Quantum spin correlations in Møller scattering of relativistic electron beams

Marta Włodarczyk,[1,*] Paweł Caban,[2,†] Jacek Ciborowski,[1,‡] Michał Dragowski,[1,§] and Jakub Rembieliński[2,‖]
[1] *Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland*
[2] *Department of Theoretical Physics, University of Lodz, Pomorska 149/153, 90-236 Łódź, Poland*

(Received 9 November 2016; published 6 February 2017)

The relativistic spin correlation function was calculated for a pair of electrons originating from Møller scattering of two polarized electron beams. The results were discussed in view of a possible measurement of the correlation function and the corresponding probabilities. The special case of scattering off a stationary target (both polarized and unpolarized) was also analyzed. It was shown that the Clauser-Horne-Shimony-Holt (CHSH) inequality may be violated in the relativistic energy range when both scattering electrons are highly polarized.

DOI: 10.1103/PhysRevA.95.022103

I. INTRODUCTION

Nonlocality of quantum mechanics has been a subject of debate since the early days of this theory [1]. Even though this feature has already been experimentally verified (see e.g. [2]), there is still need to further investigate the details of quantum correlations for entangled systems at relativistic energies (see, e.g., [3–16] and references therein).

The unexpected features of the spin correlation function for relativistic particles with mass have been reported by us since 2008 [15,16]. The relativistic spin correlation function in general depends on particle momenta. Our studies revealed its nonmonotonic dependence on energy in systems containing at least one massive particle. Both for fermions and for bosons the function may have local extrema in energy also in the center-of-mass frame. This finding has not yet been investigated experimentally; in all of the correlation experiments performed until now [17–19] the energy of the particles was insufficient to observe relativistic effects.

Preparation of a pure, maximally entangled state in the relativistic energy range is a significant experimental challenge. Therefore, the correlation function for a pair of electrons originating from Møller scattering ($e^-e^- \to e^-e^-$) was studied [20] for experimental reasons. Another advantage of this process is that it is well understood and the final spin state can be determined, which makes it well suited for the calculation of the theoretical predictions for the correlation function. Although before the Møller scattering the two-particle state is separable, the final state (for which the correlations are calculated and measured) may be entangled.

Spin projections of the final state electrons can be measured by means of Mott polarimetry [21]; its applicability was confirmed for energies up to 15 MeV [22]. It takes advantage of the dependence of a Mott scattering cross section on the spin direction due to the spin-orbit interaction and allows for measuring spin projections on directions perpendicular to particle momenta.

The correlation function and the corresponding probabilities were calculated and thoroughly discussed in a special case of a polarized electron beam scattering off atomic electrons of an unpolarized target [20], which is the simplest case that can be realized experimentally. It was found that the correlation function exhibits a nonmonotonic dependence on energy, which confirms our earlier observations for other systems. The case of a 15-MeV fully polarized beam and symmetric scattering was considered in which the correlation function reaches the value of approximately 0.08.

The Møller and Mott scattering cross sections are very low for this energy, which makes the studied interactions extremely rare. Therefore, in this paper we will focus on energy equal to 3 MeV (as the cross sections increase with decreasing energy), which is sufficient for observing relativistic effects in the correlation function. This particular value was chosen in view of an ongoing project to measure the quantum spin correlations [23].

Our principal aim was to generalize the studies presented in the previous paper [20], in search for configurations resulting in higher absolute values of the correlation function. As an intermediate step, scattering on polarized target electrons is discussed. Ultimately, the most general case of scattering of two polarized electron beams is analyzed. Using two polarized beams would additionally eliminate the problem of high rate of background originating from Mott scattering in the target, which is unavoidable in stationary target experiments (due to the fact that the cross section for Møller scattering is much lower than for Mott scattering).

II. THE FINAL STATE IN THE MØLLER SCATTERING

The initial state (before the scattering) is separable, since the colliding electrons are prepared separately. Therefore it can be represented by a tensor product of the density matrices of individual electrons:

$$\hat{\rho}^{\text{in}} = \hat{\rho}^{1\,\text{in}} \otimes \hat{\rho}^{2\,\text{in}}, \tag{1}$$

and in terms of matrix elements:

$$\rho^{\text{in}}_{(\tau_1,\tau_2),(\tau_1',\tau_2')}(q_1,q_2,q_1',q_2') = \rho^{1\text{in}}_{\tau_1\tau_1'}(q_1,q_1')\rho^{2\text{in}}_{\tau_2,\tau_2'}(q_2,q_2'), \tag{2}$$

where τ_i,τ_i', $i = 1,2$, represent indices related to the spin part of the matrix and can take values $\pm 1/2$, while q_i, q_i' denote the four-momenta of the interacting electrons.

*marta.wlodarczyk@tomtom.com
†p.caban@merlin.phys.uni.lodz.pl
‡cib@fuw.edu.pl
§michal.dragowski@fuw.edu.pl
‖jaremb@uni.lodz.pl

If one assumes that the electrons have well-determined polarization vectors $\boldsymbol{\xi}_i$ and are in sharp momentum state at the moment of interaction, the matrices $\hat{\rho}^{i\,\mathrm{in}}$ take the simple form [20]

$$\rho^{i\,\mathrm{in}}_{\tau_i\tau_i'}(q_i,q_i') = \frac{2p_i^0}{\delta^3(\mathbf{0})}\delta^3(\mathbf{q}_i - \mathbf{p}_i)\delta^3(\mathbf{q}_i' - \mathbf{p}_i)\frac{1}{2}(\mathbb{1} + \boldsymbol{\xi}_i \cdot \boldsymbol{\sigma})_{\tau_i\tau_i'}, \tag{3}$$

where $\boldsymbol{\xi}_i$ and p_i denote the polarization vector and the four-momentum of the ith electron, respectively; $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$, where σ_k are the Pauli matrices and $2p_i^0/\delta^3(\mathbf{0})$ is the normalization factor.

The final state after the scattering reads

$$\hat{\rho}^{\mathrm{out}} = \frac{\hat{M}\hat{\rho}^{\mathrm{in}}\hat{M}^\dagger}{\mathrm{Tr}\{\hat{M}\hat{\rho}^{\mathrm{in}}\hat{M}^\dagger\}}, \tag{4}$$

where M denotes the scattering amplitude. The explicit form of its matrix element (to the first order of approximation) can be found elsewhere [24] and the explicit form of (4) in the paper by Caban *et al.* [20]. It has been shown that although the initial state is separable, the degree of entanglement of the final state is nonzero and depends on both energies of the interacting particles and the scattering angle. In general, it is impossible to assign polarization vectors to the electrons after the scattering, because only their joint polarization state is well defined. One can thus use only reduced density matrices of the final state electrons to calculate their mean polarization vectors.

Even in the simplest case of a polarized beam scattering off an unpolarized target, the formulas describing the mean polarization vectors after the scattering are too complex to be presented here. In Fig. 1, transverse and longitudinal components of the polarization vectors, as well as the total length of the polarization vectors for both electrons, are shown as functions of the scattering angle of the first electron. One can see that after the scattering both Møller electrons share the initial polarization.

