Assessing the quantumness of a damped two-level system

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We perform a detailed analysis of the nonclassical properties of a damped two-level system. We compute and compare three different criteria of quantumness, the l_1 norm of coherence, the Leggett-Garg inequality, and a quantum witness based on the no-signaling in time condition. We show that all three quantum indicators decay exponentially in time as a result of the coupling to the thermal reservoir. We further demonstrate that the corresponding characteristic times are identical and given by the coherence half-life. These results quantify how violations of Leggett-Garg inequalities and nonzero values of the quantum witness are connected to the coherence of the two-level system.

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I. INTRODUCTION

The question of what genuinely distinguishes quantum from classical physics is as old as quantum theory itself [1,2]. That issue is not only of fundamental, but also of technological importance. Quantum features have indeed been shown to be a physical resource that allows performing tasks that are not possible classically [3]. Prominent examples include quantum cryptography [4], quantum teleportation [5], and quantum computing [6]. However, the boundary between the classical and the quantum worlds is notoriously blurry [7,8]. Characterizing nonclassicality is, as a result, a challenging exercise. The topic recently has attracted renewed interest in the fields of quantum biology [9,10], quantum computation [11,12], and quantum thermodynamics [13,14].

An inherent property of quantum mechanics is the coherent superposition of different states [15]. A popular example of such a quantum superposition is given by Schrödinger's famous cat that can simultaneously be both dead and alive [2]. Cat states are nowadays routinely created and studied in the laboratory [16,17]. A rigorous theoretical framework of quantum coherence as a physical resource has been developed lately [18–22] (see Ref. [23] for a review). In particular, different quantifiers of coherence, such as coherence measures and monotones, have been introduced [23]. A commonly used example of a coherence monotone is the l_1-norm $C_{l_1}(\rho) = \sum_{i\neq j} |\rho_{ij}|$, which is simply the sum of the modulus of the nondiagonal matrix elements of the density operator ρ of a system [19]. A nonvanishing $C_{l_1}(\rho)$ indicates the presence of quantum coherence in the considered basis.

Another approach to identify quantumness is to impose classical constraints that are violated by quantum theory. For instance, the classical assumptions of realism and locality lead to Bell's inequality [24]. After early successful observations of the violation of the Bell inequality by quantum systems [25–27], loophole-free experiments have been reported recently [28–30]. The violation of Bell's inequality is related to the existence of nonclassical spatial correlations between two parties. It is therefore not well suited to detect quantum behavior of a single system. The Leggett-Garg inequality, on the other hand, may be seen as a temporal analog of Bell's inequality [31,32]. It is based on the classical notions of macroscopic realism, i.e., the assumption that macroscopic systems remain in a well-defined state at all times, and of noninvasive measurability, i.e., the possibility to measure in principle the state of a system without perturbing it. A violation of the Leggett-Garg inequality reveals the presence of nonclassical temporal correlations in the dynamics of an individual system. Such a violation has experimentally been seen in an increasing number of systems in the past years from superconducting qubits to neutrinos [33–47]. It should, however, be pointed out that, although the nonviolation of Bell's inequalities is necessary and sufficient for local realism [48], the nonviolation of Leggett-Garg inequalities is a necessary but not sufficient condition for macroscopic realism [49].

A third witness for nonclassical behavior has been proposed latterly, based on the classical assumption of no-signaling in time, i.e., the idea that a measurement does not change the outcome statistics of a later measurement [50,51] (it has also been called a non-disturbing-measurement condition [40,52]). In essence, the criterion compares the dynamics of the population of a quantum system in the presence and in the absence of a measurement. A deviation between the two dynamics then signifies quantumness. Advantages of the quantum witness are that: (i) its implementation only requires two time measurements, in contrast to the three measurements usually needed to test the Leggett-Garg inequality and that (ii) it involves one-point expectations rather than two-point correlations. In addition, it provides a necessary and sufficient condition for macrorealism [49]. Experimental implementations with a single atom [46] and a superconducting flux qubit have been reported [53].

In this article, we compare the above three criteria of nonclassicality by applying them to a damped two-level system, a paradigmatic model of quantum optics [15], and condensed-matter physics [54]. We begin by solving the Markovian master equation of the dissipative two-level system in the Heisenberg picture and by computing the two-time correlation function of the Pauli operator σ_x in Sec. II. We then evaluate the coherence monotone $C_{l_1}(\rho)$ in the σ_z basis in Sec. III, derive the Leggett-Garg inequalities in Sec. IV, and calculate the quantum witness in Sec. V. We find that all three quantum indicators show an exponential decay of the quantum properties of the two-level system induced by the coupling to the external thermal reservoir. Remarkably, we establish that all three characteristic times are equal and given by the coherence half-life [55,56]. These findings clarify how violations of the Leggett-Garg inequalities and nonvanishing values of the quantum witness are related to the quantum coherence of the two-level system.

