neq i}| \geqslant 0.\end{aligned} \tag{11}$$

Phase rigidity. The phase rigidity is a quantitative measure for the biorthogonality of the eigenfunctions. It is defined by [17]

$$r_k \equiv \frac{\langle\Phi_k^*|\Phi_k\rangle}{\langle\Phi_k|\Phi_k\rangle} = A_k^{-1} \tag{12}$$

by taking into account the normalization (9). In Hermitian systems, the eigenfunctions are orthogonal and $r_k = 1$. In systems with well-separated resonance states, it follows that $r_k \approx 1$; however it is never $r_k = 1$ [17,36,37]. Hermitian quantum physics is, in this case. a reasonable approximation for the description of the open quantum system. In approaching an exceptional point, it follows that $r_k \to 0$ [17].

The phase rigidity is experimentally studied on microwave billiards [26]. The variation of r_k in approaching the EP is found indeed. The experimental result agrees with the relation (20) discussed below [21].

Our calculations show an interesting unexpected property for two nearby states with similar values of their widths γ_i [35,38]: $r_k \approx 1$ at maximum width bifurcation. These results will be discussed below in detail. An analogous result is found for two nearby states with level repulsion that is caused by an EP [35].

Mixing of the eigenfunctions via the environment (EM). The Schrödinger equation for the basic wave functions Φ_i^0 with the Hamiltonian (4) is

$$\left(\mathcal{H}_0^{(2)} - \varepsilon_i\right)\left|\Phi_i^0\right\rangle = 0, \tag{13}$$

while the Schrödinger equation with the full Hamiltonian (3) reads

$$(\mathcal{H}^{(2)} - \mathcal{E}_i)|\Phi_i\rangle = 0. \tag{14}$$

Equation (14) can be rewritten as a Schrödinger equation with a source term,

$$\left(\mathcal{H}_0^{(2)} - \mathcal{E}_i\right)|\Phi_i\rangle = -\begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}|\Phi_i\rangle. \tag{15}$$

Now we can use the standard representation of the Φ_i in the $\{\Phi_n^0\}$,

$$\Phi_i = \sum b_{ij}\Phi_j^0, \quad b_{ij} = \left\langle\Phi_j^{0*}\middle|\Phi_i\right\rangle, \tag{16}$$

under the condition that the b_{ij} are normalized by $\sum_j (b_{ij})^2 = 1$, i.e.,

$$\begin{aligned}\sum_j (b_{ij})^2 &= \mathrm{Re}\left(\sum_j (b_{ij})^2\right)\\ &= \sum_j \{[\mathrm{Re}(b_{ij})]^2 - [\mathrm{Im}(b_{ij})]^2\} = 1.\end{aligned} \tag{17}$$

We are interested in the probability of EM, which is defined by

$$\sum_j |b_{ij}|^2 = \sum_j \{[\mathrm{Re}(b_{ij})]^2 + [\mathrm{Im}(b_{ij})]^2\}. \tag{18}$$

From (17) and (18) it follows that

$$\sum_j |b_{ij}|^2 \geqslant 1. \tag{19}$$

In the neighborhood of an EP, $\sum_j |b_{ij}|^2 \gg 1$; in approaching an EP, $\sum_j |b_{ij}|^2 \to \infty$ [38].

When the maximum width bifurcation (or level repulsion) is parametrically reached, the eigenfunctions Φ_i are almost orthogonal, however the EM contained in the wave functions of the eigenstates is strong [38].

Eigenfunctions of $\mathcal{H}^{(2)}$ at an EP. According to analytical and numerical results [8,22–24], we have

$$\Phi_1^{\mathrm{cr}} \to \pm i\,\Phi_2^{\mathrm{cr}}, \quad \Phi_2^{\mathrm{cr}} \to \mp i\,\Phi_1^{\mathrm{cr}}, \tag{20}$$

where Φ_i^{cr} are the eigenfunctions at an EP. The EP is however a point in the continuum of scattering wave functions and is therefore of measure zero. Hints to the existence of an EP can be found in observable values. These are, above all, an avoided level crossing and width bifurcation, both of which are caused by an EP.

We mention here that the relations (20) are in agreement with experimental results obtained on microwave billiards [25]. These results [21] are confirmed independently from one another by different authors (see, e.g., [8,22–24]).

The eigenfunction Φ_i of the non-Hermitian Hamilton operator $\mathcal{H}$ is the main part of the wave function of the resonance state i inside the localized part of the system. The wave function of the resonance state including its tail is given in Eq. (42) in [17].

C. Numerical results

We refer to the analytical results obtained and discussed in [35] for the eigenvalues and eigenfunctions of $N = 2$ states. In the analytical studies ω is assumed to be either real or imaginary, which is of course seldom realized in realistic systems. Nevertheless, the results of these studies provide some insight into the basic features of the eigenvalues and eigenfunctions of a non-Hermitian operator, above all near an EP.