In Fig. 2(a) one can also see that their joint polarization vector (i.e., the sum of the mean polarization vectors of both electrons) is also not a unit vector, which indicates a certain degree of entanglement. The maximum of entanglement corresponds to the symmetric scattering in which both electrons have the same mean polarization due to their indistinguishability. In Fig. 2(b) the dependence of the length of the joint polarization vector on the beam kinetic energy is shown for symmetric scattering. It suggests that decreasing the beam energy results in a more entangled state after the scattering.

Negativity N, which is a proper entanglement measure, is defined as

$$N(\hat{\rho}^{\mathrm{out}}) = \Sigma_i \frac{|\lambda_i| - \lambda_i}{2}, \tag{5}$$

where λ_i are the eigenvalues of the density matrix $\hat{\rho}^{\mathrm{out}}$. It has already been shown [20] that it reaches its maximum for the symmetric scattering angle independent of the beam energy. On the other hand, in case of symmetric scattering, the degree of entanglement increases with the decrease of the beam kinetic energy, T. In the limit of $T \longrightarrow 0$, the final state becomes a pure singlet state.

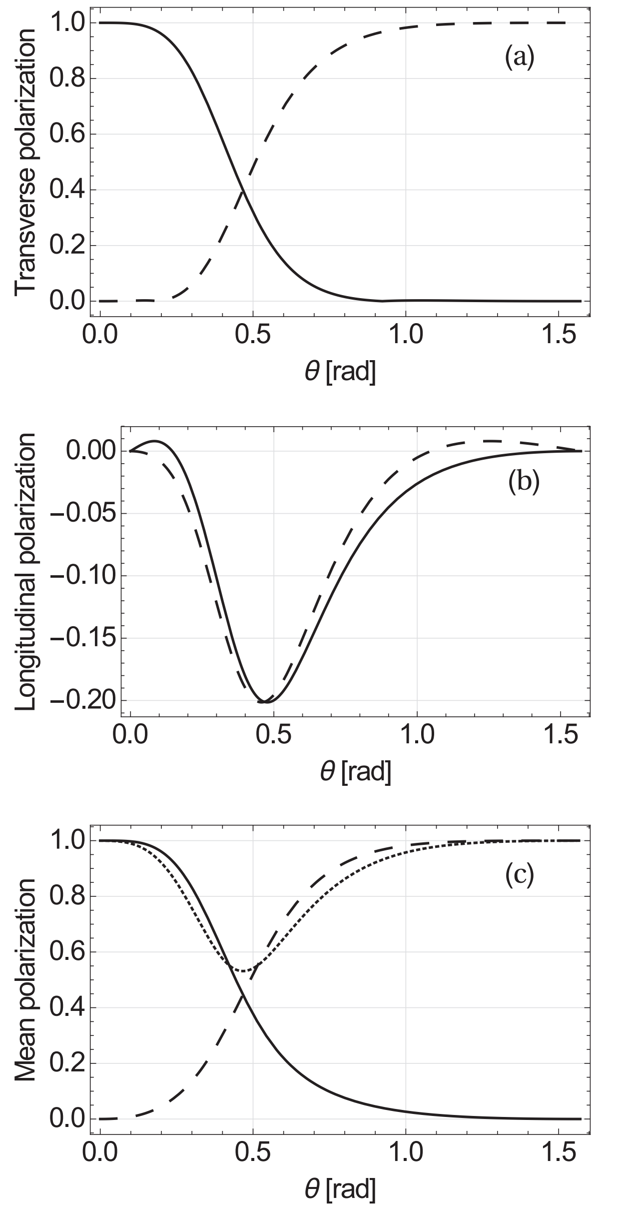

FIG. 1. The dependence of transverse (a) and longitudinal (b) polarization vector component and mean polarization vector length (c) for two Møller electrons on the scattering angle in the case of scattering of a 100% transversely polarized 3-MeV beam off an unpolarized target. The initial beam polarization vector is perpendicular to the beam direction and in the Møller scattering plane. The scattering angle θ is the angle between the initial beam direction and the direction of the electron whose polarization is plotted with a solid line; the polarization of the other electron is plotted with a dashed line. The length of the joint polarization vector (the sum of the mean polarization vectors of both electrons) is represented by a dotted line.

III. THE CORRELATION FUNCTION AND THE PROBABILITIES

Given observables $\hat{A}$ and $\hat{B}$ one can define the correlation function $\mathcal{C}(A,B)$ as

$$\mathcal{C}(A,B) = \Sigma_{a,b} ab\, P_{ab}, \tag{6}$$

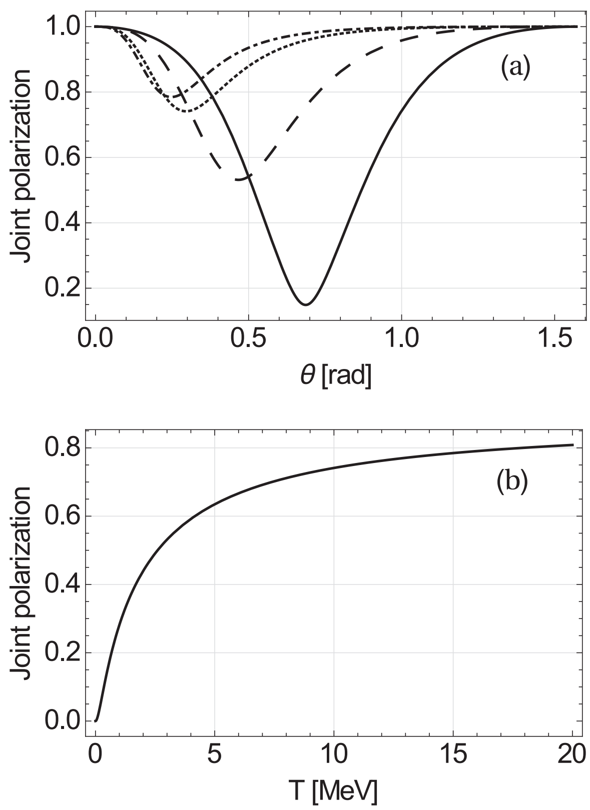

FIG. 2. Joint polarization vector length as a function of the scattering angle (a) for several fixed kinetic energies: 0.5 MeV (solid line), 3 MeV (dashed line), 10 MeV (dotted line), and 15 MeV (dotted-dashed line), and as a function of the beam kinetic energy (b) assuming symmetric scattering.

where P_{ab} denote the joint probabilities of obtaining a and b as a result of a measurement of observables $\hat{A}$ and $\hat{B}$, respectively. The probabilities P_{ab} can be calculated using the formula

$$P_{ab} = \mathrm{Tr}\{\hat{\Pi}_a \otimes \hat{\Pi}_b \hat{\rho}^{\text{out}}\}, \tag{7}$$

where $\hat{\Pi}_{a/b}$ are the projectors from the spectral decomposition of the $\hat{A}/\hat{B}$ observable corresponding to the eigenvalues a/b.