II. DAMPED TWO-LEVEL SYSTEM

We consider a two-level system weakly coupled to a bath of harmonic oscillators at temperature T, e.g., a two-level atom interacting with a thermal radiation field. In the Born-Markov approximation, the time evolution in the Heisenberg picture of a system observable $X(t)$ is governed by a Lindblad master equation of the form [57,58] (we set $\hbar = 1$ throughout for simplicity)

$$\begin{aligned}\frac{dX}{dt} &= \mathcal{L}[X]\\ &= i\frac{\omega}{2}[\sigma_z,X] + \frac{\partial X}{\partial t}\\ &\quad + \frac{\gamma_0}{2}n(\omega,T)(\sigma_-[X,\sigma_+] + [\sigma_-,X]\sigma_+)\\ &\quad + \frac{\gamma_0}{2}[n(\omega,T)+1](\sigma_+[X,\sigma_-] + [\sigma_+,X]\sigma_-). \end{aligned} \tag{1}$$

Here ω is the frequency of the two-level system, γ_0 is the spontaneous decay rate, and $n(\omega,T) = [\exp(\omega/k_B T) - 1]^{-1}$ is the thermal occupation number. We denote the total transition rate by $\gamma = \gamma_0[2n(\omega,T)+1]$.

The master equation (1) may conveniently be solved by using a superoperator formalism [59,60]. By choosing a basis consisting of the Pauli matrices σ_i $(i = x,y,z)$ and the identity operator I, the matrix representation of the Liouvillian superoperator $\mathcal{L}$ reads

$$\frac{d}{dt}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ I\end{pmatrix} = \mathcal{L}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ I\end{pmatrix} = \begin{pmatrix}-\frac{\gamma}{2} & -\omega & 0 & 0\\ \omega & -\frac{\gamma}{2} & 0 & 0\\ 0 & 0 & -\gamma & -\gamma_0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ I\end{pmatrix}. \tag{2}$$

This solution may be used to compute two-point correlation functions of the system with the help of the quantum regression theorem [61]. For the time-symmetrized correlation function $C(\tau) = \langle\{\sigma_x(t),\sigma_x(t+\tau)\}\rangle/2$, we find (see Appendix A)

$$C(\tau) = \exp\left(-\frac{\gamma}{2}\tau\right)\cos(\omega\tau). \tag{3}$$

The correlation function $C(\tau)$ only depends on the time lag τ and is hence stationary. It will be useful in the derivation of the Leggett-Garg inequalities in Sec. IV.

III. l_1 NORM OF COHERENCE

In this section, we assess the quantumness of the two-level system by computing the l_1 norm of coherence $C_{l_1} = \sum_{i\neq j}|\rho_{ij}|$, arguably the simplest and most intuitive coherence monotone [19]. This quantity depends on the state of the system, described by the density operator ρ, and on the basis in which the matrix elements ρ_{ij} are evaluated. We here choose the natural basis for the problem at hand given by the eigenbasis of the Hamiltonian of the two-level system $H = (\omega/2)\sigma_z$. In the σ_z basis, the density-matrix ρ may be expressed in terms of the expectation values of the Pauli operators [57],

$$\rho = \frac{1}{2}\begin{pmatrix} I + \langle\sigma_z\rangle & \langle\sigma_x\rangle - i\langle\sigma_y\rangle\\ \langle\sigma_x\rangle + i\langle\sigma_y\rangle & I - \langle\sigma_z\rangle\end{pmatrix}. \tag{4}$$

We accordingly obtain the coherence monotone,

$$C_{l_1}(\tau) = \sqrt{\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2}. \tag{5}$$

Equation (5) may be computed from the solution (2) of the master equation (1). We select the initial state of the two-level system to be maximally coherent. As shown in Ref. [19], an example of such a state is given by

$$|\Psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle, \tag{6}$$

where d is the dimension of the system. We choose the eigenstate $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ of the Pauli operator σ_x, which is obviously of this form for $d = 2$. As a result, we obtain the l_1 norm,

$$C_{l_1}(\tau) = \exp(-\gamma\tau/2). \tag{7}$$

In general, the l_1 norm of coherence satisfies the inequality $C_{l_1} \leqslant d - 1$ [23]. The coherence monotone (7) takes its maximum possible value at $\tau = 0$ from which it decays exponentially with a characteristic time scale given by the coherence time $t_c = 2/\gamma$. The latter quantity is defined as the time at which coherence is reduced to $1/e$ times its initial value [57]. It is also advantageous to introduce the coherence half-life $\tau_c = 2\ \ln 2/\gamma$, defined as the time at which coherence decays to half its initial value [55,56].