In Fig. 1 we show numerical results obtained for systems under more realistic conditions in which ω is complex. The energies e_i are parameter dependent, while γ_i and ω are parameter independent. The difference between the widths γ_i of the two states as well as ω is chosen in such a manner that an EP occurs. In both cases, the phase rigidity r_i approaches

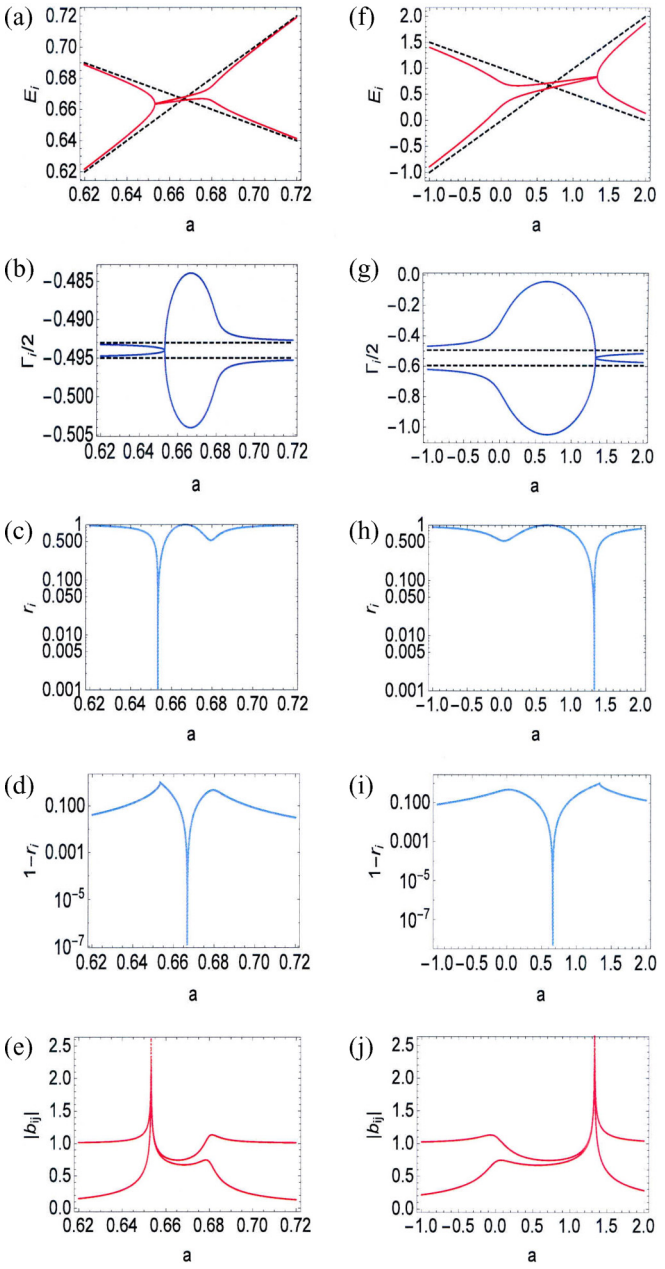

FIG. 1. Plot of eigenvalues $\mathcal{E}_i \equiv E_i + \frac{1}{2}\Gamma_i$, phase rigidity r_i and $1 - r_i$, and mixing $|b_{ij}|$ of the eigenfunctions Φ_i of the Hamiltonian $\mathcal{H}^{(2)}$ as a function of a. The value ω is independent of a: (a)–(e) $\omega = 0.01(i + \frac{1}{10})$ and (f)–(j) $\omega = 0.5(i + \frac{1}{10})$. The parameters are (a)–(e) $e_1 = 1 - a/2$, $e_2 = a$, $\gamma_1/2 = -0.495$, and $\gamma_2/2 = -0.493$ and (f)–(j) $e_1 = 1 - a/2$, $e_2 = a$, $\gamma_1/2 = -0.495$, and $\gamma_2/2 = -0.595$. The dotted lines show e_i and $\gamma_i/2$ in (a,b,f,g).

zero at the EP and is near one at maximum width bifurcation. The mixing $|b_{ij}|$ of the two eigenfunctions increases without limit in approaching the EP and is finite, $|b_{ij}| < 1$, in the parameter region of the maximum width bifurcation. The sign of a second EP can be seen in the eigenvalues as well as in the eigenfunctions at $a = 0.68$ in Figs. 1(a)–1(e) and $a = 0.0$ in Figs. 1(f)–1(j). Similar results are obtained when the energies e_i (as well as the ω) are chosen to be parameter independent, while the γ_i are parameter dependent (see, e.g., [35]).

In our calculations, we choose the coupling strength ω between the system and the environment to be parameter independent in order to exclude formally its influence onto the dynamics of the open quantum system. This allows us to fix the role of nonlinear processes.

To summarize the results of Fig. 1, we state the following. The phase rigidity and mixing of the eigenfunctions in approaching an EP are, respectively,

$$r_i \to 0, \quad |b_{ij}| \to \infty, \tag{21}$$

and in general

$$1 > r_i \geqslant 0, \quad |b_{ij}| > 1. \tag{22}$$

The phase rigidity and mixing of the wave functions between two EPs are, respectively,

$$r_i \to 1, \quad |b_{ij}| < 1, \tag{23}$$

in approaching the maximum width bifurcation. In the analytically solvable case with imaginary ω we have [38]

$$|b_{ij}| \approx 0.7, \tag{24}$$

meaning that the eigenfunctions Φ_i are almost orthogonal and strongly mixed in the set of basic wave functions $\{\Phi_k^0\}$ in approaching the maximum width bifurcation.

Similar results are obtained when $\mathrm{Re}(\omega) \gg \mathrm{Im}(\omega)$. The difference from the results shown in Fig. 1 is that now level repulsion is the main effect caused by the EP [38].

We remark here that the evolution from $r_k = 0$ at the EP to $r_k \approx 1$ at the maximum width bifurcation is driven exclusively by the nonlinear source term of the Schrödinger equation (see Sec. IV) since $\omega = \text{const}$ in our calculations. When $\omega = 0$, it is $\mathcal{E}_i = \varepsilon_i$. In this case, there are no EPs, as mentioned in Sec. II A.