In order to calculate the spin correlation function, the relativistic spin-projection observables $\mathbf{a}\hat{\mathbf{S}}$ and $\mathbf{b}\hat{\mathbf{S}}$ should be used, where $\mathbf{a}$ and $\mathbf{b}$ are unit vectors on which the particle spin is projected and $\hat{\mathbf{S}}$ is the relativistic spin operator. In the case of a two-fermion state, the correlation function is a linear combination of four probabilities: P_{++}, P_{+-}, P_{-+}, and P_{--}, where $\pm$ denotes positive and negative spin projection. Therefore, a relativistic spin operator is needed to calculate the correlation function and the corresponding probabilities in the framework of relativistic quantum mechanics. Spin, however, is not a self-contained geometrical object in relativistic quantum mechanics and only the spin-square operator can be uniquely defined in terms of the generators of the Poincaré group:

$$\hat{\mathbf{S}}^2 = -\frac{1}{m^2}\hat{W}^\mu \hat{W}_\mu, \tag{8}$$

where m is the particle mass, $\hat{W}^\mu$ is the Pauli-Lubanski four vector: $\hat{W}^\mu = \frac{1}{2}\epsilon^{\nu\alpha\beta\mu}\hat{P}_\nu \hat{J}_{\alpha\beta}$, $\hat{P}_\mu$ is the four-momentum operator, and $\hat{J}_{\alpha\beta}$ are the generators of the Lorentz group.

A well-defined relativistic operator should possess certain properties: (i) not convert positive (negative) energy states into negative (positive) energy states, (ii) be a pseudovector, (iii) have eigenvalues independent of the direction on which spin is projected, and (iv) have a proper nonrelativistic limit. The Newton-Wigner spin operator [25] of the form

$$\hat{\boldsymbol{S}} = \frac{1}{m}\left(\hat{\boldsymbol{W}} - \hat{W}^0 \frac{\hat{\boldsymbol{P}}}{\hat{P}^0 + m}\right) \tag{9}$$

satisfies the above conditions and is the only operator that, additionally, (i) is a linear combination of the Pauli-Lubanski four-vector components and (ii) fulfills the standard commutation relations. In one-particle subspace, the projection of the Newton-Wigner operator on an arbitrary direction $\mathbf{n}$ takes the form

$$\mathbf{S}(k_i,\mathbf{n}) = \frac{1}{2m}\Bigg[\left(m\mathbf{n} + \frac{\mathbf{n}\cdot\mathbf{k}_i}{m+k_i^0}\mathbf{k}_i\right)\boldsymbol{\gamma}\gamma^5 - (\mathbf{n}\cdot\mathbf{k}_i)\gamma^0\gamma^5 + i((\mathbf{n}\times\mathbf{k}_i)\boldsymbol{\gamma})\gamma^0 - \left(k_i^0\mathbf{n} - \frac{\mathbf{n}\cdot\mathbf{k}_i}{m+k_i^0}\mathbf{k}_i\right)\boldsymbol{\gamma}\gamma^0\gamma^5\Bigg]. \tag{10}$$

where γ^μ are the Dirac matrices and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. It has been shown [26] that the correlation function calculated in the relativistic quantum mechanics framework using (9) has the same form as that obtained within the quantum field theory framework. It has also been shown [8] that the Newton-Wigner operator corresponds to the Stern-Gerlach procedure of spin-projection measurement for Dirac particles.

Mott polarimetry has been established as the standard experimental method of determining the spin projection for electrons of energies up to several MeV [21]. It is, however, sensitive only to the spin component perpendicular to the electron momentum. Therefore, in this specific case the spin-projection operator reduces to

$$\mathbf{S}(k_i,\mathbf{n}) = \frac{1}{2m}\left[m\mathbf{n}\cdot\boldsymbol{\gamma}\gamma^5 + i((\boldsymbol{n}\times\mathbf{k}_i)\boldsymbol{\gamma})\gamma^0 - k_i^0\mathbf{n}\cdot\boldsymbol{\gamma}\gamma^0\gamma^5\right]. \tag{11}$$

The correlation functions in this paper will be calculated using the spin-projection observable in the above form.

As was shown in Sec. II, the degree of entanglement of the state after the Møller scattering is maximal for symmetric scattering. Therefore all the following results will be shown for the symmetric configuration.

In Sec. III A the results for the unpolarized target are recalled and further analyzed for different configurations of the $\mathbf{a}$ and $\mathbf{b}$ vectors. Results regarding the relativistic correlation function and the probabilities are presented for electrons originating from Møller scattering of a polarized beam off a polarized target (Sec. III B) and off another electron beam (Sec. III C). The calculations are much more complicated when both electrons have nonzero polarization. Further increase in the number of parameters describing the system occurs in the case of scattering of two electron beams (see Appendix A). Therefore the calculations were performed numerically using MATHEMATICA and the FEYNCALC package. The results are presented in graphical form due to the complexity of the resulting formulas.

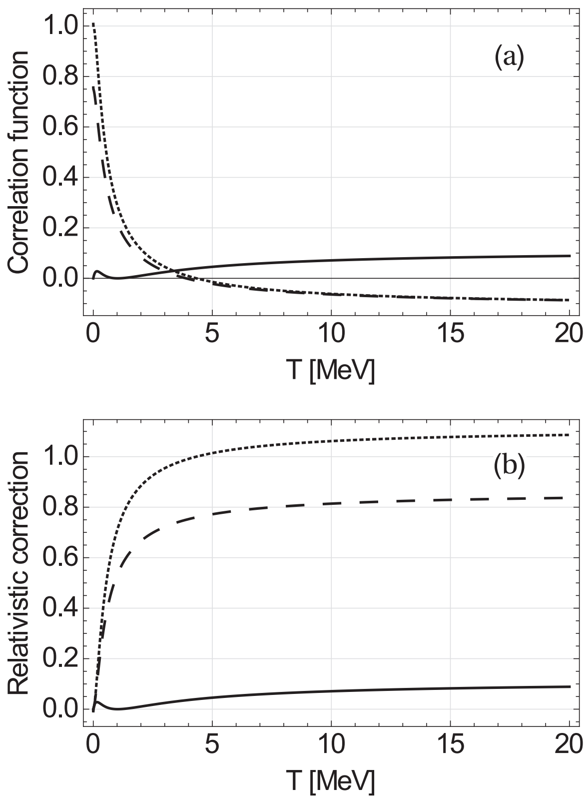

FIG. 3. The correlation function (a) and the relativistic correction (b) as functions of the beam kinetic energy for symmetric Møller scattering and different configurations of the **a** and **b** vectors: (i) both in the Møller scattering plane (solid line), (ii) both perpendicular to the Møller scattering plane (dashed line), and (iii) at angles equal to 60° and 120° to the Møller scattering plane, respectively (dotted line).

A. Scattering of a polarized beam off an unpolarized target

The relativistic correlation function for a pair of electrons originating from Møller scattering of a polarized electron beam off an unpolarized target has been calculated and analyzed by Caban *et al.* [20]. It has been shown that in this simplest case the correlation function does not depend on beam polarization and in general is a nonmonotonic function of the beam kinetic energy. For symmetric scattering, when both Møller electrons have equal energies and are equally polarized, local extrema in energy occur.

In Fig. 3(a) one can see the dependence of the correlation function on the beam kinetic energy for symmetric Møller scattering and for different configurations of the **a** and **b** vectors: (i) both in the Møller scattering plane (solid line), (ii) both perpendicular to the Møller scattering plane (dashed line), and (iii) at angles equal to 60° and 120° to the Møller scattering plane, respectively (dotted line). The absolute value of the correlation function is small (of the order of 0.1) in the range of relativistic energies (over a few MeV). Nevertheless, the difference between the nonrelativistic and relativistic value (the relativistic correction) can be quite large (even of the order of 1) for vectors **a** and **b** outside the Møller scattering plane, which can be seen in Fig. 3(b).

