Improved bound for quantum-speed-limit time in open quantum systems by introducing an alternative fidelity

Abbas Ektesabi,[1,*] Naghi Behzadi,[2,†] and Esfandyar Faizi[1,‡]
[1]*Physics Department, Azarbaijan Shahid Madani University, Tabriz, Iran*
[2]*Research Institute for Fundamental Sciences, University of Tabriz, Tabriz, Iran*
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In this paper, we introduce an alternative quantum fidelity for quantum states which perfectly satisfies all of Jozsa's axioms and is zero for orthogonal states. By employing this fidelity, we derive an improved bound for quantum-speed-limit time in open quantum systems in which the initial states can be chosen as either pure or mixed. This bound leads to the well-known Mandelstamm-Tamm–type bound for nonunitary dynamics in the case of initial pure states. However, in the case of initial mixed states, the bound provided by the introduced fidelity is tighter and sharper than the obtained bounds in the previous works.

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I. INTRODUCTION

Quantum speed limits (QSLs) are the ultimate bounds imposed by quantum mechanics on the minimal evolution time for a quantum state to become orthogonal to itself. QSLs have been widely investigated since the appearance of the first major result by Mandelstamm and Tamm [1]. They derived a QSL limit time bound for a quantum system that evolves between two pure orthogonal initial and final states under the time-independent Hamiltonian H. The bound is given by $\tau \geqslant \pi\hbar/(2\Delta E)$, where ΔE is the variance of the energy. Later Margolus and Levitin [2] provided a different QSL time bound for a closed system read, $\tau \geqslant \pi\hbar/(2E)$, where E is the mean energy with respect to the ground state. Both the Mandelstamm-Tamm and Margolus-Levitin bounds are attainable in closed quantum systems for initial pure states, while for general mixed states they can be rather loose. Since any system is coupled to an environment, an analogous bound for an open quantum system is highly desirable. Taddei *et al.* [3] extended the Mandelstamm-Tamm–type bound to both unitary and nonunitary processes described by positive nonunitary maps by using quantum Fisher information for time estimation. However, for the case of the initial mixed states, it is hard to evaluate the bound due to minimization of the quantum Fisher information in the enlarged system-environment space. Later Deffner and Lutz [4] extended both Mandelstamm-Tamm and Margolus-Levitin bounds to an open quantum system by exploiting Caushy-schwarz and von Neumann trace inequality, respectively. They showed that the non-Markovian effect leads to the faster quantum evolution. However, their bound is derived from pure initial states and cannot be applied to the mixed initial states. Also del Campo *et al.* [5] derived an analytical and computable QSL bound for open quantum systems by exploiting the relative purity. Relative purity can make a distinction between two initial pure states; however, it may fail as a distance measure between two initial mixed states. Recently Sun *et al.* [6] derived another quantum speed limit bound for open quantum systems by employing an alternative fidelity introduced in Ref. [7] where the initial states can be chosen as either pure or mixed. However, their bound is not tight, and the alternative fidelity which they used as a distance measure is not monotone under quantum operations and fails to satisfy one of Jozsa's four natural axioms [8].

In this paper, we first propose an alternative definition of quantum fidelity between quantum states which perfectly satisfies all of Jozsa's four axioms. In addition, our preliminary numerical calculations show that the proposed fidelity is monotone under quantum operations. Also this fidelity is zero when two density matrices are orthogonal, the criterion which cannot be satisfied by some previously introduced fidelities [12–14]. By employing this fidelity and applying Caushy-schwarz inequality, we derive a QSL time bound for open quantum systems where the initial state can be chosen as either pure or mixed. This bound leads to the Mandelstamm-Tamm–type bound for nonunitary dynamics in the case of initial pure states. However, in the case of initial mixed states, the obtained bound is tighter and sharper than the bounds provided by previous work.

The work is organized as follows. In Sec. II we introduce an alternative fidelity and discuss its basic properties. In Sec. III we derive the QSL time bound by exploiting the new fidelity. Section IV is devoted to demonstrating the performance of QSL time bound obtained by the introduced fidelity by considering a two-level atomic system coupled resonantly to a leaky vacuum reservoir. Finally, the paper is ended with a brief conclusion.