IV. LEGGETT-GARG INEQUALITY

Let us next characterize the nonclassicality of the two-level system by using the Leggett-Garg inequality. Consider a dichotomous observable Q, which can take the values ± 1 and which is measured at three consecutive times t_1, t_2, and t_3. Based on the classical assumptions of macroscopic realism and noninvasive measurability, one can derive the inequality [31,32],

$$K_3 = \langle Q(t_2)Q(t_1)\rangle + \langle Q(t_3)Q(t_2)\rangle - \langle Q(t_3)Q(t_1)\rangle \leqslant 1. \tag{8}$$

Quantum theory violates the above inequality. A value of the Leggett-Garg function K_3 above unity is thus a signature of nonclassical behavior. For $Q = \sigma_x$ and equally spaced measurements with separation $\tau/2$, we find

$$K_3(\tau) = -C(\tau) + 2C(\tau/2), \tag{9}$$

where $C(\tau)$ is the quantum correlation function (3). Equation (9) is shown in Fig. 1 for the case of an isolated two-level system ($\gamma = 0$). We observe that the function K_3 takes on values that are larger than one (shaded area), revealing quantum properties. However, owing to the oscillatory nature of K_3, we also note values that are smaller than one, even though the dynamics of the system is coherent in the absence of the thermal

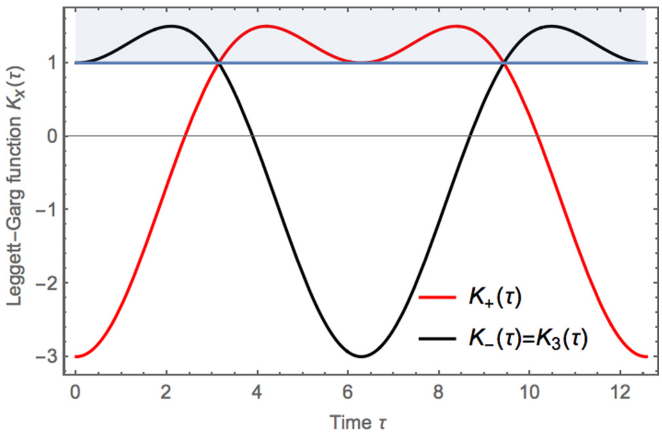

FIG. 1. Leggett-Garg functions $K_+(\tau)$ (red) and $K_3(\tau) = K_-(\tau)$ (black) Eqs. (9)–(11) as a function of the time τ for an isolated two-level system ($\gamma = 0$). The blue shaded area corresponds to the classically forbidden regime indicated by a violation of the Leggett-Garg inequalities (9)–(11). The frequency is set to $\omega = 1$ as in the other figures.

reservoir. The dynamics of the system may thus be quantum, even though the Leggett-Garg inequality is not violated.

The above-mentioned problem can here be solved by considering Leggett-Garg-type inequalities introduced in Ref. [62]. Based on the classical assumption of macroscopic realism and the condition of stationarity, the following inequalities hold for Markovian systems:

$$K_+(\tau) = -C(\tau) - 2C(\tau/2) \leqslant 1 \tag{10}$$

$$K_-(\tau) = -C(\tau) + 2C(\tau/2) \leqslant 1. \tag{11}$$

These inequalities are violated by unitary quantum dynamics; note that $K_-(\tau) = K_3(\tau)$ [63]. As seen in Fig. 1, the two inequalities for K_+ and K_- are complementary: One being violated when the other one is not and vice versa. The addition of a second Leggett-Garg function therefore allows a complete detection of the nonclassical properties of the two-level system for $\gamma = 0$. An experimental violation of

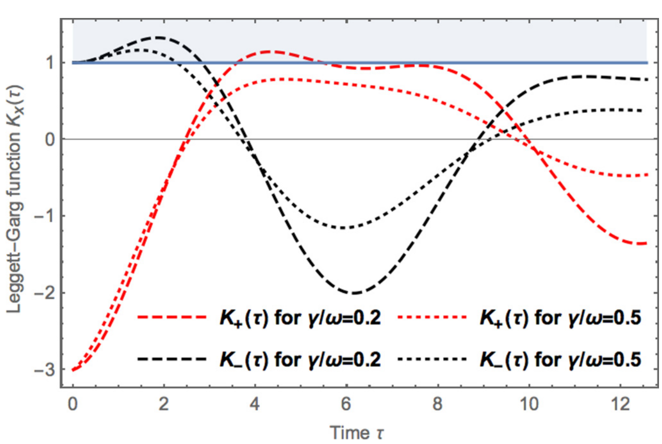

FIG. 2. Leggett-Garg functions $K_+(\tau)$ and $K_-(\tau)$ Eqs. (10) and (11) as a function of the time τ for a damped two-level system for two values of the damping coefficient γ. Above a critical value of the time τ, violations of the Leggett-Garg inequalities (10) and (11) are not possible, and the dynamics will be classical.