Unlike the correlation function, the probabilities do depend on beam polarization, as shown in Fig. 4 for configuration

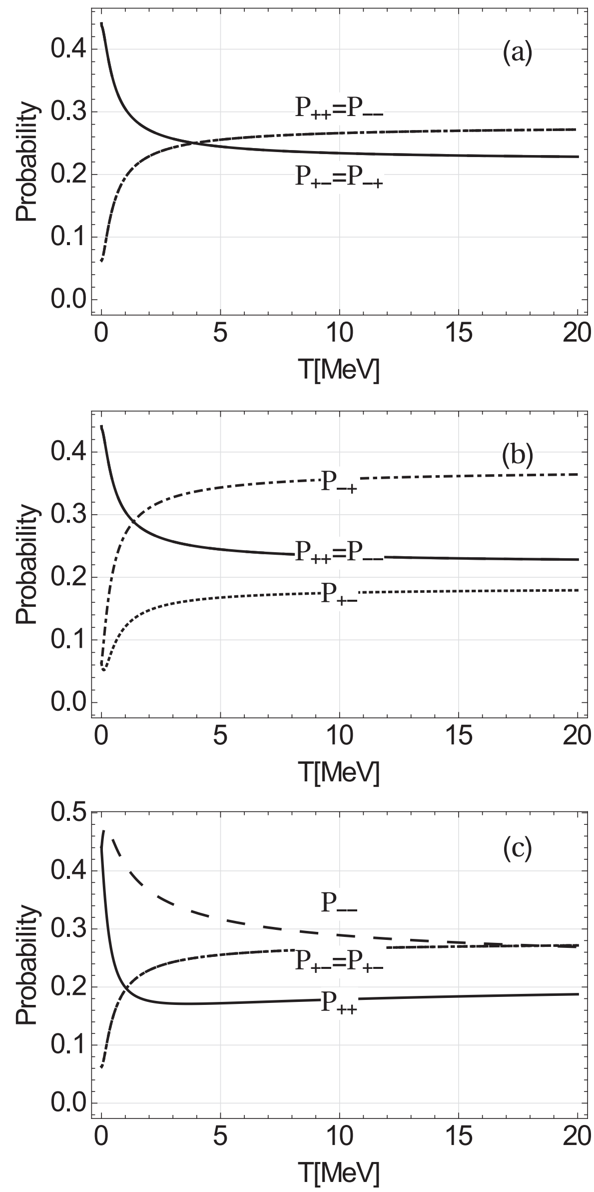

FIG. 4. Probabilities as functions of the kinetic energy for unpolarized beam (a), beam transversely polarized at 85%, polarization vector in the Møller scattering plane (b), and beam longitudinally polarized at 85% (c) for an unpolarized target, symmetric scattering and the **a** and **b** vectors at angles equal to 60° and 120° to the Møller scattering plane, respectively.

(iii) and for different beam polarization directions. Their dependence on energy is different than in the case of an unpolarized beam.

B. Scattering of a polarized beam off a polarized target

Before moving to the most general case of colliding beams, we analyze the case of a beam scattering off a polarized target. In this specific case, the following was observed: (i) the correlation function is described by exactly the same formula as in the case of an unpolarized target when the beam and the target polarization vectors are perpendicular to each other and (ii) in the case when the aforementioned vectors

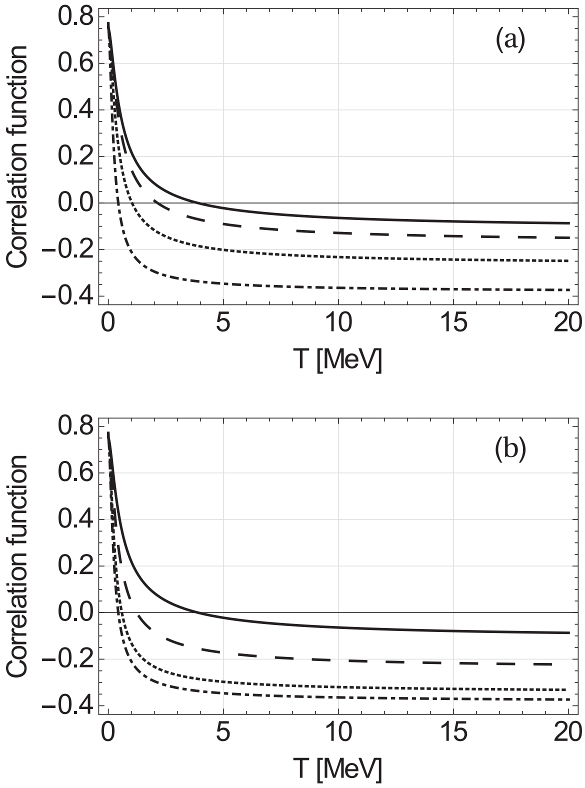

FIG. 5. The correlation function as a function of the beam kinetic energy for beam transversely polarized at 85%, polarization vector in the Møller scattering plane, for different degrees of target polarization (parallel target and beam polarization vectors): 0% (solid line), 20% (dashed line), 50% (dotted line), and 85% (dot-dashed line) (a); and different angles between the beam and the target polarization vectors: 90° (solid line), 60° (dashed line), 30° (dotted line), and 0° (dot-dashed line) (b). The **a** and **b** vectors are at angles equal to 60° and 120° to the Møller scattering plane, respectively.

are not perpendicular, the correlation function increases with increasing degree of target polarization.

Figure 5 illustrates the dependence of the correlation function on energy for a beam polarized transversely at 85% (this value was chosen for experimental reasons [23]) polarization vector in the Møller scattering plane and the **a** and **b** vectors at angles 60° and 120° to the Møller scattering plane, respectively. In Fig. 5(a) the correlation function is plotted for target electrons polarized in the same direction and for several different degrees of the target polarization. One can see that in the considered configuration the difference between the nonrelativistic and the relativistic value is bigger for higher degrees of target polarization. In Fig. 5(b) the case of both electrons polarized at 85% is shown for different angles between the polarization vectors. The absolute value of the relativistic correlation function and the relativistic correction increase with the decrease of the angle between the polarization vectors. This outcome leads to the conclusion that polarizing the target in the direction parallel to the beam polarization direction could result in higher absolute values of the correlation function in the relativistic energy range and, as will be shown in Sec. IV, allow for violation of the Bell-type inequalities.