II. PROPERTIES OF THE ALTERNATIVE FIDELITY

In order to derive a QSL time bound for open quantum systems, we should use a distance measure between two quantum states. Among the distance measures, the Bures fidelity is the most important one for quantum computation and quantum information processing [9–11]. This fidelity for two general mixed states ρ and σ is given by

$$F(\rho,\sigma) = [\mathrm{Tr}(\sqrt{\rho^{1/2}\sigma\rho^{1/2}})]^2. \tag{1}$$

It is well known from Ref. [8] that any generic notion of fidelity defined for mixed states, such as the Bures fidelity,

*ektesabi@azaruniv.edu
†n.behzadi@tabrizu.ac.ir
‡efaizi@azaruniv.edu

should satisfy Jozsa's four natural axioms:

(A_1) $0 \leqslant F(\rho,\sigma) \leqslant 1$ and $F(\rho,\sigma) = 1$ if and only if $\rho = \sigma$.

(A_2) F is symmetry under swapping of two states, i.e., $F(\rho,\sigma) = F(\sigma,\rho)$.

(A_3) $F(\rho,\sigma)$ is invariant under unitary transformations on the state space.

(A_4) If one of the states is pure ($\sigma = |\psi\rangle\langle\psi|$), the fidelity reduces to $F(\rho,|\psi\rangle\langle\psi|) = \langle\psi|\rho|\psi\rangle$.

Also, the Bures fidelity has additional properties such as multiplicativity under tensor product, monotonicity under quantum operations, and concavity. It should be noted that monotonicity under quantum operations is the most important property from a quantum information point of view and QSLs [15]. Therefore, the Bures fidelity in Eq. (1) is, fundamentally, the most suitable fidelity where the properties of other constructed fidelities are compared with the properties of this one. However, due to the difficulty in calculating the Bures fidelity, attempts have been made to find an alternative fidelity to avoid this difficulty. Wang *et al.* [7] proposed an alternative fidelity in terms of their Hilbert-Schmidt inner product and their purities, which reads as

$$F_1(\rho,\sigma) = \frac{\mathrm{Tr}(\rho\sigma)}{\sqrt{\mathrm{Tr}(\rho^2)\mathrm{Tr}(\sigma^2)}}. \quad (2)$$

Recently Sun *et al.* [6] derived an analytical and computable quantum speed limit bound for open quantum systems by exploiting F_1 in Eq. (2). However, one can easily find that F_1 fails to satisfy the fourth axiom, and thus it may introduce some defects into derivation of a quantum speed limit (as will be seen in Sec. III). By comparing with the additional properties of the Bures fidelity, F_1 is multiplicative; however, it is not concave and it is not monotone under quantum operations [11]. Thus F_1 is not a suitable distance measure. Another fidelity was defined by Miszczak *et al.* [12] and Mendonça *et al.* [13], which reads as

$$F_2(\rho,\sigma) = \mathrm{Tr}(\rho\sigma) + \sqrt{1-\mathrm{Tr}(\rho^2)}\sqrt{1-\mathrm{Tr}(\sigma^2)}. \quad (3)$$

Also Chen *et al.* [14] defined the following fidelity, which is essentially the same as F_2:

$$F_3(\rho,\sigma) = \frac{1-r}{2} + \frac{1+r}{2}F_2(\rho,\sigma), \quad (4)$$

where $r = 1/(d-1)$ and d is the dimension of the Hilbert space. F_2 and F_3 satisfy all of Jozsa's four axioms. Although F_2 and F_3 are identical in $d = 2$ and they reduce to an equivalent Bures fidelity (1), for two orthogonal density matrices such as

$$\rho = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|),$$
$$\sigma = \tfrac{1}{2}(|2\rangle\langle 2| + |3\rangle\langle 3|), \quad (5)$$

defined on a four-dimensional Hilbert space spanned by $\{|n\rangle, n = 0,1,2,3\}$, $F_2(\rho,\sigma)$ and $F_3(\rho,\sigma)$ fail to be zero for $d > 2$ (in this case $d = 4$) [7]. Also, F_2 is concave; however, F_2 and F_3 are not multiplicative and not monotone under quantum operations [13]. So F_2 and F_3 are not suitable distance measures.

