Regular patterns in the information flow of local dephasing channels

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Consider local dephasing processes of a qubit that interacts with a structured reservoir of frequency modes or a thermal bath, with Ohmic-like spectral density (SD). It is known that non-Markovian evolution appears uniquely above a temperature-dependent critical value of the Ohmicity parameter and non-Markovianity can be induced by properly engineering the external environment. In the same scenario, we find regular patterns in the flow of quantum information: Alternate directions appear in correspondence with periodic intervals of the Ohmicity parameter α_0. The information flows back into the system over long times at zero temperature for $2+4n<\alpha_0<4+4n$, where $n=0,1,2,\ldots,$ and at nonvanishing temperatures for $3+4n<\alpha_0<5+4n$. Under special conditions, backflow of information appears also for nonvanishing, even natural values of the Ohmicity parameter, at zero temperature, and for odd natural values at nonvanishing temperatures. Otherwise, the long-time information flows into the environment. In the transition from vanishing to arbitrary nonvanishing temperature, the long-time backflow of information is stable for $3+4n<\alpha_0<4+4n$, while it is reversed for $2+4n<\alpha_0<3+4n$ and $4+4n<\alpha_0<5+4n$. The patterns in the information flow are not altered if the low-frequency Ohmic-like profiles of the SDs are perturbed with additional factors that consist in arbitrary powers of logarithmic forms. Consequently, the flow of information can be controlled, directed, and reversed over long times by engineering a wide variety of reservoirs that includes and continuously departs from the Ohmic-like structure at low frequencies. Non-Markovianity and recoherence appear according to the same rules along with the backflow of information.

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I. INTRODUCTION

In open quantum systems the loss, revival, or maintenance of quantum correlations is deeply related to the structure of the external environment [1,2]. Several studies have been performed on the connection among non-Markovian dynamics, flow of quantum information, and the nature of the coupling between the system and the environment [3–7]. Non-Markovianity was usually interpreted via memory effects and persistent interactions between system and environment. In recent years new definitions and measures of non-Markovianity have been proposed; see Refs. [8,9] for a review. Non-Markovianity can be explained in terms of the flow of quantum information, which is defined in various ways: via Fisher information [10], fidelity [11] or mutual information [12], to name a few. The trace-distance measure introduced in Ref. [13] estimates the relative distinguishability of two arbitrary quantum states. In Markovian processes this measure diminishes monotonically in time. This behavior can be seen as a loss of quantum information by the open system, while in non-Markovian dynamics the memory effects can be interpreted as a flow of quantum information from the external environment back into the open system.

The dephasing process of a qubit (two-level system) that interacts with a structured reservoir of frequency modes is a referential scenario for the study of phenomena such as dissipation, decoherence, recoherence, non-Markovianity, and information flow [1,2,14–17]. The open dynamics can be described in terms of the spectral density (SD) of the system. This fundamental function depends on the couplings between the open system and the frequency modes of the external environment [14,18–21]. This formalism is adopted also for the description of the open dynamics in fermionic environments [22–25]. For the system of a qubit that interacts with a bosonic reservoir the measures of non-Markovianity mentioned above suggest the same conditions for the appearance of non-Markovian dynamics [26] and are easily evaluated from the dephasing rate and the dephasing factor of the system [3–6,27]. Persistent negative values of the dephasing rate or, equivalently, a decreasing dephasing factor indicate backflow of information into the open system and witness non-Markovianity.

A remarkable analysis of the dependence of non-Markovianity on the low-frequency part of the environmental spectrum was performed in Ref. [3]. Nonconvexity properties that involve the SD provide conditions for the appearance of non-Markovian dynamics. For Ohmic-like SDs with exponential cutoff, the transition from Markovian to non-Markovian dynamics is found in correspondence with a critical value of the Ohmicity parameter. The temporary backflow of information and recoherence manifest uniquely for values of the Ohmicity parameters that are larger than such a critical value. This value is equal to 2 at zero temperature, grows monotonically by increasing the temperature, and becomes 3 at infinite

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temperature. See Ref. [3] for details. Great efforts have been made for the experimental observation of these phenomena: Simulations of open system dynamics have been performed with trapped ions [28] and transitions from Markovian to non-Markovian dynamics have been obtained in an all-optical experiments [29], to name a few.

As a continuation of the scenario described above, here we consider the local dephasing process of a qubit that interacts either with a structured reservoir of frequency modes or with a thermal bath. In addition to the Ohmic-like condition, the SDs under study include removable logarithmic singularities at low frequencies [30,31] and are arbitrarily shaped at higher frequencies. We study the decoherence and recoherence processes by evaluating the dephasing factor and we investigate the flow of quantum information by analyzing the dephasing rate. We also search for regular patterns in the direction of the flow of information that allow a full manipulation of the flow itself by engineering the low-frequency structure of the external environment.

The paper is organized as follows. Section II is devoted to the description of the model. In Sec. III, asymptotic coherence is related to integral properties of the SD. In Sec. IV, the SDs under study are defined in terms of removable logarithmic singularities. The decoherence and recoherence processes are studied in Sec. V, by analyzing the dephasing factor at both zero and nonvanishing temperature. Patterns in the flow of quantum information are shown in Sec. VI. A summary is given in Sec. VII. Details on the calculations are provided in the Appendix.

II. MODEL

The system of a qubit that interacts locally with a reservoir of frequency modes is described by the microscopic Hamiltonian [1–4]

$$H = \omega_0\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k \sigma_z(g_k a_k + g_k^* a_k^\dagger), \qquad (1)$$

in units where $\hbar = 1$. The transition frequency of the qubit is ω_0, while σ_z represents the z-component Pauli spin operator. The index k runs over the frequency modes. The parameter ω_k represents the frequency of the kth mode, while $b_k^\dagger$ and b_k are the raising and lowering operator, respectively, of the same mode. The coefficient g_k represents the coupling strength between the qubit and the kth frequency mode. The reduced density matrix $\rho(t)$ represents the mixed state of the qubit at the time t and is obtained by tracing the density matrix of the whole system at the time t over the Hilbert space of the external environment [1]. The model is exactly solvable [15–17].

Let the qubit be initially decoupled from the external environment that is represented by a structured reservoir of field modes or by a thermal bath. The reduced time evolution is described in the interaction picture by the master equation

$$\dot{\rho}(t) = \gamma(t)[\sigma_z\rho(t)\sigma_z - \rho(t)]. \qquad (2)$$

The function $\gamma(t)$ represents the dephasing rate and is related to the temperature of the thermal bath. At zero temperature, $T = 0$, the dephasing rate is labeled here as $\gamma_0(t)$ and reads

$$\gamma_0(t) = \int_0^\infty \frac{J(\omega)}{\omega}\sin(\omega t)d\omega. \qquad (3)$$

The function $J(\omega)$ represents the SD of the system and is defined in terms of the coupling constants g_k via the form $J(\omega) = \sum_k |g_k|^2\delta(\omega-\omega_k)$. If the external environment is initially in a thermal state, $T > 0$, the dephasing rate is represented here as $\gamma_T(t)$ and reads

$$\gamma_T(t) = \int_0^\infty \frac{J_T(\omega)}{\omega}d\omega, \qquad (4)$$

where the effective SD $J_T(\omega)$ is defined for every nonvanishing temperature as

$$J_T(\omega) = J(\omega)\coth\frac{\hbar\omega}{2k_BT} \qquad (5)$$

and k_B is the Boltzmann constant.

The quantum coherence between the states $|0\rangle$ and $|1\rangle$ of the qubit is described by the off-diagonal element $\rho_{0,1}(t)$ of the density matrix that undergoes the evolution [15–17]

$$\rho_{0,1}(t) = \rho_{1,0}^*(t) = \rho_{0,1}(0)\exp[-\Xi(t)]. \qquad (6)$$

The function $\Xi(t)$ represents the dephasing factor and depends on the temperature T of the thermal bath and on the coupling between the system and the environment. At zero temperature, $T = 0$, the dephasing factor is indicated here as $\Xi_0(t)$ and results in the form

$$\Xi_0(t) = \int_0^\infty J(\omega)\frac{1-\cos(\omega t)}{\omega^2}d\omega. \qquad (7)$$

If the external environment is initially in a thermal state, $T > 0$, the dephasing factor is represented here as $\Xi_T(t)$ and reads

$$\Xi_T(t) = \int_0^\infty \frac{J_T(\omega)}{\omega^2}[1-\cos(\omega t)]. \qquad (8)$$

Both for vanishing and nonvanishing temperature, the dephasing factor is related to the dephasing rate via the time derivative $\gamma_0(t) = \dot{\Xi}_0(t)$ and $\gamma_T(t) = \dot{\Xi}_T(t)$. According to Eq. (6), recoherence corresponds to negative values of the dephasing rate.

III. COHERENCE

The loss or persistence of coherence between the two energy eigenstates of the qubit depends on integral properties of the SDs. At zero temperature, $T = 0$, coherence is not entirely lost over long times if the second negative moment of the SD is finite,

$$\int_0^\infty \frac{J(\omega)}{\omega^2}d\omega < \infty. \qquad (9)$$

This quantity is also referred to as the total Huang-Rys factor in the framework of optical spectroscopy [19–21] and is relevant for the appearance of coherence. In fact, under the condition (9) the coherence term shows long-time persistence of residual coherence

$$\rho_{0,1}(\infty) = \rho_{0,1}(0)\exp\left[-\int_0^\infty \frac{J(\omega)}{\omega^2}d\omega\right]. \qquad (10)$$

If the external environment consists in a thermal bath, $T > 0$, and the condition

$$\int_0^\infty \frac{J_T(\omega)}{\omega^2} d\omega < \infty \tag{11}$$

holds, the coherence term tends over long times to the nonvanishing asymptotic value

$$\rho_{0,1}(\infty) = \rho_{0,1}(0) \exp\left[-\int_0^\infty \frac{J_T(\omega)}{\omega^2} d\omega\right]. \tag{12}$$

The maximum modulus of the ratio between asymptotic and initial coherence is obtained at zero temperature, $T = 0$, from Eq. (10).