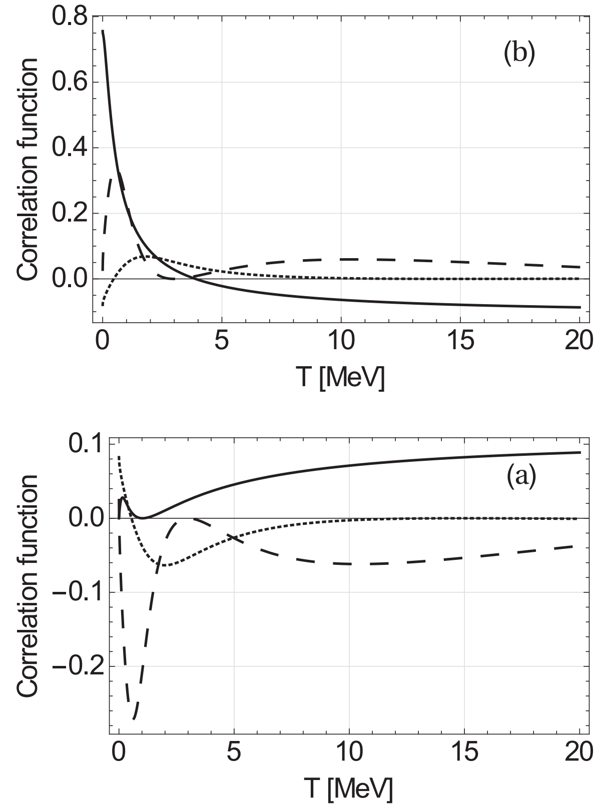

FIG. 6. The correlation function as a function of the kinetic energy of one electron for two unpolarized beams colliding at $\gamma = 10^\circ$ and the **a** and **b** vectors: in the Møller scattering plane (a), at angles equal to 60° and 120° to the Møller scattering plane (b). The kinetic energy of the second electron is equal to 0 MeV (solid line), 3 MeV (dashed line), and 15 MeV (dotted).

C. Scattering of two polarized beams

The correlation function and the probabilities for two colliding electron beams of arbitrary polarization were calculated and investigated as the most general case. Assuming symmetric scattering, this process can be parametrized by (i) polarization (degree and direction) of both beams, (ii) their kinetic energies, (iii) angle at which the beams collide γ, and (iv) the **a** and **b** vectors on which the spin of the electrons is projected. The details of the parametrization used in our calculations are given in Appendix A.

We start the discussion with the simplest case of two unpolarized beams. It can be shown that in terms of the correlation function this setup is equivalent to one beam being unpolarized. In Fig. 6 one can see the dependence of the correlation function on the beam kinetic energy for two beams colliding at $\gamma = 10^\circ$ and for two different configurations of the **a** and **b** vectors. It can be noticed that for a fixed γ the correlation function exhibits different behavior for different energies of the second beam—it can be either a monotonic function in one case or have several extrema in another. The impact of the second beam energy on the value of the relativistic correction depends on the choice of the **a** and **b** directions. In one case increasing the energy can increase and in another, decrease the difference between the nonrelativistic and the relativistic correlation function.

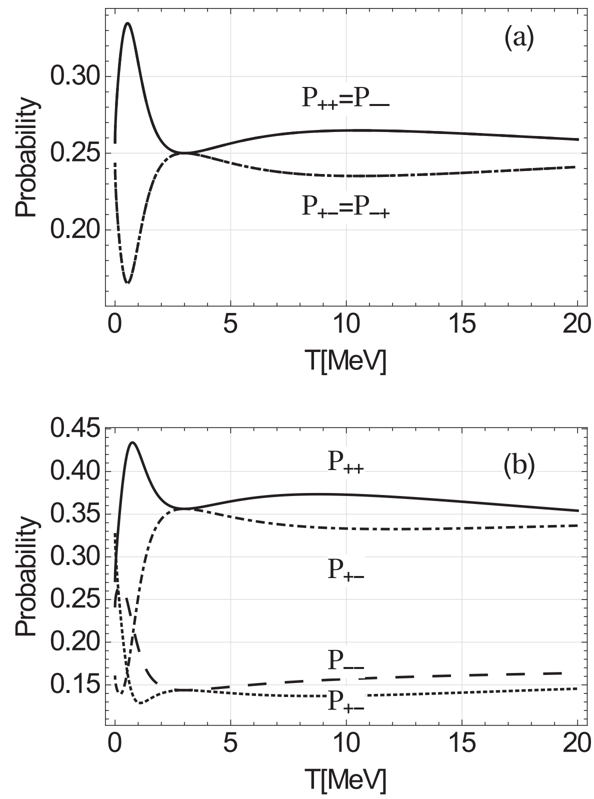

FIG. 7. The four probabilities as a function of the kinetic energy of one electron for two beams colliding at $\gamma = 10^\circ$ for two unpolarized beams (a) and for one unpolarized beam and one polarized transversely at 85%, polarization vector in the Møller scattering plane (b). The **a** and **b** vectors are at angles equal to 60° and 120° to the Møller scattering plane. The kinetic energy of the second electron is equal to 3 MeV.

Unlike the correlation function, the probabilities do depend on polarization also when only one beam is polarized, which can be seen in Fig. 7. The same effect was observed in the case of scattering off an unpolarized target. Although the correlation function was insensitive to beam polarization, the probabilities differed for different beam polarization vectors.

In the case of two polarized beams, the correlation function depends both on the degree of their polarization and on the directions in which they are polarized. In Fig. 8(a) one can see the influence of changing the degree of the second beam polarization, while both beams are polarized parallel to each other, transverse to the first beam direction and with the polarization vector in the Møller scattering plane. In Fig. 8(b) one can see the influence of changing the second beam polarization vector direction for both beams polarized at 85%. In the case of two polarized beams, unlike the case of a polarized target, the situation of perpendicular polarizations is not equivalent to one electron being unpolarized [see Fig. 8(c)].

We also investigated how the correlation function depends on the angle at which the beams collide. In Fig. 9 the dependence of the correlation function on the kinetic energy of one of the beams is shown for two different **a** and **b** configurations and for the energy of the second electron equal to 3 MeV. Both electrons are polarized in the same direction perpendicular to the first electron momentum and in the Møller scattering plane. Again, the behavior of the correlation function depends on the choice of the **a** and **b** vectors. In case of both vectors in the Møller scattering plane, small γ yields a greater difference between the relativistic and nonrelativistic case than large γ, while in the case of **a** and **b** at angles equal to 60° and 120° the opposite is true.