In this paper, we propose a new alternative definition of quantum fidelity between quantum states, which reads as

$$\mathcal{F}(\rho,\sigma) = \left[1 + \sqrt{\frac{1-\mathrm{Tr}(\rho^2)}{\mathrm{Tr}(\rho^2)}}\sqrt{\frac{1-\mathrm{Tr}(\sigma^2)}{\mathrm{Tr}(\sigma^2)}}\right]\mathrm{Tr}(\rho\sigma). \quad (6)$$

$\mathcal{F}$ satisfies all of Jozsa's axioms, and it is zero when two density matrices are orthogonal. It is not difficult to see that $\mathcal{F}$ satisfies Jozsa's axioms (A_2), (A_3), and (A_4). In the following, we prove that $\mathcal{F}$ satisfies the axiom (A_1).

Proof of axiom (A_1). We rewrite the Eq. (6) as follows:

$$\mathcal{F}(\rho,\sigma) = \mathrm{Tr}(\rho\sigma) + \sqrt{\frac{1-\mathrm{Tr}(\rho^2)}{\mathrm{Tr}(\rho^2)}}\sqrt{\frac{1-\mathrm{Tr}(\sigma^2)}{\mathrm{Tr}(\sigma^2)}}\mathrm{Tr}(\rho\sigma). \quad (7)$$

By using the Cauchy-Schwarz inequality, i.e, $|\mathrm{Tr}(\rho\sigma)| \leqslant \sqrt{\mathrm{Tr}(\rho^2)\mathrm{Tr}(\sigma^2)}$ in the second term of Eq. (7), we get

$$\mathcal{F}(\rho,\sigma) \leqslant \mathrm{Tr}(\rho\sigma) + \sqrt{1-\mathrm{Tr}(\rho^2)}\sqrt{1-\mathrm{Tr}(\sigma^2)}. \quad (8)$$

Now by considering the inequality $\mathrm{Tr}(\rho\sigma) + \sqrt{1-\mathrm{Tr}(\rho^2)}\sqrt{1-\mathrm{Tr}(\sigma^2)} \leqslant 1$ [13], we reach $\mathcal{F}(\rho,\sigma) \leqslant 1$. ■

Unlike the Bures fidelity, $\mathcal{F}$ is not multiplicative under tensor product because it is super-multiplicative:

$$\mathcal{F}(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) \geqslant \mathcal{F}(\rho_1,\sigma_1)\mathcal{F}(\rho_2,\sigma_2). \quad (9)$$

This property may introduce, though not certainly, some defects into the monotonicity property [13]; however, our numerical search shows that $\mathcal{F}$ is monotone under quantum operations. To prove the inequality (9) we can write

$$\mathcal{F}(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = \left[1 + \sqrt{\frac{1-\mathrm{Tr}(\rho_1^2)\mathrm{Tr}(\rho_2^2)}{\mathrm{Tr}(\rho_1^2)\mathrm{Tr}(\rho_2^2)}}\sqrt{\frac{1-\mathrm{Tr}(\sigma_1^2)\mathrm{Tr}(\sigma_2^2)}{\mathrm{Tr}(\sigma_1^2)\mathrm{Tr}(\sigma_2^2)}}\right]\mathrm{Tr}(\rho_1\sigma_1)\mathrm{Tr}(\rho_2\sigma_2) \quad (10)$$

and

$$\mathcal{F}(\rho_1,\sigma_1)\mathcal{F}(\rho_2\sigma_2) = \left\{1 + \sqrt{\frac{[1-\mathrm{Tr}(\rho_1^2)][1-\mathrm{Tr}(\sigma_1^2)]}{\mathrm{Tr}(\rho_1^2)\mathrm{Tr}(\sigma_1^2)}}\right\}\left\{1 + \sqrt{\frac{[1-\mathrm{Tr}(\rho_2^2)][1-\mathrm{Tr}(\sigma_2^2)]}{\mathrm{Tr}(\rho_2^2)\mathrm{Tr}(\sigma_2^2)}}\right\}\mathrm{Tr}(\rho_1\sigma_1)\mathrm{Tr}(\rho_2\sigma_2). \quad (11)$$

By defining $r_i := \mathrm{Tr}(\rho_i^2)$ and $s_i := \mathrm{Tr}(\sigma_i^2)$, we have to show that

$$\sqrt{(1-r_1r_2)(1-s_1s_2)} \geqslant \sqrt{(1-r_1)(1-s_1)}\sqrt{r_2s_2} + \sqrt{(1-r_2)(1-s_2)}\sqrt{r_1s_1} + \sqrt{(1-r_1)(1-s_1)(1-r_2)(1-s_2)}. \quad (12)$$