Residual coherence persists over long times if the dephasing factor does not diverge asymptotically, while coherence is fully lost if the dephasing factor diverges [see Eq. (6)]. Consequently, the dependence of coherence on the structure of the SD can be analyzed via the dephasing factor itself. We intend to study the short- and long-time behavior of the dephasing factor for a large variety of SDs in conditions where the second negative moment is either finite, Eq. (9), or infinite,

$$\int_0^\infty \frac{J(\omega)}{\omega^2} d\omega = \infty. \tag{13}$$

Similarly, at nonvanishing temperatures, $T > 0$, we consider both the condition (11) and the following one:

$$\int_0^\infty \frac{J_T(\omega)}{\omega^2} d\omega = \infty. \tag{14}$$

Details on the structure of the SDs under study are provided below.

IV. SPECTRAL DENSITIES WITH REMOVABLE LOGARITHMIC SINGULARITIES

The fast development of quantum technologies allows the engineering of the most various environments. According to the remarkable analysis performed in Refs. [32,33], an impurity that is trapped in a double-well potential and is surrounded by a cold gas reproduces, under suitable conditions, a qubit that interacts with an Ohmic-like environment. The Ohmicity parameter increases by enhancing the scattering length that is related to the boson-boson coupling [33]. In the case where the gas is free and one dimensional, the SD changes from sub-Ohmic to Ohmic and to super-Ohmic by increasing the scattering length. In the two-dimensional noninteracting condition the spectrum is Ohmic and the super-Ohmic regime is obtained if the magnitude of the interaction decreases. The SD is super-Ohmic in the noninteracting condition if the gas is three dimensional. We refer to [33] for details.

In light of the above observation we focus on SDs that include the Ohmic-like condition at low frequencies and are arbitrarily shaped at higher frequency. We intend to analyze the feature of the open dynamics, the flow of quantum information, non-Markovianity, and recoherence of the qubit. We evaluate the accuracy of the results obtained for the experimentally feasible Ohmic-like SDs by perturbing the power laws of the Ohmic-like profiles with additional factors that are represented by arbitrary powers of logarithmic forms. Positive (negative) values of the first logarithmic power enhance (reduce) the power-law profiles. In this way, we consider a wide variety of SDs that cover and continuously depart from the Ohmic-like condition [30].

For the sake of convenience, the SDs $J(\omega)$ are described via the dimensionless auxiliary function $\Omega(\nu)$. This function is defined for every $\nu \geqslant 0$ by the scaling property $J(\nu\Delta)/\Delta = \Omega(\nu)$ in terms of a general scale frequency Δ of the system. At nonvanishing temperatures the auxiliary function $\Omega_T(\nu)$ of the effective SD $J_T(\omega)$ is $\Omega_T(\nu) = \Omega(\nu)\coth(\hbar\Delta\nu/2k_B T)$. In this way, the action of the thermal bath is represented by a transformed SD. We consider two general classes of SDs, which are defined below.

A. Spectral densities with natural powers of logarithmic forms

The first class of SDs under study is defined by auxiliary functions $\Omega(\nu)$ that are continuous for every $\nu > 0$ and exhibit the following asymptotic behavior [34] as $\nu \to 0^+$:

$$\Omega(\nu) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{n_j} c_{j,k} \nu^{\alpha_j} (-\ln\nu)^k, \tag{15}$$

where $0 \leqslant n_j < \infty$, $\alpha_j < \alpha_{j+1}$ for every $j \geqslant 0$, and $\alpha_j \uparrow +\infty$ as $j \to +\infty$. Furthermore, we consider $\alpha_0 \geqslant 0$, and $n_0 = 0$ if $\alpha_0 = 0$. The power α_0 is referred to as the Ohmicity parameter [2,14]. In fact, if $n_0 = 0$, the corresponding SDs are super-Ohmic for $\alpha_0 > 1$, Ohmic for $\alpha_0 = 1$, and sub-Ohmic for $1 > \alpha_0 > 0$, as $\omega \to 0^+$. The singularity in $\nu = 0$ is removable by defining $\Omega(0)$ as the finite limit of $\Omega(\nu)$ as $\nu \to 0^+$. Notice that Eq. (15) describes a large variety of low-frequency asymptotic forms that include exponential and stretched exponential functions, power laws, and natural powers of logarithmic forms. The summability of the SD is guaranteed by the constraint $\Omega(\nu) = O(\nu^{-1-\chi_0})$ as $\nu \to +\infty$, where $\chi_0 > 0$. Additionally, the Mellin transforms $\hat{\Omega}(s)$ and $\hat{\Omega}_T(s)$ of the auxiliary functions $\Omega(\nu)$ and $\Omega_T(\nu)$, and the meromorphic continuations [34,35] are required to decay sufficiently fast as $|\operatorname{Im} s| \to +\infty$. See the Appendix for details.

B. Spectral densities with arbitrary powers of logarithmic forms

In light of the asymptotic analysis performed in Refs. [35,36], the second class of SDs under study is described by auxiliary functions with the following asymptotic expansion as $\nu \to 0^+$:

$$\Omega(\nu) \sim \sum_{j=0}^{\infty} w_j \nu^{\alpha_j} (-\ln\nu)^{\beta_j}. \tag{16}$$

The powers β_j are real valued, arbitrarily positive or negative, or vanishing, while $\alpha_0 > 0$. The logarithmic singularity in $\nu = 0$ is removed by setting $\Omega(0) = 0$. Let the parameter $\bar{n}$ be the least natural number such that $\alpha_{k-1} + 1 \leqslant \bar{n} < \alpha_k + 1$, where the index k is a nonvanishing natural number. The function $\Omega^{(\bar{n})}(\nu)$ is required to be continuous on the interval $(0,\infty)$. The integral $\int_0^\infty \Omega(\nu)\exp[-\imath\xi\nu]d\nu$ must converge uniformly for all sufficiently large values of the variable ξ and the integral $\int \Omega^{(\bar{n})}(\nu)\exp[-\imath\xi\nu]d\nu$ has to converge at $\nu = +\infty$ uniformly for all sufficiently large values of the variable ξ. The auxiliary function is required to be differentiable k

times and the asymptotic expansion at $\nu \to 0^+$,

$$\Omega^{(k)}(\nu) \sim \sum_{j=0}^{\infty} w_j \frac{d^k}{d\nu^k}[\nu^{\alpha_j}(-\ln \nu)^{\beta_j}],$$

is required to hold for every $k = 0,1,\dots,\bar{n}$. Furthermore, for every $k = 0,\dots,\bar{n}-1$, the function $\Omega^{(k)}(\nu)$ has to vanish as $\nu \to +\infty$. We refer to [35,36] for details.

If compared to the first class of SDs, introduced in Sec. IV A, the second class is characterized by more constraints but includes arbitrary positive or negative, or vanishing, real-valued powers of logarithmic forms. In both the classes under study the auxiliary functions $\Omega(\nu)$ are non-negative and summable, due to physical grounds, and, apart from the constraints reported above, arbitrarily shaped at high frequencies [30].

V. DEPHASING FACTOR

We start the analysis of the dephasing factor by considering a structured reservoir of frequency modes as the external environment. The SDs under study belong to the first class (Sec. IV A). Over short times, $t \ll 1/\Delta$, the dephasing factor increases quadratically in time

$$\Xi_0(t) \sim \frac{l_0}{2} t^2, \tag{17}$$

where $l_0 = \int_0^\infty J(\omega)d\omega$. This behavior is independent of the low- or high-frequency structure of the SD. Instead, the evolution of the dephasing factor over long times, $t \gg 1/\Delta$, is various and is determined by the low-frequency structure of the SD, given by Eq. (15). If $\alpha_0 = n_0 = 0$ the dephasing factor grows linearly in time for $t \gg 1/\Delta$,

$$\Xi_0(t) \sim \frac{\pi c_{0,0}}{2} \Delta t. \tag{18}$$

If $0 < \alpha_0 < 1$ we find for $t \gg 1/\Delta$ the divergent behavior

$$\Xi_0(t) \sim c_{0,n_0} r_1 (\Delta t)^{1-\alpha_0} \ln^{n_0}(\Delta t), \tag{19}$$

where $r_1 = \sin(\pi\alpha_0/2)\Gamma(\alpha_0)/(1-\alpha_0)$. The above expression provides power laws for $n_0 = 0$,

$$\Xi_0(t) \sim c_{0,n_0} r_1 (\Delta t)^{1-\alpha_0}. \tag{20}$$

If $\alpha_0 = 1$ we obtain over long times, $t \gg 1/\Delta$, the divergent logarithmic form

$$\Xi_0(t) \sim \frac{c_{0,n_0}}{n_0+1} \ln^{n_0+1}(\Delta t). \tag{21}$$

If $\alpha_0 > 1$ the dephasing factor tends to the finite asymptotic value

$$\Xi_0(\infty) = \int_0^\infty \frac{J(\omega)}{\omega^2} d\omega. \tag{22}$$

If α_0 is not an even natural number, the relaxations involve logarithmic forms

$$\Xi_0(t) \sim \Xi_0(\infty) + c_{0,n_0} r_1 (\Delta t)^{1-\alpha_0} \ln^{n_0}(\Delta t). \tag{23}$$

The above expression turns into pure inverse power laws for $n_0 = 0$,

$$\Xi_0(t) \sim \Xi_0(\infty) + c_{0,n_0} r_1 (\Delta t)^{1-\alpha_0}. \tag{24}$$

If $\alpha_0 > 1$ and $\alpha_0 = 2m_0$, where m_0 and n_0 are nonvanishing natural numbers, we find

$$\Xi_0(t) \sim \Xi_0(\infty) + c_{0,n_0} r_1' (\Delta t)^{1-2m_0} \ln^{n_0-1}(\Delta t), \tag{25}$$

where $r_1' = \pi(-1)^{m_0} n_0 (2m_0-2)!/2$. The above relaxations become inverse power laws for $n_0 = 1$,

$$\Xi_0(t) \sim \Xi_0(\infty) + c_{0,n_0} r_1' (\Delta t)^{1-2m_0}. \tag{26}$$

If α_0 is an even natural number and n_0 vanishes, consider the least nonvanishing index k_0 such that either α_{k_0} does not take even natural values or $\alpha_{k_0} = 2m_{k_0}$, where the natural numbers m_{k_0} and n_{k_0} do not vanish. The function $\Xi_0(t)$ is obtained in the former case from Eqs. (23) and (24) by replacing the power α_0 with α_{k_0} and n_0 with n_{k_0}, or in the latter case from Eqs. (25) and (26) by replacing the power m_0 with m_{k_0} and n_0 with n_{k_0}. We consider SDs such that the index k_0 exists with the required properties.