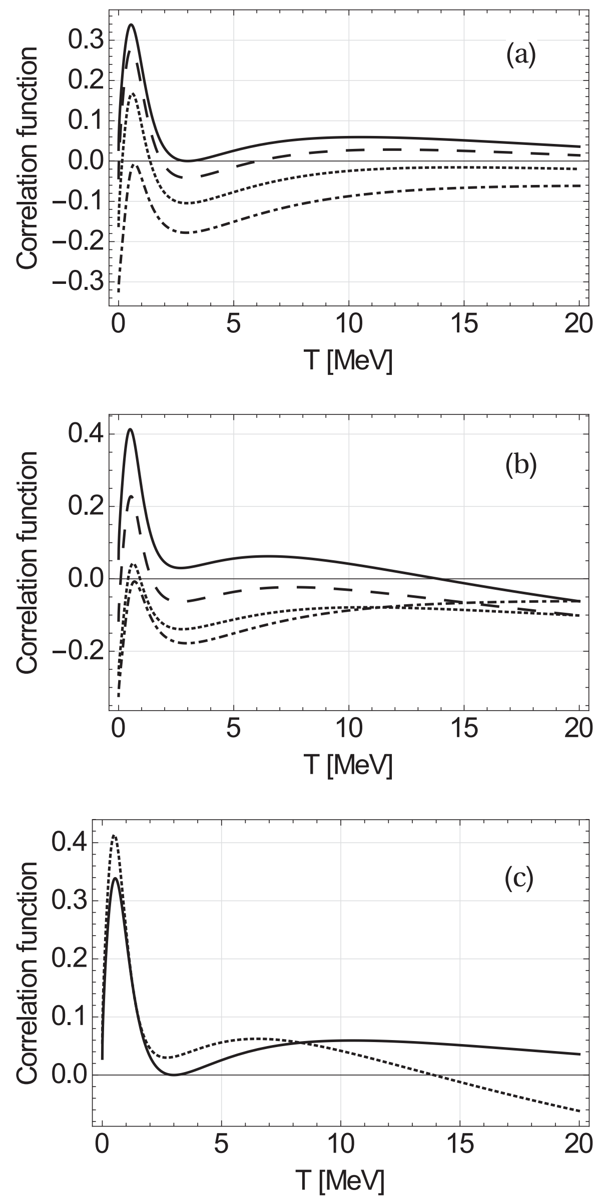

FIG. 8. The correlation function as a function of the kinetic energy of one electron for two beams colliding at $\gamma = 10^\circ$. One beam is polarized in 85% transversely to the beam direction, polarization vector in the Møller scattering plane. The other beam is: polarized at 85%, the angle between the polarization vectors is equal to 90° (solid line), 60° (dashed line), 30° (dotted line), and 0° (dot-dashed line) (a); polarized in the same direction as the first one at 0% (solid line), 20% (dashed line), 50% (dotted line), and 85% (dot-dashed line) (b). Figure (c) illustrates the difference between the case of one beam being unpolarized (solid line) and polarized at 85% in the direction perpendicular to the first beam polarization (dotted line).

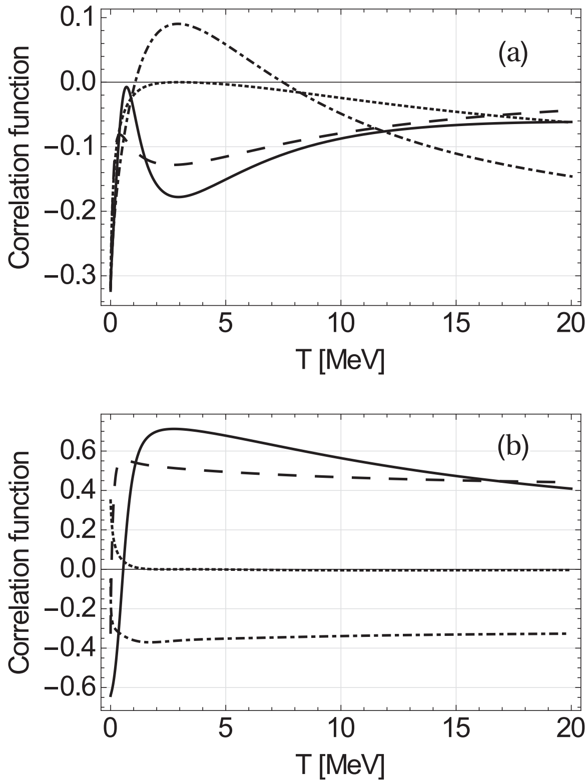

FIG. 9. The correlation function as a function of the kinetic energy of one of the beams for different angles between the beams (γ): 10° (solid line), 45° (dashed line), 90° (dotted-dashed line), and 120° (dotted line); and the **a** and **b** vectors at angles equal to 60° and 120° to the Møller scattering plane (a) and in the Møller scattering plane (b). Both beams are polarized at 85% in a direction transverse to the first beam direction, polarization vectors in the Møller scattering plane. The kinetic energy of the second beam is 3 MeV.

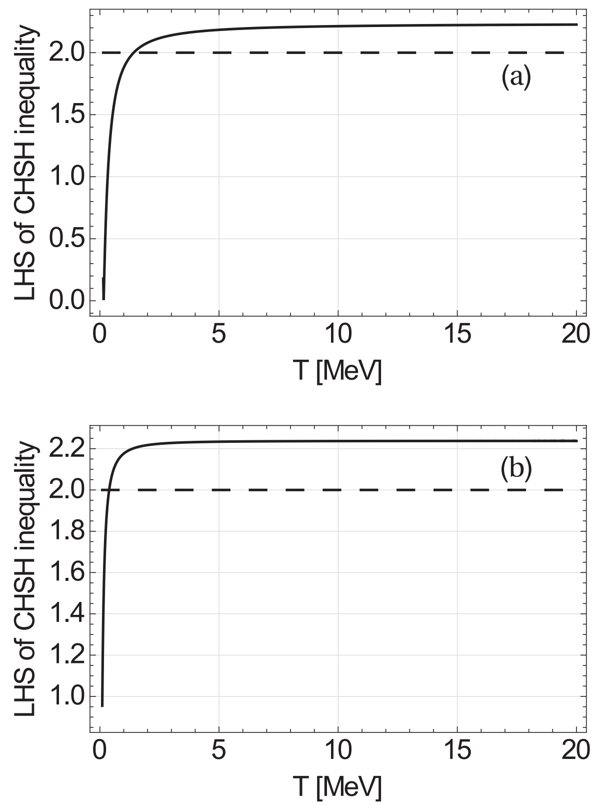

FIG. 10. The left-hand side of the CHSH inequality as a function of the beam kinetic energy in the case of a polarized target (a) and two polarized beams (b). In both cases the electrons are polarized at 90% in the same direction, perpendicular to the momentum of one of them and in the Møller scattering plane. The vectors on which the spin is projected are at angles equal to $\alpha_a = 0°$, $\alpha_b = 45°$, $\alpha_c = 90°$, and $\alpha_d = 135°$ to the Møller scattering plane. In case of two beams, the energies of both beams are equal and the angle $\gamma = 180°$.

IV. THE BELL-TYPE INEQUALITIES

It has been shown that in the case of relativistic spin correlations, the Bell-type inequalities can be violated for maximally entangled states [16] and that the degree of violation depends on energy. One of the aims of the present paper was to verify whether the Bell-type inequalities can also be violated in the state originating from the Møller scattering (i.e., not maximally entangled).

The most commonly used Bell-type inequality in the case of spin-1/2 fermions is the Clauser-Horne-Shimony-Holt (CHSH) inequality of the form [27]

$$|\mathcal{C}(\mathbf{a},\mathbf{b}) + \mathcal{C}(\mathbf{c},\mathbf{b}) + \mathcal{C}(\mathbf{c},\mathbf{d}) - \mathcal{C}(\mathbf{a},\mathbf{d})| < 2, \qquad (12)$$

where $\mathcal{C}(\mathbf{m},\mathbf{n})$ denotes the correlation function for two spin-projection observables $\mathbf{m}\hat{\mathbf{S}}$ and $\mathbf{n}\hat{\mathbf{S}}$. In the simplest case of scattering of a polarized beam off an unpolarized target, the correlation function is small enough to fulfill the above inequality for electrons of energies over few MeV. However, in the case of a polarized target and two colliding polarized beams it is possible to find configurations in which the CHSH inequality is violated. For two opposing beams of equal energy the CHSH inequality is violated for beam polarization over 85% when both electrons are polarized in the same direction, perpendicular to the momentum and in the Møller scattering plane, and the vectors on which the spin is projected are at angles equal to $\alpha_a = 0°$, $\alpha_b = 45°$, $\alpha_c = 90°$, and $\alpha_d = 135°$ to the Møller scattering plane. The left-hand side of the inequality (12) is greater than 2 in the relativistic energy range (see Fig. 10) both for a target (a) and for two beams of equal energy colliding at $\gamma = 180°$ (b). (The degree of polarization was chosen equal to 90% for illustrative purposes.)