To this aim, we define two vectors

$$X = \begin{pmatrix} \sqrt{r_1}\sqrt{1-r_2} \\ \sqrt{r_2}\sqrt{1-r_1} \\ \sqrt{1-r_1}\sqrt{1-r_2} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \sqrt{s_1}\sqrt{1-s_2} \\ \sqrt{s_2}\sqrt{1-s_1} \\ \sqrt{1-s_1}\sqrt{1-s_2} \end{pmatrix}, \tag{13}$$

with

$$\begin{aligned} \langle X|Y\rangle &= \sqrt{(1-r_1)(1-s_1)}\sqrt{r_2 s_2} \\ &+ \sqrt{(1-r_2)(1-s_2)}\sqrt{r_1 s_1} \\ &+ \sqrt{(1-r_1)(1-s_1)(1-r_2)(1-s_2)} \end{aligned} \tag{14}$$

and

$$\langle X|X\rangle = (1 - r_1 r_2) \quad \text{and} \quad \langle Y|Y\rangle = (1 - s_1 s_2). \tag{15}$$

Now, by using the Cauchy-Schwarz inequality

$$\sqrt{\langle X|X\rangle\langle Y|Y\rangle} \geqslant \langle X|Y\rangle, \tag{16}$$

the inequality (12) is satisfied. ■

The fact that $\mathcal{F}$ is supermultiplicative and not multiplicative may be a sign that it may not have the monotonicity property of a fidelity. However, the preliminary numerical search favors the validity of monotonicity property of $\mathcal{F}$ and shows that it is monotonically increasing under completely positive trace-preserving (CPTP) maps. For example, the counterexample used to show that F_2 in Eq. (3) does not behave monotonically under CPTP maps in Ref. [13] satisfies the desired monotonicity property of $\mathcal{F}$. By denoting ϱ and ς as the two two-qubit density matrices

$$\varrho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \varsigma = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

and by considering the quantum operations of tracing over the first or second qubit, we have

$$\mathcal{F}(\mathrm{Tr}_1(\varrho),\mathrm{Tr}_1(\varsigma)) = 1 > 0 = \mathcal{F}(\varrho,\varsigma) \tag{18}$$

and

$$\mathcal{F}(\mathrm{Tr}_2(\varrho),\mathrm{Tr}_2(\varsigma)) = 0 = \mathcal{F}(\varrho,\varsigma). \tag{19}$$

Therefore, Eqs. (18) and (19) show that $\mathcal{F}$ is monotonically increasing and satisfies the desired monotonicity property under this map. On the other hand, Ozawa in Ref. [16] showed by the following counterexample that the Hilbert-Schmit distance, i.e., $D_{HS}(\rho,\sigma) = ||\rho - \sigma||_{HS}^2 = \mathrm{Tr}[(\rho-\sigma)^2]$ where ρ and σ are density matrices, is not monotone. Let us consider

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

with

$$A^\dagger A + B^\dagger B = I_4. \tag{21}$$

The map $\mathcal{E}$ with Kraus operators A and B is completely positive and trace preserving such that $D_{HS}(\varrho,\varsigma) = 1$ while $D_{HS}(\mathcal{E}(\varrho),\mathcal{E}(\varsigma)) = 2$, i.e., $D_{HS}(\mathcal{E}(\varrho),\mathcal{E}(\varsigma)) > D_{HS}(\varrho,\varsigma)$. So by this counterexample, the Hilbert-Schmit distance is not monotone. In addition, $F_2(\varrho,\varsigma) = \frac{1}{2}$ and $F_2(\mathcal{E}(\varrho),\mathcal{E}(\varsigma)) = 0$, i.e. $F_2(\mathcal{E}(\varrho),\mathcal{E}(\varsigma)) < F_2(\varrho,\varsigma)$ indicating another argument that F_2 is not monotone (and also is not F_3). However, for the fidelity $\mathcal{F}$, we have $\mathcal{F}(\mathcal{E}(\varrho),\mathcal{E}(\varsigma)) = \mathcal{F}(\varrho,\varsigma) = 0$, i.e., this example does not violate the monotonicity of $\mathcal{F}$. Also in this direction, we examined the monotonicity of $\mathcal{F}$ under CPTP maps with different Kraus operators instead of A and B which satisfy Eq. (21), and quantum operations of tracing over the first and second qubit for various pairs of density matrices. Consequently, we did not find a counterexample for violation of the monotonicity property of $\mathcal{F}$.