A. The dephasing factor at zero temperature for the second class of spectral densities

At this stage we focus on SDs that belong to the second class (Sec. IV B) and that are characterized by a finite negative second moment [Eq. (9)]. This condition requires that the Ohmicity parameter α_0 is larger than unity, $\alpha_0 > 1$. Over short times, $t \ll 1/\Delta$, the dephasing factor grows quadratically in time according to Eq. (17), independently of the low- or high-frequency structure of the SD. Over long times, $t \gg 1/\Delta$, the dephasing factor relaxes to the asymptotic value $\Xi_0(\infty)$ according to arbitrarily positive or negative, or vanishing powers of logarithmic forms

$$\Xi_0(t) \sim \Xi_0(\infty) + w_0 (\Delta t)^{1-\alpha_0} [r_1 \ln^{\beta_0}(\Delta t) + \bar{r}_1 \ln^{\beta_0-1}(\Delta t)], \tag{27}$$

with $\bar{r}_1 = \beta_0 \sin(\pi\alpha_0/2)[\Gamma^{(1)}(\alpha_0-1) + \pi\Gamma(\alpha_0-1)/2]$. If the Ohmicity parameter α_0 is not an even natural number, the dominant part of the above relaxation is

$$\Xi_0(t) \sim \Xi_0(\infty) + w_0 r_1 (\Delta t)^{1-\alpha_0} \ln^{\beta_0}(\Delta t) \tag{28}$$

and provides the inverse power laws

$$\Xi_0(t) \sim \Xi_0(\infty) + w_0 (\Delta t)^{1-\alpha_0} \tag{29}$$

for $\beta_0 = 0$. If the Ohmicity parameter α_0 takes even natural values, Eq. (27) gives

$$\Xi_0(t) \sim \Xi_0(\infty) + w_0 \bar{r}_1 (\Delta t)^{1-\alpha_0} \ln^{\beta_0-1}(\Delta t) \tag{30}$$

and turns into the inverse power laws

$$\Xi_0(t) \sim \Xi_0(\infty) + w_0 \bar{r}_1 (\Delta t)^{1-\alpha_0} \tag{31}$$

for $\beta_0 = 1$. By comparing the above relaxations with those obtained for the first class of SDs under study, we observe that, at zero temperature, the long-time evolution of the dephasing factor exhibits for both the classes the same dependence on the low-frequency structure of the SD.

B. Thermal bath

Let the external environment consist of a thermal bath, $T > 0$. If the SDs belong to the first class under study

(Sec. IV A) with $\alpha_0 > 0$, the dephasing factor is approximated again by a quadratic function of time for $t \ll 1/\Delta$,

$$\Xi_T(t) \sim \frac{l_T}{2} t^2, \tag{32}$$

where $l_T = \int_0^\infty J_T(\omega) d\omega$. This behavior is again independent of the low- or high-frequency structure of the SD. Instead, the evolution of the dephasing factor over long times, $t \gg 1/\Delta$, exhibits various behaviors dependent on the low-frequency structure of the SD. If $0 < \alpha_0 < 2$ and $\alpha_0 \neq 1$ the dephasing factor diverges for $t \gg 1/\Delta$ as

$$\Xi_T(t) \sim c_{0,n_0} r_T (\Delta t)^{2-\alpha_0} \ln^{n_0}(\Delta t), \tag{33}$$

where $r_T = 2k_B T \cos(\pi\alpha_0/2)\Gamma(\alpha_0 - 2)/\hbar\Delta$. Again, power laws appear under the above conditions for $n_0 = 0$,

$$\Xi_T(t) \sim c_{0,n_0} r_T (\Delta t)^{2-\alpha_0}. \tag{34}$$

If $\alpha_0 = 1$ the dephasing factor diverges for $t \gg 1/\Delta$ as

$$\Xi_T(t) \sim c_{0,n_0} r'_T \Delta t \ln^{n_0}(\Delta t), \tag{35}$$

where $r'_T = \pi k_B T/\hbar\Delta$. The divergence becomes linear in time for $\alpha_0 = 1$ and $n_0 = 0$,

$$\Xi_T(t) \sim c_{0,n_0} r'_T \Delta t. \tag{36}$$

If $\alpha_0 = 2$ the dephasing factor grows for $t \gg 1/\Delta$ according to natural powers of logarithmic forms

$$\Xi_T(t) \sim c_{0,n_0} r''_T \ln^{n_0+1}(\Delta t), \tag{37}$$

where $r''_T = 2k_B T/\hbar\Delta(n_0 + 1)$. If $\alpha_0 > 2$ the dephasing factor converges for $t \gg 1/\Delta$ to the asymptotic value

$$\Xi_T(\infty) = \int_0^\infty \frac{J_T(\omega)}{\omega^2} d\omega. \tag{38}$$

If α_0 is not an odd natural number and $\alpha_0 > 2$, the relaxations to the asymptotic value are

$$\Xi_T(t) \sim \Xi_T(\infty) + c_{0,n_0} r_T (\Delta t)^{2-\alpha_0} \ln^{n_0}(\Delta t), \tag{39}$$

and turn into inverse power laws for $n_0 = 0$,

$$\Xi_T(t) \sim \Xi_T(\infty) + c_{0,n_0} r_T (\Delta t)^{2-\alpha_0}. \tag{40}$$

If $\alpha_0 = 2m_1 + 1$, where m_1 and n_0 are nonvanishing natural numbers, we find

$$\Xi_T(t) \sim \Xi_T(\infty) + c_{0,n_0} r'''_T (\Delta t)^{1-2m_1} \ln^{n_0-1}(\Delta t), \tag{41}$$

where $r'''_T = \pi(-1)^{m_1} k_B T n_0 (2m_1 - 2)!/\hbar\Delta$. The above form becomes an inverse power law for $n_0 = 1$,

$$\Xi_T(t) \sim \Xi_T(\infty) + c_{0,n_0} r'''_T (\Delta t)^{1-2m_1}. \tag{42}$$

If α_0 is an odd natural number and n_0 vanishes, consider the least nonvanishing index k_1 such that either α_{k_1} does not take odd natural values or $\alpha_{k_1} = 2m_{k_1} + 1$, where the natural numbers m_{k_1} and n_{k_1} do not vanish. The function $\Xi_T(t)$ is obtained in the former case from Eqs. (39) and (40) by replacing the power α_0 with α_{k_1} and n_0 with n_{k_1}, or in the latter case from Eqs. (41) and (42) by replacing the power m_1 with m_{k_1} and n_0 with n_{k_1}. We consider SDs such that the index k_1 exists with the required properties.

C. The dephasing factor at nonvanishing temperatures for the second class of spectral densities

We consider SDs such that the auxiliary functions $\Omega_T(\nu)$ belong to the second class under study (Sec. IV B) and exhibit a finite second negative moment [Eq. (11)]. This constraint requires $\alpha_0 > 2$. Over short times, $t \ll 1/\Delta$, the dephasing factor grows quadratically in time according to Eq. (32), independently of the low- or high-frequency structure of the SD. As far as the long-time evolution is concerned, a variety of logarithmic relaxations of the dephasing factor to the asymptotic value $\Xi_T(\infty)$ are obtained for $t \gg 1/\Delta$,

$$\Xi_T(t) \sim \Xi_T(\infty) + w_0(\Delta t)^{2-\alpha_0}[r_T \ln^{\beta_0}(\Delta t) + \bar{r}_T \ln^{\beta_0-1}(\Delta t)], \tag{43}$$

where

$$\bar{r}_T = \frac{\beta_0 k_B T}{\hbar\Delta}\left[\pi \sin\left(\frac{\pi\alpha_0}{2}\right)\Gamma(\alpha_0 - 2) - 2\cos\left(\frac{\pi\alpha_0}{2}\right)\Gamma^{(1)}(\alpha_0 - 2)\right].$$

If the Ohmicity parameter α_0 is not an odd natural number, the dominant part of the above relaxation is

$$\Xi_T(t) \sim \Xi_T(\infty) + w_0 r_T (\Delta t)^{2-\alpha_0} \ln^{\beta_0}(\Delta t). \tag{44}$$

If the Ohmicity parameter α_0 takes odd natural values, Eq. (43) gives for $t \gg 1/\Delta$ the asymptotic form

$$\Xi_T(t) \sim \Xi_T(\infty) + w_0 \bar{r}_1 (\Delta t)^{2-\alpha_0} \ln^{\beta_0-1}(\Delta t) \tag{45}$$

and provides the power laws

$$\Xi_T(t) \sim \Xi_T(\infty) + w_0 \bar{r}_1 (\Delta t)^{2-\alpha_0} \tag{46}$$

for $\beta_0 = 1$.

The comparison between the above relaxations and those obtained for the first class of SDs suggests that, at nonvanishing temperatures, the long-time evolution of the dephasing factor exhibits for both classes under study the same dependence on the low-frequency structure of the SD.

Numerical computations have been performed by adopting the toy SDs

$$J_1(\omega) = q_1\Delta\left(\frac{\omega}{\Delta}\right)^\alpha \exp\left(-\lambda\frac{\omega}{\Delta}\right)\ln^2\frac{\omega}{\Delta}, \tag{47}$$

$$J_2(\omega) = q_2\Theta\left(\frac{\omega}{\Delta}\left(\frac{1}{2} - \frac{\omega}{\Delta}\right)\right)\Delta\left(\frac{\omega}{\Delta}\right)^\alpha \ln^\beta\frac{\omega}{\Delta}, \tag{48}$$

where $\Theta(\nu)$ represents the Heaviside step function. The first example of SDs [Eq. (47)] is tailored at low frequencies by a quadratic logarithmic term and power laws, exhibits an exponential cutoff at high frequencies, and belongs to the first class under study (Sec. IV A). The second example of SDs [Eq. (48)] is tailored at low frequencies by arbitrary powers of logarithmic forms and power laws, with finite support $[0, \Delta/2]$, and belongs to the second class under study (Sec. IV B). Plots of the coherence term are drawn in Fig. 1. The persistence of asymptotic coherence is confirmed by the long-time nonvanishing behavior of the curves. Numerical analysis of the dephasing factor are displayed in Figs. 2–4. The short-time quadratic growth is confirmed by the parallel asymptotic lines, with slope 2, appearing in Fig. 2. The