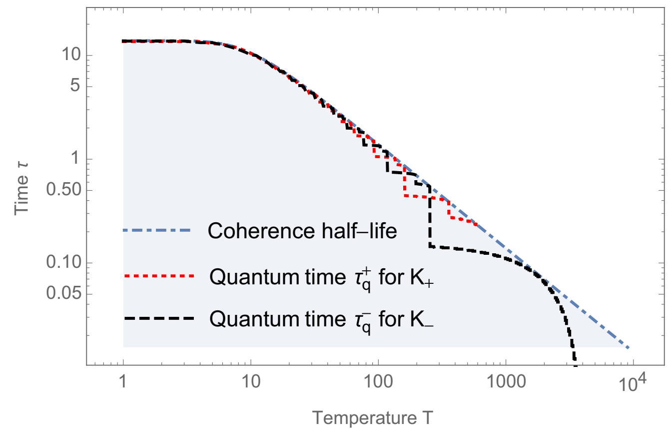

FIG. 3. Quantum times $\tau_q^\pm$ (red dotted and black dashed lines) Eq. (12) defined as the maximum time τ for which a violation of the Leggett-Garg inequalities (10) and (11) is possible as a function of the temperature T of the thermal reservoir. Both quantum times $\tau_q^\pm$ are bounded from above by the decoherence half-life τ_c (blue dot-dashed line) Eq. (7). Parameters are here $\omega = 20$ and $\gamma_0 = 0.01$.

the two inequalities (10) and (11) has been described in Refs. [38,39,45].

Figure 2 displays the Leggett-Garg functions $K_+(\tau)$ and $K_-(\tau)$ for increasing values of the coupling constant γ. The interaction with the thermal reservoir leads to a damping of the oscillations of the two functions. After a certain maximal measurement spacing, no further violations of the Leggett-Garg inequalities (10) and (11) will be observed, and the dynamics of the two-level system will be classical. We accordingly introduce a quantum time $\tau_q^\pm$, defined as the largest measurement time τ, between the first and the last measurements for which a violation of the inequalities (10) and (11) may occur

$$\tau_q^\pm = \max\{\tau | K_\pm(\tau) \geqslant 1\}. \tag{12}$$

The times $\tau_q^\pm$ characterize the quantum-to-classical transition of the two-level system as quantum features are only possible for times smaller than $\tau_q^\pm$.

Figure 3 shows the numerically determined quantum times $\tau_q^\pm$ as a function of the temperature T of the thermal reservoir. As expected the quantum times decrease with increasing temperature, indicating that nonclassical properties (shaded area) are destroyed faster when the system is coupled to a hot environment. We remark that the times $\tau_q^\pm$ are step functions (owing to the oscillatory nature of the Leggett-Garg functions K_+ and K_-) and that the decay with temperature of the height of the steps is precisely given by the coherence half-life $\tau_c = 2 \ln 2/\gamma$ of the coherence monotone C_{l_1} Eq. (7). We may hence conclude that violations of the Leggett-Garg inequalities (10) and (11) occur until the coherence induced by a first measurement has decayed to half its initial value at the time of a subsequent measurement.

V. QUANTUM WITNESS

We finally analyze the quantum properties of the damped two-level system by employing the quantum witness which has been introduced in two slightly different ways in Refs. [50,51].

Following Ref. [50], we consider a d-level system and denote by $p_n(t_0)$ its probability to be at $t = t_0$ in the classical state n $(1 \leqslant n \leqslant d)$. The probability to find the system in state m at time t is [50]

$$\bar{p}_m(t) = \sum_{n=1}^{d} \Omega_{mn}(t,t_0) p_n(t_0), \tag{13}$$

where the propagator $\Omega_{mn}(t,t_0) = p(m,t|n,t_0)$ gives the probability of a transition from state n to state m in time $t - t_0$ (the bar emphasizes that state n is classical). The quantum witness is then defined as [50]

$$\mathcal{W}_q = \left| p_m(t) - \sum_{n=1}^{d} \Omega_{mn}(t,t_0) p_n(t_0) \right|. \tag{14}$$

A nonzero value of the quantum witness $\mathcal{W}_q > 0$ reveals the nonclassicality of the initial state. Compared to the Leggett-Garg inequality (8), the condition of noninvasive measurability is here replaced by the requirement to perform an ideal state preparation of each state n and m.