III. EIGENVALUES AND EIGENFUNCTIONS OF A 3 × 3 NON-HERMITIAN HAMILTONIAN

Let us consider the Hamiltonian

$$\mathcal{H}^{(N)} = \begin{pmatrix} \varepsilon_1 & \omega_{12} & \cdots & \omega_{1N} \\ \omega_{21} & \varepsilon_2 & \cdots & \omega_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{N1} & \omega_{N2} & \cdots & \varepsilon_N \end{pmatrix}, \tag{25}$$

where $\varepsilon_i \equiv e_i + i/2\gamma_i$ are the energies and widths of the N states, $\omega_{i\,k\neq i}$ are the complex coupling matrix elements of the states i and k via the common environment, and the $\omega_{i\,k=i}$ denote the self-energy of the state i, which is mostly assumed to be included in the ε_i in our calculations. The values ω_{ik} for different i and k usually differ from one another. It is however a well-known fact from numerical calculations [17] that a resonance state becomes trapped by another nearby state when its width is somewhat smaller than that of the nearby state. Finally, the widths of most relatively-short-lived states of the system are similar to one another. These states determine the evolution of the system.

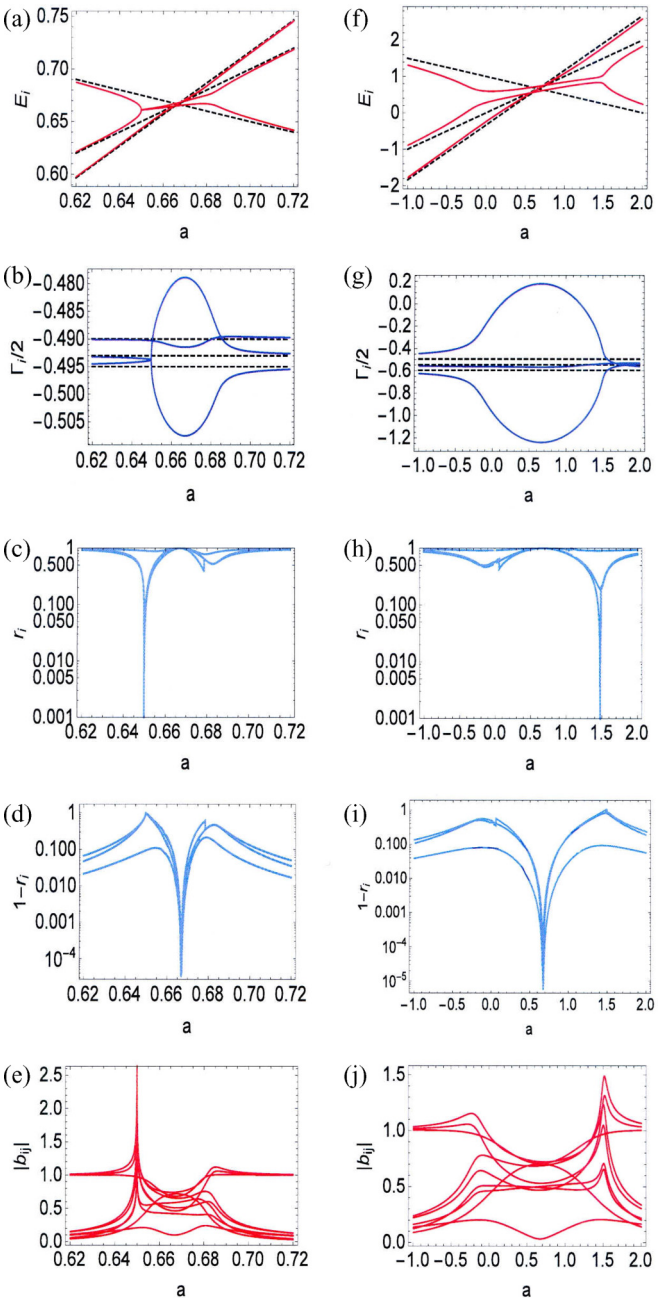

FIG. 2. Plot of eigenvalues $\mathcal{E}_i \equiv E_i + \frac{1}{2}\Gamma_i$, phase rigidity r_i and $1 - r_i$, and mixing $|b_{ij}|$ of the eigenfunctions Φ_i of the Hamiltonian $\mathcal{H}^{(3)}$ as a function of a. The value ω is independent of a: (a)–(e) $\omega = 0.01(i + \frac{1}{10})$ and (f)–(j) $\omega = 0.5(i + \frac{1}{10})$. The parameters are (a)–(e) $e_1 = 1 - a/2$, $e_2 = a$, $e_3 = -1/3 + 1.5a$, $\gamma_1/2 = -0.495$, $\gamma_2/2 = -0.493$, and $\gamma_3/2 = -0.49$ and (f)–(j) $e_1 = 1 - a/2$, $e_2 = a$, $e_3 = -1/3 + 1.5a$, $\gamma_1/2 = -0.495$, $\gamma_2/2 = -0.595$, and $\gamma_3/2 = -0.545$. The dotted lines show e_i and $\gamma_i/2$ in (a,b,f,g).

A. Numerical results for $N = 3$

In Fig. 2 we show the numerical results obtained for the eigenvalues and eigenfunctions of $N = 3$ states by using parameters similar to those for $N = 2$ states in Fig. 1. The $\omega_{ij} \equiv \omega$ are chosen to be equal for the different i and j. Above all, they are parameter independent, similar to the corresponding ω in Fig. 1.