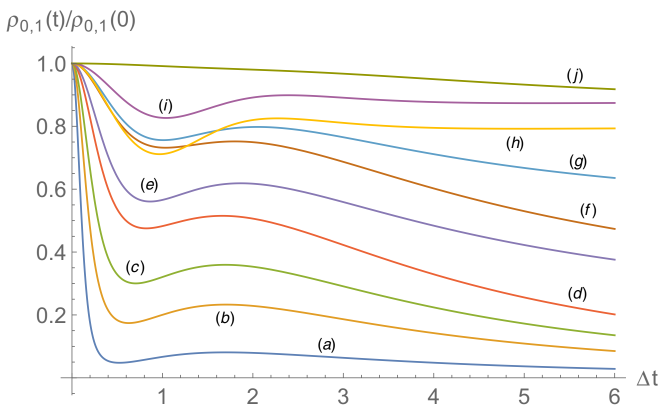

FIG. 1. Quantity $\rho_{0,1}(t)/\rho_{0,1}(0)$ versus Δt for $0 \leqslant \Delta t \leqslant 6$, SDs given by Eq. (47), $q_1 = 1$, and different values of the parameters α and λ. Curve (a) corresponds to $\alpha = 1.6$ and $\lambda = 0.3$, (b) corresponds to $\alpha = 1.6$ and $\lambda = 0.4$, (c) corresponds to $\alpha = 1.6$ and $\lambda = 0.48$, (d) corresponds to $\alpha = 1.6$ and $\lambda = 0.6$, (e) corresponds to $\alpha = 2$ and $\lambda = 0.8$, (f) corresponds to $\alpha = 2$ and $\lambda = 1$, (g) corresponds to $\alpha = 2.5$ and $\lambda = 1.2$, (h) corresponds to $\alpha = 5$ and $\lambda = 2$, (i) corresponds to $\alpha = 5$ and $\lambda = 2.2$, and (j) corresponds to $\alpha = 3$ and $\lambda = 3$.

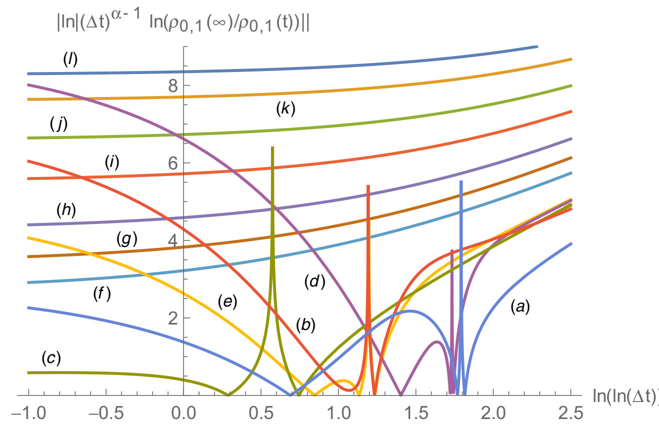

FIG. 3. Quantity $|\ln|(\Delta t)^{\alpha-1}\ln[\rho_{0,1}(\infty)/\rho_{0,1}(t)]||$ versus $\ln[\ln(\Delta t)]$ for $\exp(1/e) \leqslant \Delta t \leqslant \exp[\exp(2.5)]$, SDs given by Eq. (47), $q_1 = 1$, and different values of the parameters α and λ. Curve (a) corresponds to $\alpha = 2.4$ and $\lambda = 50$, curve (b) corresponds to $\alpha = 3.8$ and $\lambda = 30$, (c) corresponds to $\alpha = 3$ and $\lambda = 0.01$, curve (d) corresponds to $\alpha = 3.2$ and $\lambda = 250$, curve (e) corresponds to $\alpha = 3.3$ and $\lambda = 20$, curve (f) corresponds to $\alpha = 1.5$ and $\lambda = 1$, curve (g) corresponds to $\alpha = 1.4$ and $\lambda = 0.1$, curve (h) corresponds to $\alpha = 1.3$ and $\lambda = 1$, curve (i) corresponds to $\alpha = 1.2$ and $\lambda = 0.1$, (j) corresponds to $\alpha = 1.14$ and $\lambda = 0.001$, curve (k) corresponds to $\alpha = 1.1$ and $\lambda = 0.1$, and curve (l) corresponds to $\alpha = 1.08$ and $\lambda = 0.001$.

long-time logarithmic relaxations result in the asymptotic lines that are shown in Figs. 3 and 4. The former correspond to the quadratic logarithmic term of the first computed SDs. The latter refer to various noninteger logarithmic powers of the second computed SDs.

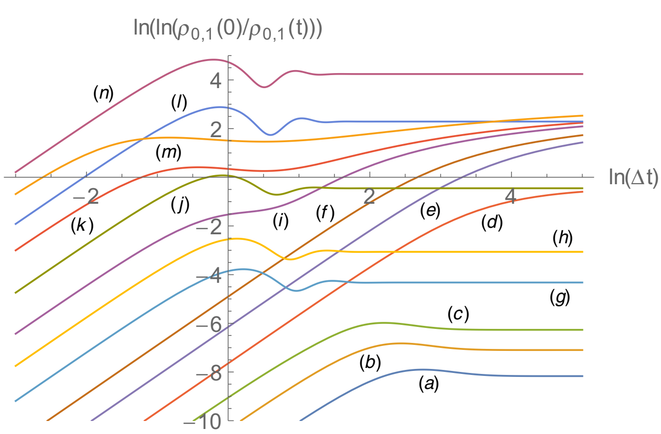

FIG. 2. Quantity $\ln\{\ln[\rho_{0,1}(0)/\rho_{0,1}(t)]\}$ versus $\ln(\Delta t)$ for $\exp(-3) \leqslant \Delta t \leqslant \exp(5)$, SDs given by Eq. (47), $q_1 = 1$, and different values of the parameters α and λ. Curve (a) corresponds to $\alpha = 5$ and $\lambda = 15$, curve (b) corresponds to $\alpha = 5$ and $\lambda = 10$, curve (c) corresponds to $\alpha = 5$ and $\lambda = 7$, curve (d) corresponds to $\alpha = 2$ and $\lambda = 22$, curve (e) corresponds to $\alpha = 1.5$ and $\lambda = 20$, curve (f) corresponds to $\alpha = 1.5$ and $\lambda = 9$, curve (g) corresponds to $\alpha = 10$ and $\lambda = 4.8$, curve (h) corresponds to $\alpha = 10$ and $\lambda = 4.3$, curve (i) corresponds to $\alpha = 1.5$ and $\lambda = 1$, curve (j) corresponds to $\alpha = 10$ and $\lambda = 3.4$, curve (k) corresponds to $\alpha = 1.5$ and $\lambda = 0.4$, curve (l) corresponds to $\alpha = 20$ and $\lambda = 6.1$, curve (m) corresponds to $\alpha = 1.5$ and $\lambda = 0.2$, and curve (n) corresponds to $\alpha = 20$ and $\lambda = 5.55$.

VI. REGULAR PATTERNS IN THE LONG-TIME INFORMATION FLOW

For the system under study the trace distance measure of non-Markovianity that is defined in Refs. [4,13] takes a simple expression in terms of the dephasing rate and dephasing factor. In this case the non-Markovianity measure results in the

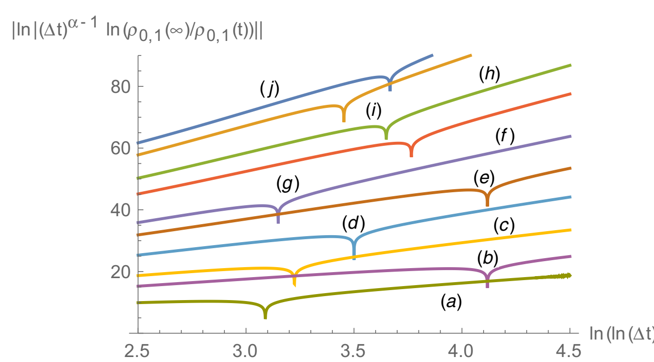

FIG. 4. Quantity $|\ln|(\Delta t)^{\alpha-1}\ln[\rho_{0,1}(\infty)/\rho_{0,1}(t)]||$ versus $\ln[\ln(\Delta t)]$ for $\exp[\exp(2.5)] \leqslant \Delta t \leqslant \exp[\exp(4.5)]$, SDs given by Eq. (48), $q_2 = 1$, different values of the parameter α, and noninteger values of the logarithmic power β. Curve (a) corresponds to $\alpha = 2.2$ and $\beta = 4.5$, curve (b) corresponds to $\alpha = 2.1$ and $\beta = 6.2$, curve (c) corresponds to $\alpha = 2.3$ and $\beta = 7.7$, curve (d) corresponds to $\alpha = 2.3$ and $\beta = 10.1$, curve (e) corresponds to $\alpha = 2.2$ and $\beta = 12.4$, curve (f) corresponds to $\alpha = 2.6$ and $\beta = 14.3$, curve (g) corresponds to $\alpha = 2.4$ and $\beta = 17.5$, curve (h) corresponds to $\alpha = 2.5$ and $\beta = 19.5$, curve (i) corresponds to $\alpha = 2.7$ and $\beta = 22.5$, and curve (j) corresponds to $\alpha = 2.6$ and $\beta = 23.8$.

form [3,27,37]

$$\mathcal{N}=\int_{\gamma(t)<0}|\gamma(t)|e^{-\Xi(t)}dt. \tag{49}$$

The open dynamics is Markovian if the dephasing rate is non-negative. On the contrary, persistent negative values of the dephasing rate are a source of non-Markovianity and are interpreted as a flow of information from the environment back into the system. Consequently, at zero temperature, $T=0$, the open dynamics is Markovian if the function $J(\omega)/\omega$ is nonincreasing. If the SD is differentiable this condition reads

$$J'(\omega)\leqslant\frac{J(\omega)}{\omega} \tag{50}$$

for every $\omega>0$. At nonvanishing temperatures, $T>0$, the open dynamics is Markovian if the function $J_T(\omega)/\omega$ is nonincreasing. If the SD is differentiable, this requirement results in the constraint

$$J'(\omega)\leqslant\left(\frac{1}{\omega}+\frac{\hbar}{k_BT}\operatorname{cosech}\frac{\hbar\omega}{k_BT}\right)J(\omega) \tag{51}$$

for every $\omega>0$. Consequently, if the open dynamics is non-Markovian, the function $J(\omega)/\omega$, for $T=0$, or the function $J_T(\omega)/\omega$, for $T>0$, is increasing in an interval of frequencies of nonvanishing measure at least. In this way, a nonvanishing contribution is provided to the integral of Eq. (49). Let the SD be differentiable for every $\omega>0$. If the open dynamics is non-Markovian the constraint (50), for $T=0$, or (51), for $T>0$, is not fulfilled for one value of the frequency at least.