Also our preliminary numerical calculations show that $\mathcal{F}$ satisfies the property of concavity, so the inequality

$$\mathcal{F}(\rho, p\sigma_1 + (1-p)\sigma_2) \geqslant p\mathcal{F}(\rho,\sigma_1) + (1-p)\mathcal{F}(\rho,\sigma_2) \tag{22}$$

is satisfied for density matrices ρ, σ_1, and σ_2 numerically. In conclusion, it is worthwhile to note the point whether a distance measure (fidelity) can be monotone while it is not multiplicative or concave. The supermultiplicativity (or loss of multiplicativity) may destroy the monotonicity of distance measures [13]. However, it is unknown whether loss of concavity has the potential to induce a defect into the monotonicity of distance measures.

III. QUANTUM-SPEED-LIMIT TIME

Now we are in a position to derive a new bound for QSL time (τ_{QSL}) by using the fidelity (6) as a distance measure introduced in the previous section. The absolute value for the time derivative of the fidelity $\mathcal{F}(\rho_0,\rho_t)$ is

$$\left|\frac{d\mathcal{F}}{dt}\right| = \left|\frac{-\mathrm{Tr}(\dot{\rho}_t\rho_t)}{[\mathrm{Tr}(\rho_t^2)]^2}\sqrt{\frac{1-\mathrm{Tr}(\rho_0^2)}{\mathrm{Tr}(\rho_0^2)}}\sqrt{\frac{\mathrm{Tr}(\rho_t^2)}{1-\mathrm{Tr}(\rho_t^2)}}\mathrm{Tr}(\rho_0\rho_t) + \left[1 + \sqrt{\frac{1-\mathrm{Tr}(\rho_0^2)}{\mathrm{Tr}(\rho_0^2)}}\sqrt{\frac{1-\mathrm{Tr}(\rho_t^2)}{\mathrm{Tr}(\rho_t^2)}}\right]\mathrm{Tr}(\rho_0\dot{\rho}_t)\right|, \tag{23}$$

where by using triangle inequality it becomes

$$\left|\frac{d\mathcal{F}}{dt}\right| \leqslant \sqrt{\frac{1-\mathrm{Tr}(\rho_0^2)}{\mathrm{Tr}(\rho_0^2)}}\sqrt{\frac{\mathrm{Tr}(\rho_t^2)}{1-\mathrm{Tr}(\rho_t^2)}}\left|\frac{\mathrm{Tr}(\dot{\rho}_t\rho_t)\mathrm{Tr}(\rho_0\rho_t)}{(\mathrm{Tr}(\rho_t^2))^2}\right| + \left[1+\sqrt{\frac{1-\mathrm{Tr}(\rho_0^2)}{\mathrm{Tr}(\rho_0^2)}}\sqrt{\frac{1-\mathrm{Tr}(\rho_t^2)}{\mathrm{Tr}(\rho_t^2)}}\right]|\mathrm{Tr}(\rho_0\dot{\rho}_t)|. \quad (24)$$

By considering the Cauchy-Schwarz inequality in the second term of Eq. (24), we get

$$\left|\frac{d\mathcal{F}}{dt}\right| \leqslant \sqrt{\frac{1-\mathrm{Tr}(\rho_0^2)}{\mathrm{Tr}(\rho_0^2)}}\sqrt{\frac{\mathrm{Tr}(\rho_t^2)}{1-\mathrm{Tr}(\rho_t^2)}}\left|\frac{\mathrm{Tr}(\dot{\rho}_t\rho_t)\mathrm{Tr}(\rho_0\rho_t)}{[\mathrm{Tr}(\rho_t^2)]^2}\right| + \sqrt{\mathrm{Tr}(\rho_0^2)\mathrm{Tr}(\dot{\rho}_t^2)} + \sqrt{\frac{1-\mathrm{Tr}(\rho_t^2)}{\mathrm{Tr}(\rho_t^2)}}\sqrt{1-\mathrm{Tr}(\rho_0^2)}\sqrt{\mathrm{Tr}(\dot{\rho}_t^2)}. \quad (25)$$