The comparison of Figs. 1 and 2 shows that the main features of the eigenvalues and eigenfunctions are the same for $N = 2$ and $N = 3$. The eigenvalues repel each other in energy and their widths bifurcate under the influence of an EP; the phase rigidities approach zero and the mixing of the wave functions increases without limit at and near an EP; the phase rigidities approach the value one and the corresponding almost orthogonal wave functions are mixed when the width bifurcation is maximum. These effects are enhanced when $N = 3$ as compared to those occurring when $N = 2$.

As in the two-level case, similar results are obtained when $\mathrm{Re}(\omega) \gg \mathrm{Im}(\omega)$. In this case, level repulsion is the main effect caused by the EP [39].

B. Third-order exceptional points

Hints at third-order EPs (at which three eigenvalues coalesce at one parameter value) cannot be found in Fig. 2. The reason is that every EP is a point in the continuum (with measure zero) that can be identified only by its influence on observable values in a finite parameter range around it. Furthermore, a third-order EP occurring in the system without any EM of its states via the environment is shielded due to EM in a realistic system. It therefore cannot be observed in an open quantum system.

According to the numerical results shown in Fig. 2, we see several second-order EPs in a critical parameter region around the value at which the conditions for a third-order EP are mathematically fulfilled. The observable effect caused by a third-order EP is some clustering of second-order EPs that occurs in a finite parameter range around the value at which the third-order EP is mathematically expected. This fact is discussed in detail in [35].

These results show the differences between a formal mathematical result and effects that can really be observed in a physical system. The point is that two states that cross at an EP lose (due to the EM of the states) their individual character in a finite parameter range around the EP and the areas of influence of various second-order EPs overlap. In this manner, they amplify collectively their impact on physical values with the result that, e.g., a third-order EP is shielded in a physical system.

IV. SCHRÖDINGER EQUATION WITH A NONLINEAR SOURCE TERM

The Schrödinger equation (14) can be rewritten as the Schrödinger equation (15) with a source term. In this equation, the coupling ω of the states i and $j \neq i$ via the common environment of scattering wave functions (EM) is contained in the source term. The source term is nonlinear [17]

$$\left(\mathcal{H}_0^{(2)} - \mathcal{E}_i\right)|\Phi_i\rangle = \sum_{k=1,2} \langle \Phi_k | W | \Phi_i \rangle \sum_{m=1,2} \langle \Phi_k | \Phi_m \rangle |\Phi_m\rangle \tag{26}$$

since $\langle \Phi_k | \Phi_m \rangle \neq 1$ for $k = m$ and $\langle \Phi_k | \Phi_m \rangle \neq 0$ for $k \neq m$ [see Eqs. (10) and (11)]. In (26) the definition $W \equiv -\left(\begin{smallmatrix} 0 & \omega \\ \omega & 0 \end{smallmatrix}\right)$ is used for convenience. The most important part of the

nonlinear contributions is contained in

$$\left(\mathcal{H}_0^{(2)} - \mathcal{E}_n\right)\left|\Phi_n\right\rangle = \langle\Phi_n|W|\Phi_n\rangle|\Phi_n|^2|\Phi_n\rangle. \tag{27}$$

Far from an EP, the source term is (almost) linear since $\langle\Phi_k|\Phi_k\rangle \to 1$ and $\langle\Phi_k|\Phi_{l\neq k}\rangle = -\langle\Phi_{l\neq k}|\Phi_k\rangle \to 0$. Near an EP however, the source term is nonlinear since $\langle\Phi_k|\Phi_k\rangle \neq 1$ and $\langle\Phi_k|\Phi_{l\neq k}\rangle = -\langle\Phi_{l\neq k}|\Phi_k\rangle \neq 0$.

Due to the EM involved in the source term, the eigenfunctions Φ_i and eigenvalues $\mathcal{E}_i$ of $\mathcal{H}^{(2)}$ contain global features. The environment of an open quantum system is the continuum of scattering wave functions that has an infinite number of degrees of freedom. It may cause therefore, among other effects, a dynamical phase transition [17,39]. The transition is nonadiabatic [6,17,39].

In order to illustrate the nonlinear effects involved in the source term of the Schrödinger equation (26) let us consider, as an example, the resonance part of the S matrix from which the resonance structure of the cross section can be calculated,

$$\sigma(E) \propto |1 - S(E)|^2. \tag{28}$$

A unitary representation of the resonance part of the S matrix in the case of two resonance states coupled to a common continuum of scattering wave functions reads [40]

$$S = \frac{\left(E - E_1 - \frac{i}{2}\Gamma_1\right)\left(E - E_2 - \frac{i}{2}\Gamma_2\right)}{\left(E - E_1 + \frac{i}{2}\Gamma_1\right)\left(E - E_2 + \frac{i}{2}\Gamma_2\right)}. \tag{29}$$

Here the influence of the EPs onto the cross section is contained in the eigenvalues $\mathcal{E}_i = E_i + i/2\Gamma_i$. The expression (29) allows us therefore to receive reliable results also when the phase rigidity is reduced, $r_k < 1$.

The expression (29) can be used to derive analytically an expression for the resonance structure of the S matrix at an EP [40],

$$S = 1 - 2i\frac{\Gamma_d}{E - E_d + \frac{i}{2}\Gamma_d} - \frac{\Gamma_d^2}{\left(E - E_d + \frac{i}{2}\Gamma_d\right)^2}, \tag{30}$$

where $E_1 = E_2 \equiv E_d$ and $\Gamma_1 = \Gamma_2 \equiv \Gamma_d$. As a result of interferences, this expression consists of three terms, one of which is explicitly nonlinear. The resonance structure (30) shows two bumps approximately at the energies ε_i of the two resonance states and an interference minimum between them. This structure resembles that of two more or less isolated resonances whose energies are ε_1 and ε_2.