In general, the asymptotic behavior of the dephasing rate depends on integral properties of the SDs. Over long times, $t\gg 1/\Delta$, the dephasing rate vanishes at zero temperature, $T=0$, if the first negative moment of the SD is finite,

$$\int_0^\infty\frac{J(\omega)}{\omega}d\omega<\infty. \tag{52}$$

This quantity is also referred to as the reorganization energy in the framework of optical spectroscopy [19,21]. Similarly, at nonvanishing temperatures, $T>0$, the dephasing rate vanishes for $t\gg 1/\Delta$ if

$$\int_0^\infty\frac{J_T(\omega)}{\omega}d\omega<\infty. \tag{53}$$

Let the external environment consist of a reservoir of frequency modes. If the SDs belong to the first class under study (Sec. IV A), the dephasing rate increases linearly over short times, $t\ll 1/\Delta$,

$$\gamma_0(t)\sim l_0 t. \tag{54}$$

This behavior is independent of the low- or high-frequency structure of the SD. Over long times, $t\gg 1/\Delta$, different forms of relaxations are obtained dependent on the low-frequency structure of the SD, given by Eq. (15). If $\alpha_0=n_0=0$ the dephasing rate tends to the nonvanishing asymptotic value for $t\gg 1/\Delta$,

$$\gamma_0(t)\sim\frac{\pi c_{0,0}\Delta}{2}. \tag{55}$$

If $\alpha_0>0$ and α_0 is not an even natural number, the dephasing rate vanishes for $t\gg 1/\Delta$ according to the relaxations

$$\gamma_0(t)\sim c_{0,n_0}g_1(\Delta t)^{-\alpha_0}\ln^{n_0}(\Delta t), \tag{56}$$

which become inverse power laws for $n_0=0$,

$$\gamma_0(t)\sim c_{0,n_0}g_1(\Delta t)^{-\alpha_0}, \tag{57}$$

where $g_1=\Delta\sin(\pi\alpha_0/2)\Gamma(\alpha_0)$. Notice that Eq. (55) is recovered from Eq. (57) as $\alpha_0\to 0^+$. If $\alpha_0=2m_2$ where m_2 and n_0 are nonvanishing natural numbers, the dephasing rate vanishes for $t\gg 1/\Delta$ as

$$\gamma_0(t)\sim c_{0,n_0}g_1'(\Delta t)^{-2m_2}\ln^{n_0-1}(\Delta t), \tag{58}$$

where $g_1'=\pi(-1)^{m_2+1}n_0(2m_2-1)!\Delta/2$. The above relaxations become inverse power laws for $n_0=1$,

$$\gamma_0(t)\sim c_{0,n_0}g_1'(\Delta t)^{-2m_2}. \tag{59}$$

If α_0 is an even natural number and n_0 vanishes, consider the least nonvanishing index k_2 such that either α_{k_2} does not take even natural values or $\alpha_{k_2}=2m_{k_2}$, where the natural numbers m_{k_2} and n_{k_2} do not vanish. The function $\gamma_0(t)$ is obtained in the former case from Eqs. (56) and (57) by replacing the power α_0 with α_{k_2} and n_0 with n_{k_2}, or in the latter case from Eqs. (58) and (59) by replacing the power m_2 with m_{k_2} and n_0 with n_{k_2}. We consider SDs such that the index k_2 exists with the required properties.

A. The dephasing rate at zero temperature for the second class of spectral densities

At this stage we analyze the time evolution of the dephasing rate at zero temperature by considering the second class of SDs under study (Sec. IV B). Over short times, $t\ll 1/\Delta$, the dephasing rate increases linearly according to Eq. (54). This behavior is independent of the low- or high-frequency structure of the SD. Over long times, $t\gg 1/\Delta$, we find various forms of logarithmic relaxations

$$\gamma_0(t)\sim\frac{w_0}{(\Delta t)^{\alpha_0}}[g_1\ln^{\beta_0}(\Delta t)-\bar{g}_1\ln^{\beta_0-1}(\Delta t)], \tag{60}$$

where

$$\bar{g}_1=\beta_0\Delta\left[\frac{\pi}{2}\cos\left(\frac{\pi\alpha_0}{2}\right)\Gamma(\alpha_0)+\sin\left(\frac{\pi\alpha_0}{2}\right)\Gamma^{(1)}(\alpha_0)\right].$$

If the Ohmicity parameter α_0 does not take even natural values, the dominant part of the above asymptotic form is

$$\gamma_0(t)\sim w_0g_1(\Delta t)^{-\alpha_0}\ln^{\beta_0}(\Delta t) \tag{61}$$

and becomes the power law

$$\gamma_0(t)\sim w_0g_1(\Delta t)^{-\alpha_0} \tag{62}$$

for $\beta_0=0$. If the Ohmicity parameter α_0 is an even natural number, Eq. (60) gives

$$\gamma_0(t)\sim -w_0\bar{g}_1(\Delta t)^{-\alpha_0}\ln^{\beta_0-1}(\Delta t) \tag{63}$$

and becomes the power law

$$\gamma_0(t)\sim -w_0\bar{g}_1(\Delta t)^{-\alpha_0} \tag{64}$$

for $\beta_0=1$. Notice the full similarity between the above expressions and those obtained for the first class of SDs under study, at zero temperature.

According to the above analysis, at zero temperature and for the first class of SDs under study, the information flows into the environment over short times, $t \ll 1/\Delta$. Over long times, $t \gg 1/\Delta$, the information flows back into the system for the values of the Ohmicity parameter $2+4n < \alpha_0 < 4+4n$, where $n = 0,1,2,\ldots$. Backflow of information is obtained also for every nonvanishing even natural value of the Ohmicity parameter if $n_0 = 0$ and $2+4n < \alpha_{k_2} \leqslant 4+4n$, where n takes natural values. Additionally, if $n_0 > 0$, long-time backflow appears for every even natural value $\alpha_0 = 4l_1$, where l_1 is a nonvanishing natural number. Along with the backflow of information, the long-time dynamics is non-Markovian, the modulus of the coherence term increases up to the nonvanishing asymptotic value, and recoherence is observed. If the Ohmicity parameter differs from the values reported above, the long-time information flows into the environment, the long-time dynamics is Markovian, and the modulus of the coherence term decreases down to the asymptotic value, as suggested by the long-time behavior of the dephasing factor. If compared to the initial condition, coherence is partially lost for $\alpha_0 > 1$ and is fully lost if $0 \leqslant \alpha_0 \leqslant 1$.

For the second class of SDs under study, at zero temperature, the information backflow, non-Markovianity, and recoherence exhibit, over long times, exactly the same dependence on the low-frequency structure of the SD as the one found for the first class. Notice that in the whole paper the analysis concerns uniquely the short- and long-time flow of information. Consequently, the dynamics can still be non-Markovian due to intermediate backflows, even if no information flows from the environment back into the system over long times.

B. Thermal bath

Let the external environment be a thermal bath, $T > 0$. For SDs that belong to the first class under study (Sec. IV A) and $\alpha_0 > 0$ the dephasing rate increases linearly over short times, $t \ll 1/\Delta$,

$$\gamma_T(t) \sim l_T t. \tag{65}$$

This behavior is independent of the low- or high-frequency structure of the SD. Over long times, the dephasing rate divergences or vanishes dependent on the low-frequency profile of the SD that is given by Eq. (15). If $0 < \alpha_0 < 1$ the dephasing rate diverges for $t \gg 1/\Delta$ according to

$$\gamma_T(t) \sim c_{0,n_0} g_T (\Delta t)^{1-\alpha_0} \ln^{n_0}(\Delta t), \tag{66}$$

which describes power laws for $n = 0$,

$$\gamma_T(t) \sim c_{0,n_0} g_T (\Delta t)^{1-\alpha_0}. \tag{67}$$

The coefficient g_T reads

$$g_T = \frac{2k_B T \cos(\pi\alpha_0/2)\Gamma(\alpha_0)}{\hbar(1-\alpha_0)}.$$

If $\alpha_0 = 1$ the dephasing rate diverges for $t \gg 1/\Delta$ as

$$\gamma_T(t) \sim \frac{c_{0,n_0}\pi k_B T}{\hbar} \ln^{n_0}(\Delta t). \tag{68}$$

If $\alpha_0 = 1$ and $n_0 = 0$ the dephasing rate converges for $t \gg 1/\Delta$ to the nonvanishing value

$$\gamma_T(t) \sim \frac{c_{0,n_0}\pi k_B T}{\hbar}. \tag{69}$$

If $\alpha_0 > 1$ and α_0 is not an odd natural number, the dephasing factor vanishes for $t \gg 1/\Delta$ according to Eq. (66). If $\alpha_0 = 1+2m_3$, where m_3 and n_0 are nonvanishing natural numbers, the dephasing rate vanishes for $t \gg 1/\Delta$ as

$$\gamma_T(t) \sim c_{0,n_0} g'_T (\Delta t)^{-2m_3} \ln^{n_0-1}(\Delta t), \tag{70}$$

where $g'_T = \pi(-1)^{1+m_3} k_B T n_0 (2m_3-1)!/\hbar$. The above relaxation provides inverse power laws for $n_0 = 1$,

$$\gamma_0(t) \sim c_{0,n_0} g'_T (\Delta t)^{-2m_3}. \tag{71}$$

If α_0 is an odd natural number and n_0 vanishes, consider the least nonvanishing index k_3 such that either α_{k_3} does not take odd natural values or $\alpha_{k_3} = 1+2m_{k_3}$, where the natural numbers m_{k_3} and n_{k_3} do not vanish. The function $\gamma_T(t)$ is obtained in the former case from Eqs. (66) and (67) by replacing the power α_0 with α_{k_3} and n_0 with n_{k_3}, or in the latter case from Eqs. (70) and (71) by substituting the power m_3 with m_{k_3} and n_0 with n_{k_3}. We consider SDs such that the index k_3 exists with the required properties.