Integration of Eq. (25) over deriving time τ gives the following inequality for the QSL time bound:

$$\tau \geqslant \tau_{QSL} = \frac{|1-\mathcal{F}_\tau|}{X_\tau}, \quad (26)$$

where $\mathcal{F}_\tau := \mathcal{F}(\rho_0,\rho_\tau)$ is the target value of the fidelity at time τ, and X_τ is defined as

$$X_\tau := \frac{1}{\tau}\int_0^\tau \left\{\sqrt{\frac{1-\mathrm{Tr}(\rho_0^2)}{\mathrm{Tr}(\rho_0^2)}}\sqrt{\frac{\mathrm{Tr}(\rho_t^2)}{1-\mathrm{Tr}(\rho_t^2)}}\left|\frac{\mathrm{Tr}(\dot{\rho}_t\rho_t)\mathrm{Tr}(\rho_0\rho_t)}{[\mathrm{Tr}(\rho_t)^2]^2}\right| + \sqrt{\mathrm{Tr}(\rho_0^2)\mathrm{Tr}(\dot{\rho}_t^2)} + \sqrt{\frac{1-\mathrm{Tr}(\rho_t^2)}{\mathrm{Tr}(\rho_t^2)}}\sqrt{1-\mathrm{Tr}(\rho_0^2)}\sqrt{\mathrm{Tr}(\dot{\rho}_t^2)}\right\} dt. \quad (27)$$

Equation (26) provides an expression for lower bound of QSL time and can be used to consider for either Markovian or non-Markovian dynamics. It is interesting to note that in the case of initial pure states, we have $\mathrm{Tr}(\rho_0^2) = 1$, and therefore Eq. (27) turns into

$$X_\tau = \frac{1}{\tau}\int_0^\tau \sqrt{\mathrm{Tr}(\dot{\rho}_t^2)}\, dt, \quad (28)$$

and the target fidelity becomes

$$\mathcal{F}_\tau = \mathrm{Tr}(\rho_0\rho_\tau). \quad (29)$$

Substituting Eqs. (28) and (29) into Eq. (26) yields

$$\tau \geqslant \tau_{QSL} = \frac{\tau|1-\mathrm{Tr}(\rho_0\rho_\tau)|}{\int_0^\tau \sqrt{\mathrm{Tr}(\dot{\rho}_t^2)}\, dt}, \quad (30)$$

which is the well-known Mandelstamm-Tamm–type bound for nonunitary dynamics in the case of initial pure states, the case which was obtained initially in Ref. [4] by using the Bures angle as a metric. Note that for a fidelity which does not satisfy Jozsa's fourth axiom, the obtained QSL time bound, by the help of Caushy-Schwarz inequality, fails to reduce to the Mandelstamm-Tamm–type bound in the case of initial pure states. Also, it is expected that this deficiency causes occurrence of difficulties in derivation of a tighter bound in the case of initial mixed states. For example, F_1 reflects both of these undesirable issues in its respective QSL time bound. In the next section, we examine our bound (26) by a concrete open quantum system as a physical model in which the initial state of the system is generally mixed.

IV. PHYSICAL MODEL

To investigate the performance of the bound (26) for QSL time, we consider a two-level quantum system which is resonantly coupled to a leaky vacuum reservoir. The whole Hamiltonian of the system and the reservoir can be written as

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \sum_k \hbar\omega_k a_k^\dagger a_k + \sum_k \hbar(g_k a_k\sigma_+ + g_k^* a_k^\dagger \sigma_-), \quad (31)$$

where σ_z is the Pauli matrix and σ_+ (σ_-) is the Pauli raising (lowering) operator for the atom with transition frequency ω_0. a_k ($a_k^\dagger$) is the annihilation (creation) operator for the kth field mode with frequency ω_k, and g_k is the coupling constant between the kth field mode and the system. The dynamics of the system can be described by

$$L_t(\rho_t) = \gamma_t\left(\sigma_-\rho_t\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-,\rho_t\}\right). \quad (32)$$