Many years ago, the resonance structure of the cross section with two resonance states was calculated as a function of the coupling strength between the system and environment [33]. These calculations were performed by using the standard expression for the S matrix with the energies ε_i replaced by the eigenvalues $\mathcal{E}_i$. The results show a double-hump structure at the EP (Fig. 9 in [33]) that corresponds exactly to the expression (30) obtained analytically. Our conclusion from these results is that nonlinear terms determine the resonance structure of the cross section in the neighborhood of an EP.

V. S MATRIX: RESONANCE STRUCTURE IN THE ONE-CHANNEL CASE

According to other works, the resonance structure of the S matrix is well understood when all resonance states are coupled to one and the same decay channel (this is the so-called one-channel case). The resonance structure is calculated by means of the Hermitian formalism, in which no EPs are involved. The basic results of theoretical and experimental studies agree under the condition that the resonances do not overlap, i.e. that they are well separated from one another in the cross section.

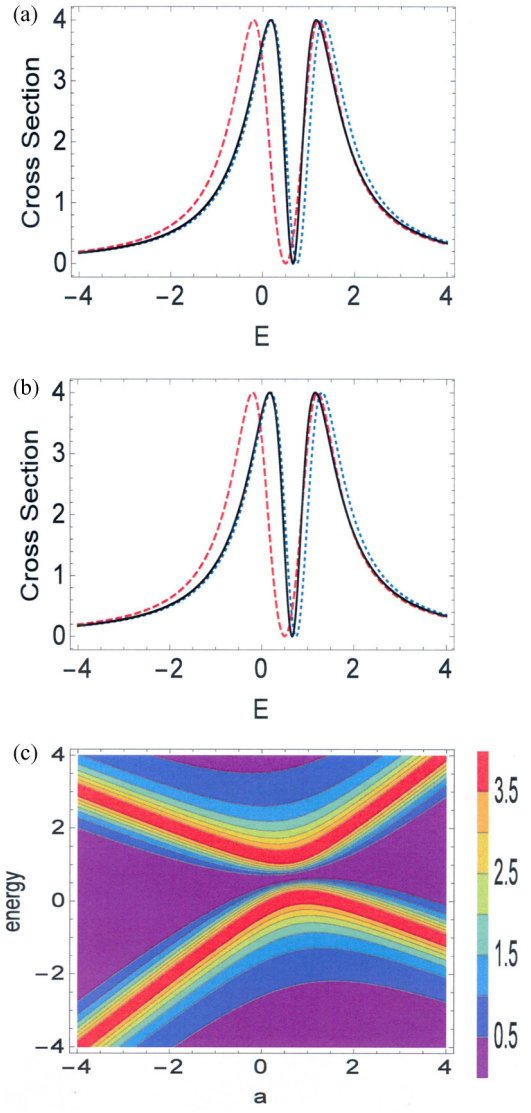

FIG. 3. Cross section with two resonance states. The parameters are the same as in Figs. 1(a)–1(e), but $\omega = 0$ in (b). In (a) and (b) $a = 0$ (dashed red line), $a = 1$ (dotted blue line), and $a = 0.653\,333$ (solid black line). (c) A 2D contour plot with $\omega = 0.01(i + \frac{1}{10})$.

In order to attain a better understanding of the resonance structure of the cross section also in this simple case, we calculate it with and without taking into account EM of the resonance states. In the first case ($\omega \neq 0$) EPs are involved, while in the second case ($\omega = 0$) EPs do not appear. The Hamiltonian is non-Hermitian in both cases. We compare the resonance structure of the S matrix obtained in the two cases.

We performed calculations with different values of ω and for different sets of resonance states. In Figs. 3–5 we show typical results. They are obtained by choosing the values of ω to be the same as in Figs. 1(a)–1(e) ($N = 2$) and in Figs. 2(a)–2(e)

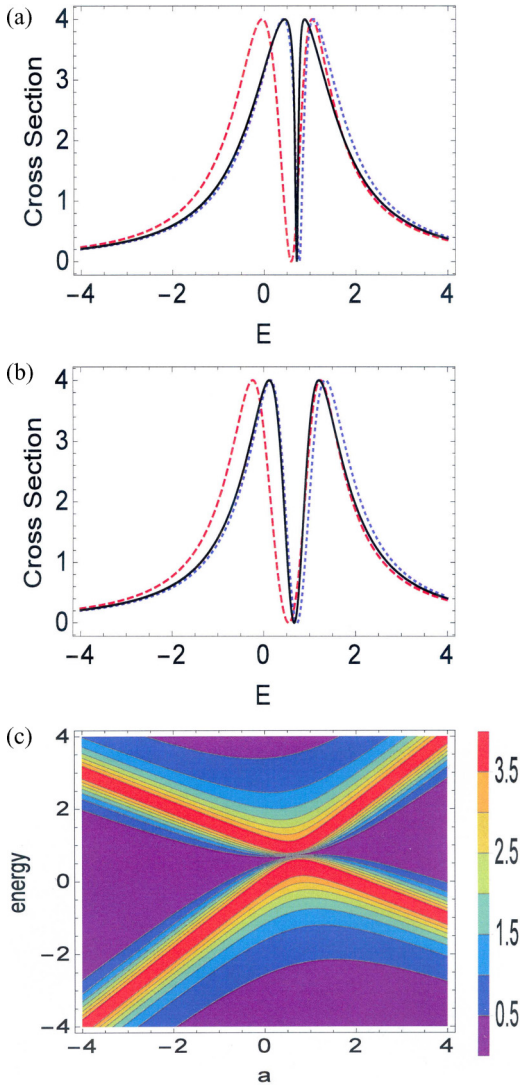

FIG. 4. Cross section with two resonance states. The parameters are the same as in Figs. 1(f)–1(j), but $\omega = 0$ in (b). In (a) and (b), $a = 0$ (dashed red line), $a = 1$ (dotted blue line), and $a = 0.6502$ (solid black line). (c) A 2D contour plot with $\omega = 0.5(i + \frac{1}{10})$.