C. The dephasing rate at nonvanishing temperatures for the second class of spectral densities

Let the external environment consist of a thermal bath, $T > 0$, and the auxiliary functions $\Omega_T(\nu)$ belong to the second class under study (Sec. IV B). Over short times, $t \ll 1/\Delta$, the dephasing rate increases linearly according to Eq. (65), if $\alpha_0 > 0$. Except for this requirement, the behavior is independent of the low- or high-frequency structure of the SD. Over long times, $t \gg 1/\Delta$, the dephasing rate vanishes according to arbitrary powers of logarithmic forms

$$\gamma_T(t) \sim w_0 (\Delta t)^{1-\alpha_0}[g_T \ln^{\beta_0}(\Delta t) + \bar{g}_T \ln^{\beta_0-1}(\Delta t)], \tag{72}$$

where

$$\bar{g}_T = \frac{k_B T \beta_0}{\hbar}\left[2\cos\left(\frac{\pi\alpha_0}{2}\right)\Gamma^{(1)}(\alpha_0-1) - \pi\sin\left(\frac{\pi\alpha_0}{2}\right)\Gamma(\alpha_0-1)\right].$$

If the Ohmicity parameter α_0 does not take odd natural values, the dominant part of the above relaxation is

$$\gamma_T(t) \sim w_0 g_T (\Delta t)^{1-\alpha_0} \ln^{\beta_0}(\Delta t) \tag{73}$$

and becomes the power law

$$\gamma_T(t) \sim w_0 g_T (\Delta t)^{1-\alpha_0} \tag{74}$$

if $\beta_0 = 0$. If the Ohmicity parameter α_0 is an odd natural number, Eq. (72) gives

$$\gamma_T(t) \sim w_0 \bar{g}_T (\Delta t)^{1-\alpha_0} \ln^{\beta_0-1}(\Delta t) \tag{75}$$

and becomes the power law

$$\gamma_T(t) \sim w_0 \bar{g}_T (\Delta t)^{1-\alpha_0} \tag{76}$$

if $\beta_0 = 1$. Again, full similarity is observed between the above expressions and the long-time evolution of the dephasing rate

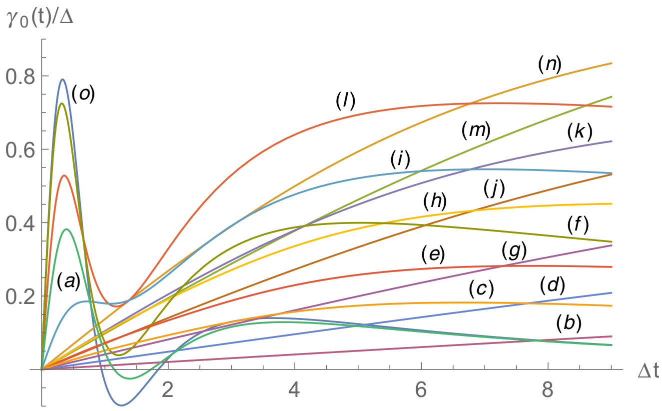

FIG. 5. Ratio $\gamma_0(t)/\Delta$ versus Δt for $0 \leqslant \Delta t \leqslant 9$, SDs given by Eq. (47), $q_1 = 1$, and different values of the parameters α and λ. Curve (a) corresponds to $\alpha = 2$ and $\lambda = 1.1$, curve (b) corresponds to $\alpha = 0.9$ and $\lambda = 40$, curve (c) corresponds to $\alpha = 1.5$ and $\lambda = 3$, curve (d) corresponds to $\alpha = 0.8$ and $\lambda = 27$, curve (e) corresponds to $\alpha = 1.3$ and $\lambda = 2.9$, curve (f) corresponds to $\alpha = 1.3$ and $\lambda = 0.7$, curve (g) corresponds to $\alpha = 0.8$ and $\lambda = 17$, curve (h) corresponds to $\alpha = 1.1$ and $\lambda = 2.9$, curve (i) corresponds to $\alpha = 1.1$ and $\lambda = 1.2$, (j) corresponds to $\alpha = 0.8$ and $\lambda = 10$, curve (k) corresponds to $\alpha = 0.9$ and $\lambda = 4.8$, curve (l) corresponds to $\alpha = 1$ and $\lambda = 0.7$, curve (m) corresponds to $\alpha = 0.7$ and $\lambda = 9.5$, curve (n) corresponds to $\alpha = 0.8$ and $\lambda = 4.5$, and (o) corresponds to $\alpha = 2$ and $\lambda = 0.9$.

that is obtained at nonvanishing temperatures for the first class of SDs under study.

Numerical computations of the dephasing rate are displayed in Figs. 5–8. The short-time linear growth is shown in Fig. 6.

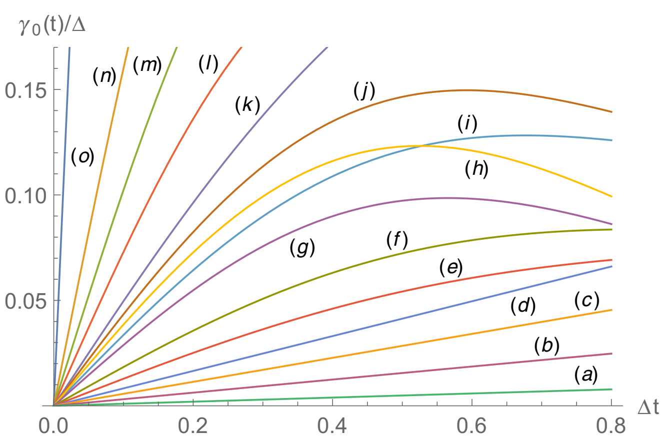

FIG. 6. Ratio $\gamma_0(t)/\Delta$ versus Δt for $0 \leqslant \Delta t \leqslant 0.8$, SDs given by Eq. (48), $q_1 = 1$ and different values of the parameters α and λ. Curve (a) corresponds to $\alpha = 1.5$ and $\lambda = 12$, curve (b) corresponds to $\alpha = 1.5$ and $\lambda = 5$, curve (c) corresponds to $\alpha = 0.8$ and $\lambda = 12.5$, curve (d) corresponds to $\alpha = 0.7$ and $\lambda = 11.5$, curve (e) corresponds to $\alpha = 1.6$ and $\lambda = 1.9$, curve (f) corresponds to $\alpha = 1.6$ and $\lambda = 1.7$, curve (g) corresponds to $\alpha = 2$ and $\lambda = 1.6$, curve (h) corresponds to $\alpha = 2$ and $\lambda = 1.5$, curve (i) corresponds to $\alpha = 1.4$ and $\lambda = 1.4$, (j) corresponds to $\alpha = 1.4$ and $\lambda = 1.3$, curve (k) corresponds to $\alpha = 0.8$ and $\lambda = 1.2$, curve (l) corresponds to $\alpha = 1.3$ and $\lambda = 1.1$, curve (m) corresponds to $\alpha = 1.3$ and $\lambda = 1$, curve (n) corresponds to $\alpha = 1$ and $\lambda = 0.8$, and (o) corresponds to $\alpha = 1$ and $\lambda = 0.5$.

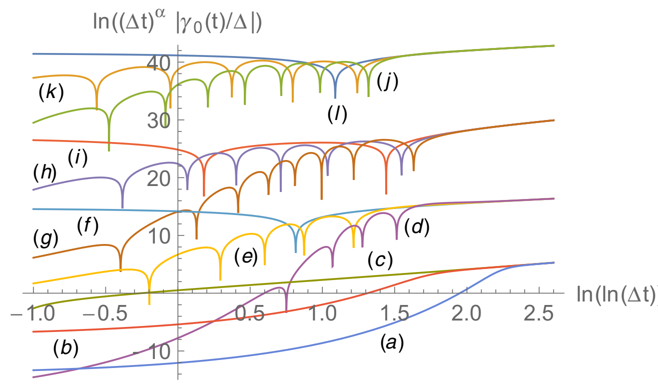

FIG. 7. Quantity $\ln[(\Delta t)^\alpha|\gamma_0(t)/\Delta|]$ versus $\ln[\ln(\Delta t)]$ for $\exp(1/e) \leqslant \Delta t \leqslant \exp[\exp(2.6)]$, SDs given by Eq. (47), $q_1 = 1$, and different values of the parameters α and λ. Curve (a) corresponds to $\alpha = 1$ and $\lambda = 10\,000$, curve (b) corresponds to $\alpha = 1$ and $\lambda = 200$, curve (c) corresponds to $\alpha = 1$ and $\lambda = 0.01$, curve (d) corresponds to $\alpha = 10$ and $\lambda = 20$, curve (e) corresponds to $\alpha = 10$ and $\lambda = 4$, curve (f) corresponds to $\alpha = 10$ and $\lambda = 0.01$, curve (g) corresponds to $\alpha = 15$ and $\lambda = 5$, curve (h) corresponds to $\alpha = 15$ and $\lambda = 2$, curve (i) corresponds to $\alpha = 15$ and $\lambda = 0.01$, curve (j) corresponds to $\alpha = 20$ and $\lambda = 2$, curve (k) corresponds to $\alpha = 20$ and $\lambda = 1$, and curve (l) corresponds to $\alpha = 20$ and $\lambda = 0.001$.

The long-time logarithmic relaxations result in the asymptotic lines of Figs. 7 and 8. The former refer to the quadratic logarithmic term of the first computed SDs (47). The latter correspond to various noninteger logarithmic powers of the second computed SDs (48).