The spectral density of the reservoir is assumed to have the Lorentzian form

$$J(\omega) = \frac{1}{2\pi}\frac{\gamma_0\lambda^2}{(\omega_0-\omega)^2+\lambda^2}, \quad (33)$$

where γ_0 is the coupling strength and λ is the width of the Lorentzian function. The density matrix of the system at time t can be obtained analytically [17] as

$$\rho(t) = \begin{pmatrix} \rho_{11}(0)|G(t)|^2 & \rho_{10}(0)G(t) \\ \rho_{01}(0)G(t)^* & 1-\rho_{11}(0)|G(t)|^2 \end{pmatrix}, \quad (34)$$

where the function $G(t)$ is defined as the solution of the the integro-differential equation

$$\frac{d}{dt}G(t) = -\int_0^t dt_1 f(t-t_1)G(t_1), \quad (35)$$

with the initial condition $G(0) = 1$, and the correlation kernel $f(t-t_1)$ related to the spectral density of the reservoir as

$$f(t-t_1) = \int d\omega J(\omega)e^{i(\omega_0-\omega_k)(t-t_1)}. \quad (36)$$

Using the Laplace transformation and its inverse, $G(t)$ can be given by

$$G(t) = e^{-\lambda t/2}\left[\cosh\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sinh\left(\frac{dt}{2}\right)\right], \quad (37)$$

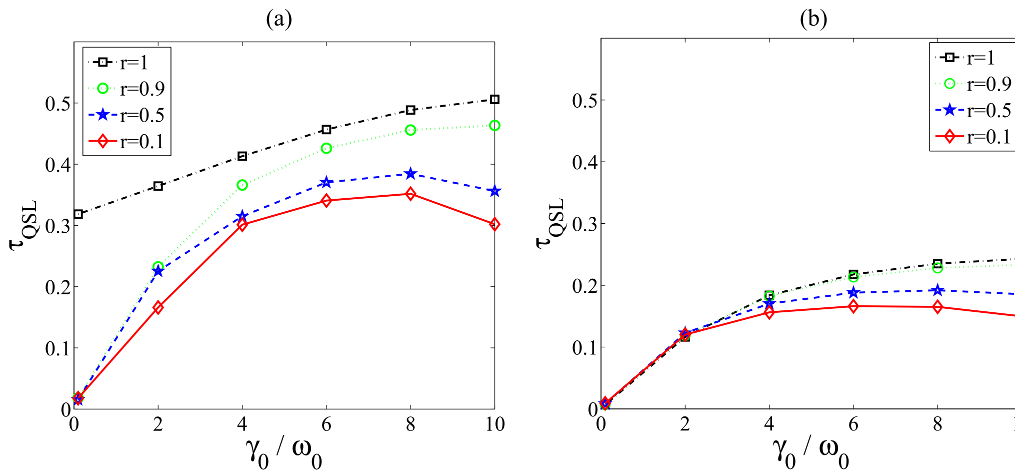

FIG. 1. QSL time, τ_{QSL}, as a function of the coupling strength γ_0 (in units of ω_0) for initial states (38) with different mixed coefficients $r = 0.1, 0.5, 0.9, 1$. (a) The bounds derived from (26) and (b) the bounds obtained from Ref. [6]. $\lambda = 1$ (in units of ω_0) and $\tau = 1$.

with $d = \sqrt{\lambda^2 - 2\gamma_0\lambda}$. Also, the time-dependent decay rate in Eq. (32) is given by $\gamma_t = -\text{Im}(\frac{\dot{G}(t)}{G(t)})$. The dynamics is Markovian in the weak-coupling regime $\gamma_0 < \lambda/2$ and becomes non-Markovian for strong coupling $\gamma_0 > \lambda/2$. In this work, we consider a mixed initial state of Werner type

$$\rho_0 = \frac{1-r}{2} I + r|\psi\rangle\langle\psi|, \tag{38}$$

where I is a 2×2 identity matrix, $0 \leqslant r \leqslant 1$, and $|\psi\rangle = (|1\rangle + |0\rangle)/\sqrt{2}$.

In Figs. 1 and 2, we present the QSL time bounds as a function of the coupling strength γ_0 for the initial states of Eq. (38) with different mixed coefficients r. Figure 1(a) represents the QSL time bounds in (26) with parameter $\lambda = 1$ and the deriving time $\tau = 1$. We can see that the larger value of r which corresponds to the higher purity introduces a higher QSL time bound. As the non-Markovianity behavior grows in term of γ_0, the lower bound decreases with respect to the mixedness of the initial state, i.e., the speed of evolution for the initial mixed states grows. Figure 1(b) sketches the obtained bound from the previous work [6] with the same condition as Fig. 1(a), which is derived from exploiting F_1, i.e., Eq. (2), as a distance measure. Obviously, the QSL bound obtained in this paper is tighter than the derived bound in the previous work [6] for both pure and mixed initial states.