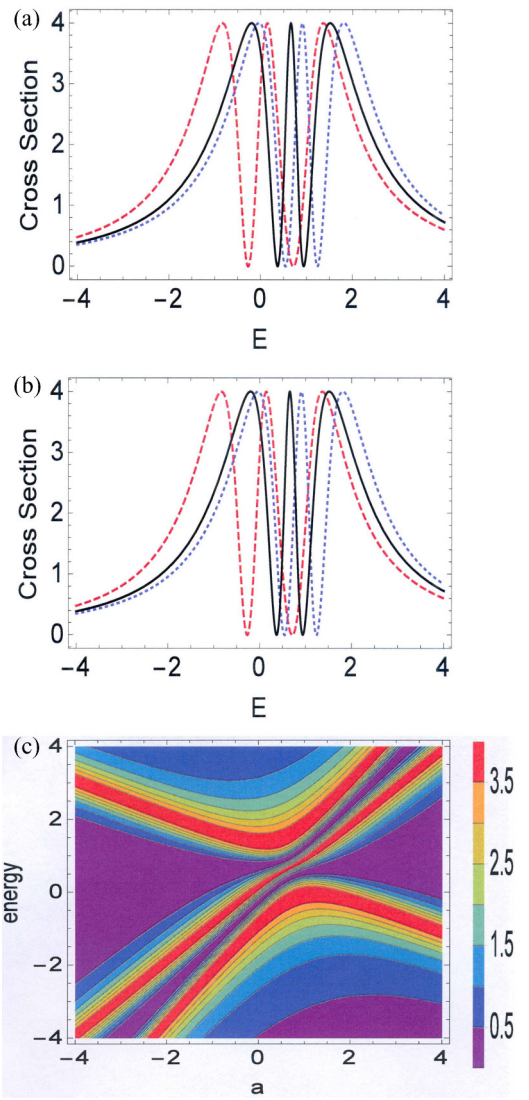

FIG. 5. Cross section with three resonance states. The parameters are the same as in Figs. 2(a)–2(e), but $\omega = 0$ in (b). In (a) and (b) $a = 0$ (dashed red line), $a = 1$ (dotted blue line), and $a = 0.6502$ (solid black line). (c) A 2D contour plot with $\omega = 0$.

($N = 3$). The ω are complex and near those known from realistic systems, so results can be obtained only numerically. The calculations are performed with the energies $\varepsilon_i \equiv e_i + \frac{i}{2}\gamma_i$ chosen in Figs. 1 and 2. All states are coupled to one and the same continuum.

A. Numerical results: Resonance structure with $\omega \neq 0$

Using (28) and (29) for the S matrix, we calculated the resonance structure of the cross section with, respectively, two and three resonance states under different conditions by taking into account EM. In all cases with $N = 2$ resonance states we see a double-hump structure, while the cross section shows a triple-hump structure when $N = 3$. For examples see Figs. 3(a), 4(a), and 5(a).

In Fig. 3(a) the coupling of the states to the continuum is relatively weak; see the corresponding eigenvalue pictures in Figs. 1(a) and 1(b). The resonance part of the S matrix shows the typical two-hump structure.

In Fig. 4(a) the difference between the two widths $\gamma_i/2$ is relatively large; see the corresponding eigenvalue pictures Figs. 1(f) and 1(g). In order to see the influence of an EP, also the coupling strength $|\omega|$ has to be relatively large in this case. According to Fig. 1(g), it is $\sum_{i=1,2} \Gamma_i \approx \Gamma_2$, where Γ_2 is the width of the short-lived state 2. As can be seen from Fig. 4(a), the cross section shows the characteristic double-hump structure not only in the very neighborhood of the EP but also beyond this value.

Using Eqs. (28) and (29) for the S matrix, we are able to reproduce the double-hump structure of the cross section as a function of the coupling strength ω, which is shown in Fig. 9 of Ref. [33]. The calculations in [33] are performed on the basis of the standard S-matrix theory, however with the energies ε_i replaced by the eigenvalues $\mathcal{E}_i$. The role of the interference of

the different contributions to the resonance structure is also shown in Fig. 9 in [33].

In Fig. 5(a) we show the results with $N = 3$ for the case that the widths of the three states are similar to one another and ω is relatively small; see the corresponding eigenvalue pictures Figs. 2(a) and 2(b). The cross section shows the typical three-hump structure at different values of the parameter a near the region with several neighbored EPs as well as beyond it.

The two-dimensional (2D) contour plots of the resonance structure of the cross section with two levels, calculated with $\omega \neq 0$, are shown in Figs. 3(c) and 4(c). In both cases, the cross section falls steeply to its minimum value between the two EPs. Here the eigenfunctions are (almost) orthogonal and mixed in the set of basic wave functions $\{\Phi_n^0\}$ (see Fig. 1). As can be seen from Fig. 4(c), the minimum value appears at the value of the maximum width bifurcation [see Figs. 1(a)–1(e)]. The 2D contour plot of the resonance structure of the cross section with three levels, calculated with $\omega \neq 0$, is the same as that calculated with $\omega = 0$ (see Sec. V B).