The above results show that at nonvanishing temperatures and for the first class of SDs under study, the information flows

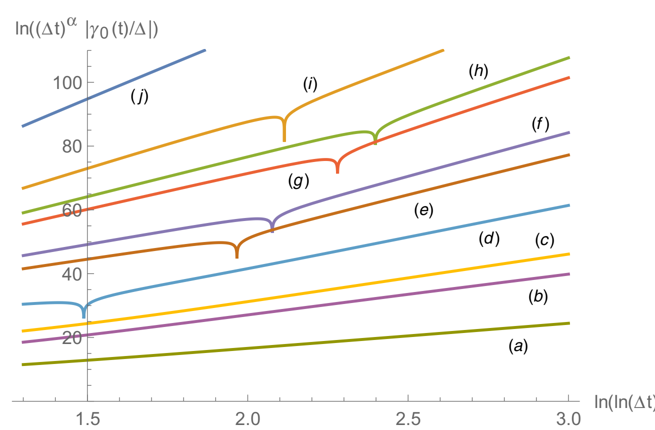

FIG. 8. Quantity $\ln[(\Delta t)^\alpha|\gamma_0(t)/\Delta|]$ versus $\ln[\ln(\Delta t)]$ for $\exp[\exp(1.3)] \leqslant \Delta t \leqslant \exp[\exp(3)]$, SDs given by Eq. (48), $q_2 = 1$, different values of the parameter α, and noninteger values of the logarithmic power β. Curve (a) corresponds to $\alpha = 1.1$ and $\beta = 8.1$, curve (b) corresponds to $\alpha = 2.1$ and $\beta = 13.4$, curve (c) corresponds to $\alpha = 1.9$ and $\beta = 15.4$, curve (d) corresponds to $\alpha = 2.2$ and $\beta = 20.5$, curve (e) corresponds to $\alpha = 2.6$ and $\beta = 25.8$, curve (f) corresponds to $\alpha = 2.6$ and $\beta = 28.1$, curve (g) corresponds to $\alpha = 3.1$ and $\beta = 33.9$, curve (h) corresponds to $\alpha = 2.9$ and $\beta = 35.9$, curve (i) corresponds to $\alpha = 1.9$ and $\beta = 41.7$, and curve (j) corresponds to $\alpha = 5.9$ and $\beta = 50.7$.

into the environment over short times, $t \ll 1/\Delta$. Over long times, $t \gg 1/\Delta$, the information flows back into the system for the values of the Ohmicity parameter $3+4n < \alpha_0 < 5+4n$, where $n = 0,1,2,\ldots$. Backflow of information is obtained also for every odd natural value of the Ohmicity parameter if $n_0 = 0$ and $3+4n < \alpha_{k_3} \leqslant 5+4n$, where n takes natural values. Additionally, if $n_0 > 0$, long-time backflow appears for every odd natural value $\alpha_0 = 1+4l_2$, where l_2 is a nonvanishing natural number. If the Ohmicity parameter differs from the values reported above, the long-time information spreads into the environment under Markovian evolution and the modulus of the coherence term decreases down to the asymptotic value. If compared to the initial condition, coherence is partially lost if $\alpha_0 > 2$. Coherence is fully lost if $0 < \alpha_0 \leqslant 2$. By considering the second class of SDs, the information backflow, non-Markovianity, decoherence, and recoherence exhibit over long times, at nonvanishing temperatures, exactly the same dependence on the low-frequency structure of the SD and on the temperature as the one found for the first class.

Consider the transition from vanishing to an arbitrary nonvanishing temperature. For both classes of the SDs under study, the backflow of information is stable for $3+4n < \alpha_0 < 4+4n$. Additionally, the backflow is stable for the values $3+4n$ under the special conditions mentioned above. The backflow is inverted for $2+4n < \alpha_0 < 3+4n$ and $4+4n < \alpha_0 < 5+4n$, where $n = 0,1,2,\ldots$. In the same transition, the long-time recoherence process is unaffected in the former conditions and is destroyed in the latter ones.

VII. CONCLUSION

We have considered the local dephasing process of a qubit that interacts with a structured reservoir of frequency modes or a thermal bath. We have studied the coherence between the two energy eigenstates of the qubit and the flow of quantum information by analyzing the dephasing factor and dephasing rate over short and long times. The SDs under study are obtained by introducing in the low-frequency Ohmic-like structure additional factors that are represented by powers of logarithmic forms $J(\omega) \overset{\propto}{\sim} \Delta(\omega/\Delta)^{\alpha_0}[-\ln(\omega/\Delta)]^{\beta_0}$ for $\omega \ll \Delta$. For the first class of SDs, the Ohmicity parameter α_0 takes arbitrarily non-negative real values, while the logarithmic power β_0 is natural valued. For the second class, the Ohmicity parameter takes arbitrarily positive real values, while the logarithmic power is real valued, arbitrarily positive or negative, or vanishing. The logarithmic singularities are removable and enhance, for positive logarithmic powers, or reduce, for negative logarithmic powers, the low-frequency power-law profiles of the physically feasible Ohmic-like SDs. Over higher frequencies the SDs are arbitrarily tailored.

The full loss or persistence of coherence, over long times, is determined by integral properties of the SD and corresponds to a divergent or a convergent dephasing factor, respectively. For both classes of SDs under study the dephasing factor increases quadratically and the dephasing rate grows linearly over short times, both at zero and at an arbitrary nonvanishing temperature. Over long times, the dephasing factor and the dephasing rate exhibit various relaxations to the asymptotic values that are described by logarithmic and power laws. The dependence on the low-frequency structure of the SD is the same for both classes of SDs. For the second class, the relaxations are arbitrarily faster or slower, or coincide with inverse power laws, due to the arbitrariness of the logarithmic powers.

The information flows into the environment over short times, at both vanishing and nonvanishing temperature. Over long times, we have found that regular patterns appear in the direction of the flow of information, back into the system or forth into the environment, dependent on the Ohmicity parameter α_0 of the SD, regardless of the logarithmic form factors. At zero temperature, the long-time information flows from the environment back into the system in correspondence with the periodic intervals $2+4n < \alpha_0 < 4+4n$ for every $n = 0,1,2,\ldots$. Under special conditions, backflow of information appears at zero temperature also for nonvanishing even natural values of the Ohmicity parameter. At nonvanishing temperatures, backflow of information is obtained over the periodic intervals $3+4n < \alpha_0 < 5+4n$. Under special conditions, backflow of information is found at nonvanishing temperatures also for odd natural values of the Ohmicity parameter. In the transition from vanishing to an arbitrary nonvanishing temperature, the backflow of information stably persists over the intervals $3+4n < \alpha_0 < 4+4n$ and, under the special conditions mentioned above, for the values $\alpha_0 = 3+4n$. Instead, the backflow is inverted over the intervals $2+4n < \alpha_0 < 3+4n$ and $4+4n < \alpha_0 < 5+4n$. Non-Markovianity and recoherence of the qubit appear along with the backflow of information. Consequently, the transition from vanishing to an arbitrary nonvanishing temperature does not destroy the recoherence process for $3+4n < \alpha_0 < 4+4n$ and, under special conditions, for the values $\alpha_0 = 3+4n$.

Argumentation on the experimental setting is beyond the purposes of the present paper. Still, it is worth mentioning that the present results apply to the Ohmic-like SDs of trapped impurity atoms that are immersed in a Bose-Einstein condensate environment. Furthermore, if the low-frequency power-law profiles of the Ohmic-like SDs are enhanced or reduced via arbitrary positive or negative powers of logarithmic form factors, the direction of the information flow is not altered by the logarithmic terms and depends uniquely on the Ohmicity parameter of the Ohmic-like term of the SD. Consequently, the patterns in the information flow remain stable with respect to the mentioned logarithmic perturbations of the Ohmic-like SDs. We believe that the present analysis provides further scenarios for the implementation of a stable control of the flow of quantum information and the appearance of non-Markovian dynamics and recoherence via the engineering reservoir approach.

APPENDIX: DETAILS

The evolution of the reduced density matrix $\rho(t)$ is given by the master equation (2). The off-diagonal elements of the reduced density matrix are described by Eq. (6) in terms of the dephasing factor $\Xi(t)$. This function is given by Eq. (7), for $T = 0$, and Eq. (8), for $T > 0$. If the second negative moment of the SD is finite, the expression $\int_0^\infty J(\omega)\cos(\omega t)/\omega^2 d\omega$ vanishes over long times due to the Riemann-Lebesgue lemma. In this way, Eq. (10) is obtained.

For $T>0$, according to the Riemann-Lebesgue lemma, if the second negative moment of the effective SD is finite, the expression $\int_0^\infty J_T(\omega)\cos(\omega t)/\omega^2 d\omega$ vanishes over long times. In this way, Eq. (12) is obtained.

The asymptotic behavior of the function $\Xi_0(t)$ is studied in the dimensionless variables $\nu=\omega/\Delta$ and $\tau=\Delta t$ by considering the function $F_0(\tau)$, which is defined as $F_0(\tau)=\Xi_0(\tau/\Delta)$. According to this definition, the function reads

$$F_0(\tau)=2\int_0^\infty \frac{\Omega(\nu)}{\nu^2}\sin^2\left(\frac{\tau\nu}{2}\right)d\nu. \tag{A1}$$

The Mellin transform [34,35] of the function $F_0(\tau)$ is defined as $\hat{F}_0(s)=\int_0^\infty \tau^{s-1}F_0(\tau)\,d\tau$ and reads

$$\hat{F}_0(s)=-\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\hat{\Omega}(-1-s). \tag{A2}$$

The fundamental strip depends on the asymptotic behavior of the auxiliary function [34,35]. Consider the first class of SDs under study (Sec. IV A) and the asymptotic form (15). The fundamental strip of the Mellin transform $\hat{F}_0(s)$ is $\min\{0,\alpha_0-1\}>\mathrm{Re}\,s>-2$. The asymptotic relationship [38]

$$\left|\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\right|\sim\left|\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\right|\sim\left(\frac{\pi}{2}\right)^{1/2}|\,\mathrm{Im}\,s|^{\mathrm{Re}\,s-1/2} \tag{A3}$$

about the Gamma function $\Gamma(s)$ holds for $|\,\mathrm{Im}\,s|\to+\infty$. In the strip $\max\{-4,-2-\chi_0\}<\mathrm{Re}\,s<\min\{-1/2-\epsilon_0,\alpha_0-1\}$ and for $|\,\mathrm{Im}\,s|\to+\infty$, the Mellin transform of the function $F_0(t)$ vanishes as $\hat{F}_0(s)=o(|\,\mathrm{Im}\,s|^{-1-\epsilon_0})$, where $\epsilon_0\in(0,3/2)$. Consequently, the function $\hat{F}_0(s)$ decreases sufficiently fast in the strip as $|\,\mathrm{Im}\,s|\to+\infty$ and the singularity in $s=-2$ provides the asymptotic expansion (17) of the dephasing factor at short times. As far as the long-time evolution is concerned, let the strip $\mu_0\leqslant\mathrm{Re}\,s\leqslant\delta_0$ exist such that the function $\hat{\Omega}(-1-s)$, or the meromorphic continuation, vanishes as

$$\hat{\Omega}(-1-s)=O(|\,\mathrm{Im}\,s|^{-\zeta_0}) \tag{A4}$$

for $|\,\mathrm{Im}\,s|\to+\infty$, where $\zeta_0>1/2+\delta_0$. The parameters μ_0 and δ_0 fulfill the constraints $\mu_0\in(-2,\min\{0,\alpha_0-1\})$ and $\delta_0\in(\alpha_0-1,0)$ for $0\leqslant\alpha_0<1$ or $\delta_0\in(\alpha_{k_4},\alpha_{k_5})$ for $\alpha_0\geqslant1$. The parameter α_{k_4} coincides with the positive power α_0 if α_0 is not an even natural number or if $\alpha_0=2m_0$ and $n_0>0$; otherwise α_{k_4} coincides with the parameter α_{k_0} that is defined in Sec. V. The index k_4 is the least natural number that is larger than k_3 and such that α_{k_4} is not an even natural number, or such that α_{k_4} is an even natural number and $n_{k_4}>0$. Under the above conditions, the singularity of the function $\hat{F}_0(s)$ in $s=\alpha_0-1$ for $0\leqslant\alpha_0\leqslant1$ or in $s=0$ and $s=\alpha_{k_4}-1$ for $\alpha_0>1$ provides the asymptotic forms given by Eqs. (18)–(26).