Also, we reexamine our bound with $\lambda = 20$ and $\tau = 1$ and compare it with bound of Ref. [6], as depicted in Figs. 2(a) and 2(b). Interestingly, for a given r, it is observed that the new bound not only is again tighter than the bound of Ref. [6] but also becomes sharper than the case shown in Fig. 2(b). On the other hand, the sharp decrement of bound (26), when the environment enters the non-Markovian regime, is more apparent. Therefore, it can be treated as a better witness of non-Markovinity in this way.

Moreover, qualitative changes of the QSL time bound when the dynamics becomes non-Markovian at the critical point ($\gamma_0 = \frac{\lambda}{2}$) are more evident for larger values of λ [for comparison, see Figs. 1(a) and 2(a)]. We also note that if we take the initial state $\rho_0 = |1\rangle\langle 1|$, the bound (26) for $\lambda = 50$ is the same as the corresponding bound of the Hilbert-Schmit norm (Mandelstamm-Tamm–type bound) obtained in Ref. [4].

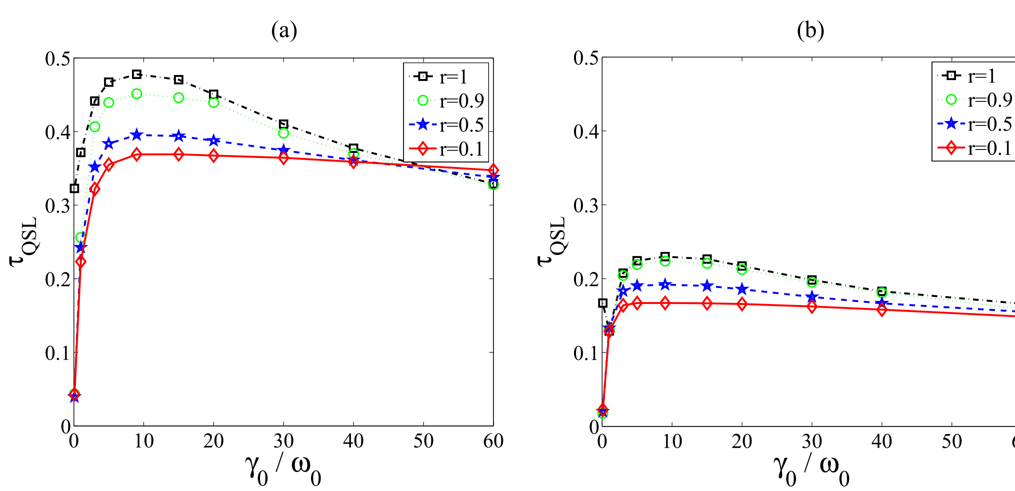

FIG. 2. QSL time, τ_{QSL}, as a function of the coupling strength γ_0 (in units of ω_0) for initial states (38) with different mixed coefficients $r = 0.1, 0.5, 0.9, 1$. (a) The bounds derived from (26) and (b) the bounds obtained from Ref. [6]. $\lambda = 20$ (in units of ω_0) and $\tau = 1$.

It should be noted that the interaction term of the Hamiltonian (31) corresponds to the amplitude damping channel for a qubit [18]. Therefore, if the system's initial state is taken as $\rho_0 = |0\rangle\langle 0|$ (the ground state of the two-level system without excitation), then according to Eq. (34), $\rho_\tau = \rho_0$. Therefore, the system with this initial state does not evolve in time, so its speed of evolution is zero (or $\tau_{QSL} = \infty$).

V. CONCLUSIONS

We have introduced an alternative fidelity which satisfies Jozsa's four axioms. Although the fidelity is not multiplicative, preliminary numerical calculations show that it is monotone and concave. Also, we have shown that the introduced fidelity is zero for any two orthogonal density matrices. Then by applying this fidelity as a distance measure between initial and time evolved final states of a quantum system, we have derived an improved bound for QSL time in open quantum systems. We have demonstrated that the improved bound leads to the well-known Mandelstamm-Tamm type bound for nonunitary dynamics in the case of initial pure states. One the other hand, we remember from Ref. [15] that a monotonic distance measure gives a tighter QSL time bound relative to the other nonmonotonic one, so in this regard it has been observed that the bound (26) derived from monotonic $\mathcal{F}$ (6) is tighter than the bound derived from nonmonotonic F_1 (2) in Ref. [6]. Finally, we have demonstrated that the improved bound decreases sharply in the non-Markovian regime.

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