In any case, the 2D contour plots of the resonance structure of the cross section should not be confused with eigenvalue trajectories that avoid crossing. Furthermore, they are not symmetric with respect to $E = 0$ corresponding to the eigenvalue pictures Figs. 1 and 2 (different from Fig. 9 for the cross section in [33], which is related to eigenvalue figures that are symmetric with respect to $E = 0$).

B. Numerical results: Resonance structure with $\omega = 0$

We compare the resonance structure of the S matrix obtained in the non-Hermitian formalism with taking into account EM ($\omega \neq 0$) to that obtained without EM (corresponding to $\omega = 0$). Typical results are shown in Figs. 3(b) and 4(b) for $N = 2$ and in Fig. 5(b) for $N = 3$. These figures have to be compared with, respectively, Figs. 3(a), 4(a), and 5(a).

As in all our calculations, the resonance structure of the S matrix is almost the same for $\omega = 0$ and $\omega \neq 0$. Differences in the resonance structure of the S matrix can be seen only when ω is large [compare Fig. 4(a) to Fig. 4(b)]. In all other cases, the resonance structure is typically the same [compare Fig. 3(a) to Fig. 3(b) and Fig. 5(a) to Fig. 5(b)].

In Fig. 5(c) we show the 2D contour plot of the cross section with three resonances, calculated with $\omega = 0$. It looks like that obtained with $\omega \neq 0$. Also the results for $N = 2$ with $\omega = 0$ are typically the same as those with $\omega \neq 0$ [which are shown in Figs. 3(c) and 4(c)].

In the 2D contour plot of the resonance structure with three levels [Fig. 5(c)], we see two second-order EPs instead of a third-order EP. This result corresponds to the discussion of third-order EPs in Sec. III B.

C. Influence of exceptional points

In Figs. 3–5 we have shown the numerical results obtained for the resonance structure of the S matrix when the system is considered with EM ($\omega \neq 0$) and without EM ($\omega = 0$). We considered the most sensitive situation where the resonance structure is influenced by two adjoining EPs. Exceptional points and EM appear only when $\omega \neq 0$. Nevertheless, the resonance structure of the cross section is almost the same in the two cases.

This result is valid not only when the number of resonances is 2, but also when it is larger than 2. This means that the resonance structure of the cross section is almost independent of EM, i.e., on the coupling of the states via one common continuum of scattering states.

This unexpected result can be explained in the following manner. The evolution of the system between the two EPs is driven exclusively by the nonlinear source term of the Schrödinger equation (15) since ω is constant in our calculations and therefore cannot be responsible for the width bifurcation. Obviously, the nonlinear source term is able, in the one-channel case, to largely conserve the resonance structure of the cross section.

Altogether, we have here some type of self-affirmation. Analytical results for the resonance structure of the cross section can be obtained, in the one-channel case (with well separated resonances), when the system is described by a Hermitian operator whose eigenvalues and eigenfunctions are smoothly parameter dependent. These results agree quite well with those of experimental observations. The description of the system as a closed system therefore seems to be justified. In addition, more complicated cases with, e.g., more than one open channel cannot be solved analytically in the standard theory. Thus, the justification of the Hermitian approach for the description of the system (with well-separated resonances) rests solely on the analytical results obtained for the one-channel case.

Our results for the one-channel case show that this case cannot be used to prove or disprove the Hermitian quantum physics. For that purpose, the study of more complicated cases is needed (see Sec. VI).

We mention here that the resonance scattering at third-order EPs is studied in [41] by using a method that is different from ours. Also in these calculations, three peaks appear in the cross section. According to [41], the sprouting out of the three levels under parameter variation depends on the particular parameter chosen. A similar result is obtained [35,39] in the framework of the formalism presented in the present paper. However, a third-order EP does not appear in our calculations [see Fig. 5(c)]. Instead we see several second-order EPs and indications of them, which agrees with the discussion in Sec. III B.

VI. S MATRIX: RESONANCE STRUCTURE IN THE TWO-CHANNEL CASE

We will not provide here new numerical results for the two-channel case. Instead we refer to results obtained a few years ago [5,34,42] for the transmission through a small system (quantum dot). In order to describe transmission, we have to consider at least two channels: the entrance and the exit channel. In [5], unexpected experimental results [1–3] on the resonance structure of the transmission could be explained. In Refs. [34,42], different calculations were performed for both a system with a small number of resonance states and a system with many states. In the latter case, the calculations were performed first in the tight-binding approach according to the formalism presented by Datta [43]. Then the non-Hermitian Hamilton operator was diagonalized and

the eigenvalues and eigenfunctions were determined. This formalism is equivalent to that used in the present paper (see also [44]). The calculations for systems with a small number of states are performed according to S-matrix theory and using tight-binding approach [45].

The results for the two-channel case are more interesting than those for the one-channel case discussed in Sec. V since the influence of EPs and EM can be seen immediately [5,34,42]. The resonance structure of the transmission can be traced back to the eigenvalues of the non-Hermitian operator, EM of the states generally cannot be neglected, and the phase rigidity is anticorrelated with the transmission probability. The last property is the most interesting one. It has no analog in the standard formalism.