At nonvanishing temperatures the asymptotic behavior of the dephasing factor is evaluated via the function $F_T(\tau)$, defined as $F_T(\tau)=\Xi_T(\tau/\Delta)$, and the Mellin transform $\hat{F}_T(s)$, which reads

$$\hat{F}_T(s)=-\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\hat{\Omega}_T(-1-s). \tag{A5}$$

The fundamental strip is $\min\{0,\alpha_0-2\}>\mathrm{Re}\,s>-2$ for $\alpha_0>0$. The relationship (A3) implies that in the strip $\max\{-4,-2-\chi_0\}<\mathrm{Re}\,s<\min\{-1/2-\epsilon_1,\alpha_0-2\}$ and for $|\,\mathrm{Im}\,s|\to+\infty$, the Mellin transform of the function $F_T(t)$ vanishes as $\hat{F}_T(s)=o(|\,\mathrm{Im}\,s|^{-1-\epsilon_1})$, where $\epsilon_1\in(0,3/2)$. Consequently, the function $\hat{F}_T(s)$ decreases sufficiently fast in the strip as $|\,\mathrm{Im}\,s|\to+\infty$ and the singularity in $s=-2$ provides the asymptotic expansion (32) of the dephasing factor at short times. As far as the long-time behavior is concerned, let the strip $\mu_1\leqslant\mathrm{Re}\,s\leqslant\delta_1$ exist such that the function $\hat{\Omega}_T(-1-s)$, or the meromorphic continuation, vanishes as

$$\hat{\Omega}_T(-1-s)=O(|\,\mathrm{Im}\,s|^{-\zeta_1}) \tag{A6}$$

for $|\,\mathrm{Im}\,s|\to+\infty$, where $\zeta_1>1/2+\delta_1$. The parameters μ_1 and δ_1 fulfill the constraints $\mu_1\in(-2,\alpha_0-2)$ and $\delta_1\in(\alpha_0-2,0)$, for $0<\alpha_0<2$, or $\mu_1\in(-2,0)$ and $\delta_1\in(\alpha_{k_6},\alpha_{k_7})$, for $\alpha_0\geqslant2$. The parameter α_{k_6} coincides with the positive power α_0 if α_0 is not an odd natural number or if $\alpha_0=1+2m_1$ and $n_0>0$; otherwise α_{k_6} coincides with the parameter α_{k_1} that is defined in Sec. V. The index k_7 is the least natural number that is larger than k_6 and such that α_{k_7} is not an odd natural number, or α_{k_7} is an odd natural number and $n_{k_7}>0$. Under the above conditions, the singularity of the function $\hat{F}_T(s)$ in $s=\alpha_0-2$, for $0<\alpha_0\leqslant2$, or in $s=0$ and $s=\alpha_{k_6}-2$, for $\alpha_0>2$, provides Eqs. (33)–(42).

For the second class of SDs under study (Sec. IV B), the short time evolution of the dephasing factor coincides, at both zero and nonvanishing temperature, with the one obtained for the first class of SDs, due to the dominated convergence of the time-series expansion. As far as the long-time evolution is concerned, the study performed in Refs. [35,36] allows the asymptotic analysis of the expression (7), for $T=0$, and (8), for $T>0$, in terms of the dimensionless variables ν and τ. In this way, the asymptotic forms (27)–(31), for $T=0$, and (43)–(46), for $T>0$, are obtained.

The dephasing rate $\gamma(t)$ is defined by Eq. (3), for $T=0$, and by Eq. (4), for $T>0$. The constraints (50) and (51) are obtained by observing that the sine transforms of non-increasing functions are non-negative. For the first class of SDs (Sec. IV A), the asymptotic behavior of the dephasing rate $\gamma_0(t)$ is studied by considering the function $G_0(\tau)$ that is defined as $G_0(\tau)=\gamma_0(\tau/\Delta)$ and reads

$$G_0(\tau)=\Delta\int_0^\infty\frac{\Omega(\nu)}{\nu}\sin(\nu\tau)d\nu. \tag{A7}$$

The Mellin transform $\hat{G}_0(s)$ is

$$\hat{G}_0(s)=\Delta\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\hat{\Omega}(-s). \tag{A8}$$

The fundamental strip is $\min\{1,\alpha_0\}>\mathrm{Re}\,s>-1$. The relationship (A3) suggests that in the strip $\max\{-3,-1-\chi_0\}<\mathrm{Re}\,s<-1/2-\epsilon_2$ and for $|\,\mathrm{Im}\,s|\to+\infty$, the Mellin transform of the function $G_0(t)$ vanishes as $\hat{G}_0(s)=o(|\,\mathrm{Im}\,s|^{-1-\epsilon_2})$, where $\epsilon_2\in(0,1/2)$. Consequently, the function $\hat{G}_0(s)$ decreases sufficiently fast in the strip as $|\,\mathrm{Im}\,s|\to+\infty$ and the singularity in $s=-1$ provides Eq. (54). As far as the long-time evolution is concerned, let the strip $\mu_2\leqslant\mathrm{Re}\,s\leqslant\delta_2$ exist such that the function $\hat{\Omega}(-s)$, or the meromorphic continuation, vanishes as

$$\hat{\Omega}(-s)=O(|\,\mathrm{Im}\,s|^{-\zeta_2}) \tag{A9}$$

for $|\,\mathrm{Im}\,s| \to +\infty$, where $\zeta_2 > 1/2 + \delta_2$. The parameters μ_2 and δ_2 fulfill the constraints $\mu_2 \in (-1, \min\{1,\alpha_0\})$ and $\delta_2 \in (\alpha_{k_8},\alpha_{k_9})$. The parameter α_{k_8} coincides with the positive power α_0 if α_0 is not an even natural number or if $\alpha_0 = 2m_2$ and $n_0 > 0$; otherwise α_{k_8} coincides with the power α_{k_2} that is defined in Sec. V. The index k_9 is the least natural number that is larger than k_8 and such that α_{k_9} is not an even natural number, or such that α_{k_9} is an even natural number and $n_{k_9} > 0$. Under the above condition, the singularity of the function $\hat{G}_0(s)$ in $s = \alpha_{k_8}$ provides the asymptotic forms given by Eqs. (55)–(59).

For nonvanishing temperatures, $T > 0$, we study the function $G_T(\tau)$ that is defined as $G_T(\tau) = \gamma_T(\tau/\Delta)$. The Mellin transform $\hat{G}_T(s)$ results in the form

$$\hat{G}_T(s) = \Delta \sin\left(\frac{\pi s}{2}\right)\Gamma(s)\hat{\Omega}_T(-s). \tag{A10}$$

The fundamental strip is $\min\{1,\alpha_0 - 1\} > \mathrm{Re}\,s > -1$, where $\alpha_0 > 0$. The relationship (A3) implies that in the strip $\max\{-3, -1-\chi_0\} < \mathrm{Re}\,s < \min\{-1/2 - \epsilon_3, \alpha_0 - 1\}$ and for $|\,\mathrm{Im}\,s| \to +\infty$, the Mellin transform of the function $G_T(t)$ vanishes as $\hat{G}_0(s) = o(|\,\mathrm{Im}\,s|^{-1-\epsilon_3})$, where $\epsilon_3 \in (0,1/2)$. Consequently, the function $\hat{G}_T(s)$ decreases sufficiently fast in the strip as $|\,\mathrm{Im}\,s| \to +\infty$ and the singularity in $s = -1$ gives Eq. (65). As far as the long-time behavior is concerned, let the strip $\mu_3 \leqslant \mathrm{Re}\,s \leqslant \delta_3$ exist such that the function $\hat{\Omega}(-s)$, or the meromorphic continuation, vanishes as

$$\hat{\Omega}_T(-s) = O(|\,\mathrm{Im}\,s|^{-\zeta_3}) \tag{A11}$$

for $|\,\mathrm{Im}\,s| \to +\infty$, where $\zeta_3 > 1/2 + \delta_3$. The parameters μ_3 and δ_3 fulfill the constraints $\mu_3 \in (-1, \min\{1,\alpha_0 - 1\})$, for $\alpha_0 > 0$, and $\delta_3 \in (\alpha_{k_{10}},\alpha_{k_{11}})$. The parameter $\alpha_{k_{10}}$ coincides with the positive power α_0 if α_0 is not an odd natural number, or if $\alpha_0 = 1 + 2m_3$ and $n_0 > 0$; otherwise $\alpha_{k_{10}}$ coincides with the power α_{k_3} that is defined in Sec. VI. The index k_{11} is the least natural number that is larger than k_{10} and such that $\alpha_{k_{11}}$ is not an odd natural number, or such that $\alpha_{k_{11}}$ is an odd natural number and $n_{k_{11}} > 0$. Under the above conditions the singularity of the function $\hat{G}_T(s)$ in $s = \alpha_{k_{10}} - 1$ provides the asymptotic forms given by Eqs. (66)–(71).

Consider the second class of SDs (Sec. IV B). The short-time evolution of the dephasing rate coincides, at both zero and nonvanishing temperature, with the one obtained for the first class of SDs, due to the dominated convergence of the time-series expansion. The long-time behavior of the dephasing rate is evaluated from the study performed in Refs. [35,36] in terms of the dimensionless variables ν and τ. In this way, we derive the expressions (60)–(64), for $T = 0$, and (72)–(76), for $T > 0$.

The direction of the flow of information, over short and long times, is determined by studying the sign of the first term of the asymptotic expansion of the dephasing rate, over short and long times, respectively. Persistent negatives values correspond to backflow of information. This concludes the demonstration of the present results.

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