V. SUMMARY AND CONCLUSIONS

The relativistic spin correlation function was calculated for the final-state electrons in Møller scattering. The calculations were performed for the most general case involving scattering of two beams of arbitrary polarization, which is a generalization of our previous work regarding the scattering on a stationary unpolarized target only [20].

This work has been done in the context of experimental investigation of relativistic spin correlations, since the discussed process can be conveniently realized experimentally, contrary to the preparation of pure, maximally entangled spin states. An experiment involving the scattering of two polarized beams would be technically challenging. Therefore the results were discussed both in case of scattering of two electron beams, as well as scattering off a target, which is more suitable for measurement.

In some configurations there might be a large difference between the nonrelativistic value of the correlation function and its value for a few MeV beam energy. Observing relativistic effects, and therefore investigation of the fundamental properties of quantum entanglement in the relativistic regime, is possible even in the simplest case of a polarized electron beam scattering off an unpolarized target.

Nevertheless, the correlation function can take much larger absolute values if both electrons are polarized, which makes this configuration better suited for experimental verification of the predictions of relativistic quantum mechanics. For a polarized target the largest absolute value of the relativistic correlation function is achieved when both scattering electrons are highly polarized in the same direction and the **a** and **b** vectors are outside the Møller scattering plane. Configurations resulting in even higher absolute values can be found for two polarized beams, but the analysis of this case is more complicated due to the increase in the number of free parameters describing the system. (The correlation function depends both on the configuration of the **a** and **b** vectors and the angle at which the beams collide.)

The analysis has shown that violation of the Bell-type inequalities is possible for relativistic electrons originating from Møller scattering of two polarized beams, contrary to the case of an electron beam scattered off an unpolarized target. Violation of the Bell-type inequalities has not yet been observed for relativistic particles with mass.

ACKNOWLEDGMENTS

This work was supported from the funds of the National Science Centre: (1) as a part of the research project, Contract/Decision No. DEC-2012/06/M/ST2/00430; (2) as a postdoc project (M.W.) Contract/Decision No. DEC-2013/08/S/ST2/00551; (3) as a part of the research project (P.C., J.R.) Contract/Decision No. 2014/15/B/ST2/00117. P.C. and J.R. were also supported by University of Lodz.

APPENDIX: PARAMETRIZATION

We consider Møller scattering of two polarized electrons with four-momenta p_1 and p_2, respectively. Let us denote the angle between their three-momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ as γ. Without loss of generality one can assume that the scattering takes place in the XZ plane and that one of the particles (e.g., the first one) propagates along the Z direction (see Fig. 11). It implies

$$\mathbf{p}_1 = \sqrt{p_1^{0^2} - m^2}\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{A1}$$

$$\mathbf{p}_2 = \sqrt{p_2^{0^2} - m^2}\begin{pmatrix}\sin\gamma\\0\\\cos\gamma\end{pmatrix}, \tag{A2}$$

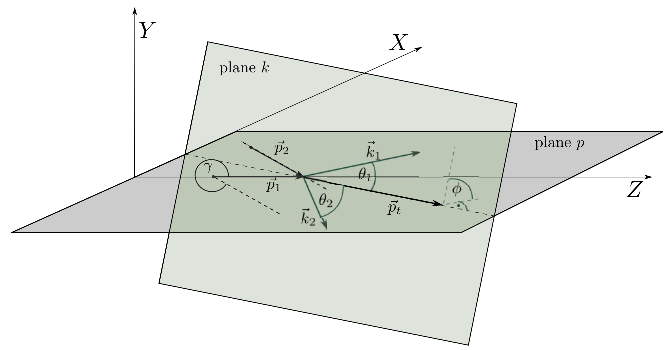

FIG. 11. The figure illustrates the kinematics of the Møller scattering for two electrons with three-momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, colliding at angle γ. The scattering takes place in the XZ plane (denoted p plane). Outgoing particles of three-momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ propagate in the k plane, at angle ϕ to the aforementioned plane p. θ_1 and θ_2 are the angles between the total momentum direction $\mathbf{p}_t$, and the $\mathbf{k}_1$ and $\mathbf{k}_2$ vectors, respectively. The three-momentum conservation law implies that the p and k planes intersect along the $\mathbf{p}_t$ direction.

where m stands for the electron mass. The total momentum $\mathbf{p}_t = \mathbf{p}_1 + \mathbf{p}_2$ has the following form:

$$\mathbf{p}_t = \begin{pmatrix}\sin\gamma\sqrt{p_2^{0^2} - m^2}\\0\\\cos\gamma\sqrt{p_2^{0^2} - m^2} + \sqrt{p_1^{0^2} - m^2}\end{pmatrix}. \tag{A3}$$

Let us now consider the particles after the scattering and denote their four-momenta as k_1 and k_2. Their three-momenta, $\mathbf{k}_1$ and $\mathbf{k}_2$, determine a plane (k) which is at an angle ϕ to the XZ scattering plane (p). Planes p and k intersect along the $\mathbf{p}_t$ direction. Now let us rotate the reference frame XYZ in the p plane so that the new $\tilde{Z}$ axis coincides with the $\mathbf{p}_t$ direction. In such a frame ($\tilde{X}Y\tilde{Z}$), the total momentum vector has an especially simple form:

$$\tilde{\mathbf{p}}_t = |\mathbf{p}_t|\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{A4}$$

where

$$|\mathbf{p}_t| = \sqrt{p_1^{0^2} + p_2^{0^2} - 2m^2 + 2\sqrt{p_1^{0^2} - m^2}\sqrt{p_2^{0^2} - m^2}\cos\gamma}, \tag{A5}$$

and the vectors $\tilde{\mathbf{k}}_1$ and $\tilde{\mathbf{k}}_2$ read

$$\tilde{\mathbf{k}}_1 = \sqrt{k_1^{0^2} - m^2}\begin{pmatrix}\sin\theta_1\cos\phi\\\sin\theta_1\sin\phi\\\cos\theta_1\end{pmatrix}, \tag{A6}$$

$$\tilde{\mathbf{k}}_2 = \sqrt{k_2^{0^2} - m^2}\begin{pmatrix}-\sin\theta_2\cos\phi\\-\sin\theta_2\sin\phi\\\cos\theta_2\end{pmatrix}, \tag{A7}$$

where θ_1 and θ_2 are the angles between the total momentum vector and $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. One can easily notice that the condition $\mathbf{p}_t = \tilde{\mathbf{k}}_1 + \tilde{\mathbf{k}}_2$ implies

$$\sin\theta_2 = \frac{\sqrt{k_1^{02} - m^2}}{\sqrt{k_2^{02} - m^2}} \sin\theta_1, \tag{A8}$$

and

$$\cos\theta_2 = \pm\frac{\sqrt{k_2^{02} - k_1^{02}\sin^2\theta_1 - m^2\cos^2\theta_1}}{\sqrt{k_2^{02} - m^2}}. \tag{A9}$$