The phase rigidity is a theoretical value characteristic of the non-Hermitian formalism. It can be traced experimentally in a microwave billiard [26]. It will however be difficult to study it directly in a realistic system. According to the above-mentioned numerical results [34,42] it is however anticorrelated with an observable value, namely, with the transmission probability.

This anticorrelation of the theoretical value (phase rigidity) with an observable one (transmission probability) allows us to really test the non-Hermitian formalism. Moreover, when this anticorrelation really exists, it is of high interest for applications.

VII. DISCUSSION OF THE RESULTS AND SUMMARY

A critical consideration of the standard Hermitian formalism for the description of open quantum systems is possible by starting from a general non-Hermitian formalism [14] that includes the Hermitian quantum physics as a limiting case. In the non-Hermitian formalism, the normalization of the eigenfunctions of the Hamiltonian can freely be chosen [23]. If it is chosen by means of (9), the non-Hermitian quantum formalism fulfills the condition to approach, on the one hand, the standard Hermitian quantum physics under certain conditions (which can be formulated) and to be, on the other hand, more general than it. The mathematical consequences are the following.

(i) The phases of the eigenfunctions relative to one another are not rigid (see Sec. II B). This fact agrees with the basic relation (20) that is valid in approaching an EP: The two eigenfunctions Φ_1 and Φ_2 of $\mathcal{H}^{(2)}$ are (almost) orthogonal to one another when the two eigenstates 1 and 2 are distant from one another, while the orthogonality is completely lost in approaching an EP.

(ii) The eigenstates contain EM and differ therefore from the original eigenstates. A mixing of the wave functions of only two states may appear at low level density such that it may be difficult to choose the basic set of "pure" wave functions. At high level density EM causes a dynamical phase transition that is nonadiabatic due to the involved nonlinear processes [39].

(iii) Some well-known unsolved puzzles of standard Hermitian quantum physics do not appear in the non-Hermitian description of open quantum systems [14]. Among others, the problem of the Schrödinger cat and the short tunneling time characterizing the decay of the states in the Hermitian quantum physics are not puzzling when the system is considered to be open. Furthermore, the nonlinear processes involved in the non-Hermitian formalism are irreversible (see the discussion around Fig. 9 in [6]).

In Secs. II, III, and V of the present paper, we have shown numerical results obtained in the framework of non-Hermitian quantum theory for a system that is coupled to one common channel [14]. We consider systems with $N = 2$ and $N = 3$ states in the most sensitive parameter range in which the dynamics of the system is determined by two EPs. We have compared the results, obtained for the same situation, with and without taking into account EM. In the first case, EPs are caused by the EM of the states. In the second case, however, neither EM nor EPs appear. Nevertheless, the resonance structure of the S matrix is almost the same in the two cases. This result does not depend on the number of states taken into account in the calculation.

In our calculations with the non-Hermitian Hamiltonian, the coupling strength ω between the system and environment is chosen to be fixed. Width bifurcation of the states therefore may be caused exclusively by the nonlinear terms contained in the Schrödinger equation at and near an EP. These nonlinear terms conserve, obviously, the resonance structure of the cross section in the one-channel case. Thus, the one-channel case does not allow us to test the non-Hermitian formalism.

The situation is completely different when the system is coupled to two (or more) channels. A prominent example is the transmission through, e.g., a quantum dot. Here, at least two different channels are involved: the entrance and exit channel. In the present paper, we do not provide new numerical results. Instead we refer to some results obtained earlier (see Sec. VI). Most interesting is the anticorrelation between phase rigidity and transmission probability that can be seen clearly in the results of different calculations.

Thus, the observation of nonanalytical effects in the transmission through a quantum dot is not in contradiction with the results known from the standard S-matrix description in the one-channel case. Quite the contrary, these effects are characteristic of the non-Hermitian theory of open quantum systems. They exist also in the one-channel case where they cannot be seen however due to their suppression by the nonlinear terms of the Schrödinger equation near EPs.

VIII. CONCLUSIONS

The results of the present paper answer the questions asked in the Introduction. Although EPs influence the dynamics of open quantum systems, they cannot be observed directly. In the one-channel case, the resonance structure of the cross section can be described well *without* taking them into account. The reason for this unexpected result is the nonlinear processes caused by the EPs. They restore, in the one-channel case, the original resonance structure of the cross section and hide the influence of the EPs on observable values. A cursory consideration allows therefore the conclusion that EPs do not play any role in open quantum systems.

This conclusion can be justified however only in the one-channel case. The two-channel case is much richer and more interesting. In the Introduction, we pointed to the so-called phase lapses observed experimentally in the transmission through a quantum dot. These unexpected results are explained

by means of the existence of EPs. We mentioned moreover in Sec. VI the results obtained theoretically for the transmission through a localized quantum system by using the tight-binding approach. We underline however that, in any case, the results of non-Hermitian quantum physics differ from those of Hermitian quantum physics only a little in a parameter range that is *not* influenced by EPs. In this parameter range, Fermi's golden rule holds.

The results for the two-channel case show, under different conditions, an anticorrelation between phase rigidity and transmission probability, i.e., between an *internal* property of the eigenfunctions of the non-Hermitian Hamilton operator and an observable value. Based on the non-Hermitian quantum theory formulated in the present paper, an experimental test of this relation will contribute to an understanding of the short tunneling time [15] that is observed experimentally.

The transmission through a small system needs to be studied in the future in more detail, theoretically as well as experimentally. On the one hand, it allows us to test the non-Hermitian quantum theory for open quantum systems [14], since it relates the theoretical value phase rigidity to the observable value of the transmission probability. On the other hand, the anticorrelation between these two values will offer the possibility of important applications.

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