The relation

$$|\mathbf{p}_t| = \sqrt{k_1^{02} - m^2}\cos\theta_1 + \sqrt{k_2^{02} - m^2}\cos\theta_2 \tag{A10}$$

implies

$$\begin{aligned} k_1^0 = {} & \frac{1}{|\mathbf{p}_t|^2\cos^2\theta_1 - \left(p_1^0 + p_2^0\right)^2}\Big\{\left(p_1^0 + p_2^0\right)\left(\sqrt{p_1^{02} - m^2}\sqrt{p_2^{02} - m^2}\cos\gamma - m^2 - p_1^0 p_2^0\right) \\ & - |\mathbf{p}_t|\cos\theta_1\sqrt{\left(\sqrt{p_1^{02} - m^2}\sqrt{p_2^{02} - m^2}\cos\gamma - m^2 - p_1^0 p_2^0\right)^2 + m^2\left[|\mathbf{p}_t|^2\cos^2\theta_1 - \left(p_1^0 + p_2^0\right)^2\right]}\Big\}. \end{aligned} \tag{A11}$$

Since $k_2^0 = p_1^0 + p_2^0 - k_1^0$, the only free parameters are θ_1 and the angle ϕ, which can take arbitrary values. After rotating $\tilde{\mathbf{k}}_1$ and $\tilde{\mathbf{k}}_2$ back to the XYZ frame, we get

$$\mathbf{k}_1 = \sqrt{k_1^{02} - m^2}\begin{pmatrix} \frac{\left(\sqrt{p_2^{02}-m^2}\cos\gamma + \sqrt{p_1^{02}-m^2}\right)\sin\theta_1\cos\phi + \sqrt{p_2^{02}-m^2}\sin\gamma\cos\theta_1}{\sqrt{p_1^{02}+p_2^{02}-2m^2+2\sqrt{p_1^{02}-m^2}\sqrt{p_2^{02}-m^2}\cos\gamma}} \\ \sin\theta_1\sin\phi \\ \frac{(\sqrt{p_2^{02}-m^2}\cos\gamma + \sqrt{p_1^{02}-m^2})\cos\theta_1 - \sqrt{p_2^{02}-m^2}\sin\gamma\sin\theta_1\cos\phi}{\sqrt{p_1^{02}+p_2^{02}-2m^2+2\sqrt{p_1^{02}-m^2}\sqrt{p_2^{02}-m^2}\cos\gamma}} \end{pmatrix} \tag{A12}$$

$$\mathbf{k}_2 = \sqrt{k_2^{02} - m^2}\begin{pmatrix} \frac{-\left(\sqrt{p_2^{02}-m^2}\cos\gamma + \sqrt{p_1^{02}-m^2}\right)\sin\theta_2\cos\phi + \sqrt{p_2^{02}-m^2}\sin\gamma\cos\theta_2}{\sqrt{p_1^{02}+p_2^{02}-2m^2+2\sqrt{p_1^{02}-m^2}\sqrt{p_2^{02}-m^2}\cos\gamma}} \\ -\sin\theta_2\sin\phi \\ \frac{(\sqrt{p_2^{02}-m^2}\cos\gamma + \sqrt{p_1^{02}-m^2})\cos\theta_2 + \sqrt{p_2^{02}-m^2}\sin\gamma\sin\theta_2\cos\phi}{\sqrt{p_1^{02}+p_2^{02}-2m^2+2\sqrt{p_1^{02}-m^2}\sqrt{p_2^{02}-m^2}\cos\gamma}} \end{pmatrix}. \tag{A13}$$

Symmetric scattering of two colliding beams corresponds to the $\theta_1 = \theta_2$ condition. In this case the formula (A11) takes the simple form

$$k_1^0 = k_2^0 = \frac{p_1^0 + p_2^0}{2} \tag{A14}$$

and

$$\cos\theta_1 = \cos\theta_2 = \frac{|\mathbf{p}_t|}{2\sqrt{\left(k_1^0\right)^2 - m^2}}. \tag{A15}$$

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán *et al.*, Nature **526**, 682 (2015).
[3] P. M. Alsing and G. J. Milburn, Quantum Inf. Comput. **2**, 487 (2002).
[4] H. Terashima and M. Ueda, Int. J. Quantum Inform. **01**, 93 (2003).
[5] A. Peres, P. F. Scudo, and D. R. Terno, Phys. Rev. Lett. **88**, 230402 (2002).
[6] A. Peres and D. R. Terno, Rev. Mod. Phys. **76**, 93 (2004).
[7] J. Rembieliński and K. A. Smoliński, Phys. Rev. A **66**, 052114 (2002).

[8] P. Caban and J. Rembieliński, Phys. Rev. A **72**, 012103 (2005).
[9] P. Caban and J. Rembieliński, Phys. Rev. A **74**, 042103 (2006).
[10] N. Friis, R. A. Bertlmann, M. Huber, and B. C. Hiesmayr, Phys. Rev. A **81**, 042114 (2010).
[11] V. Palge, V. Vedral, and J. A. Dunningham, Phys. Rev. A **84**, 044303 (2011).
[12] P. L. Saldanha and V. Vedral, Phys. Rev. A **85**, 062101 (2012).
[13] T. Debarba and R. O. Vianna, Int. J. Quant. Inf. **10**, 1230003 (2012).
[14] M. C. Palmer, M. Takahashi, and H. F. Westman, Ann. Phys. **336**, 505 (2013).
[15] P. Caban, J. Rembieliński, and M. Włodarczyk, Phys. Rev. A **77**, 012103 (2008).
[16] P. Caban, J. Rembieliński, and M. Włodarczyk, Phys. Rev. A **79**, 014102 (2009).
[17] M. Lamehi-Rachti and W. Mittig, Phys. Rev. D **14**, 2543 (1976).
[18] S. Hamieh, H. J. Wörtche, C. Bäumer, A. M. van den Berg, D. Frekers, M. N. Harakeh, J. Heyse, M. Hunyadi, M. A. de Huu, C. Polachic *et al.*, J. Phys. G: Nucl. Part. Phys. **30**, 481 (2004).
[19] H. Sakai, Phys. Rev. Lett. **97**, 150405 (2006).
[20] P. Caban, J. Rembieliński, and M. Włodarczyk, Phys. Rev. A **88**, 032116 (2013).
[21] T. J. Gay and F. B. Dunning, Rev. Sci. Instrum. **63**, 1635 (1992).
[22] J. Sromicki, K. Bodek, D. Conti, St. Kistryn, J. Lang, S. Navert, O. Naviliat-Cuncic, E. Stephan, C. Sys, J. Zejma, W. Haeberli, E. Reichert, and M. Steigerwald, Phys. Rev. Lett. **82**, 57 (1999).
[23] J. Ciborowski, P. Caban, M. Drągowski, J. Enders, Y. Fritzsche, A. Poliszczuk, J. Rembieliński, and M. Włodarczyk, Eur. Phys. J. Web Conf. (to be published).
[24] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[25] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
[26] P. Caban, J. Rembieliński, and M. Włodarczyk, Phys. Rev. A **88**, 022119 (2013).
[27] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).