The same expression may be obtained directly from the classical no-signaling in time condition [51]. Consider two observables A and B, respectively, measured at time $t = t_0$ and at time $t > 0$. The measurement outcome n of A is obtained with probability $p_n(t_0)$, whereas the measurement outcome m of B is obtained with probability $p'_m(t)$. For a joint measurement of the two observables, the probability of obtaining m in the second measurement is given by

$$p'_m(t) = \sum_{n=1}^{d} p(m,t|n,t_0) p_n(t_0), \tag{15}$$

with the conditional probability $p(m,t|n,t_0)$. In the absence of the first measurement on A, the probability of outcome m of B is denoted by $p_m(t)$. According to the classical no-signaling in time assumption, the measurement of A should have no influence on the statistical outcome of the later measurement of B and $p'_m(t) = p_m(t)$. The quantum witness then is defined as the difference $\mathcal{W}_q = |p_m(t) - p'_m(t)|$, which is identical to Eq. (14).

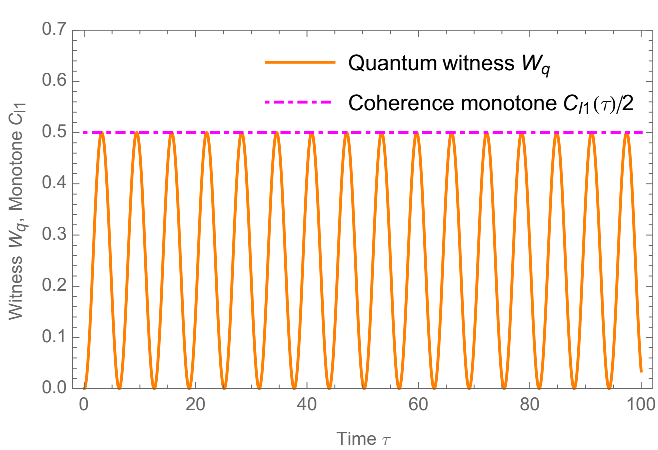

FIG. 4. Quantum witness $\mathcal{W}_q$ (orange solid line) Eq. (14) as a function of the time τ for an isolated two-level system ($\gamma = 0$). The quantum witness $\mathcal{W}_q$ is bounded from above by the coherence monotone $C_{l_1}(\tau)/2$ (purple dot-dashed line) Eq. (7).

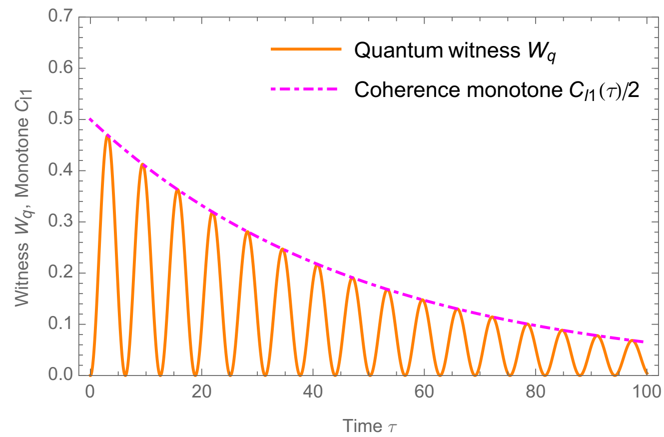

FIG. 5. Quantum witness $\mathcal{W}_q$ (orange solid line) Eq. (14) as a function of the time τ for a damped two-level system ($\gamma = 0.04$). The quantum witness $\mathcal{W}_q$ is bounded from above by the coherence monotone $C_{l_1}(\tau)/2$ (purple dot-dashed line) Eq. (7).

The quantum witness (14) may be evaluated for the dissipative two-level system by using the solution (2) of the master equation (1). The system is prepared initially in the $|+\rangle$ state at $t = 0$. At time $t = \tau/2$, a first nonselective measurement in the σ_x basis is performed or not, whereas at time $t = \tau$ the projector $\Pi_+ = |+\rangle\langle+|$ is measured. By explicitly computing the propagator $\Omega_{mn}(\tau,\tau/2)$, we find (see Appendix B)

$$\mathcal{W}_q(\tau) = \tfrac{1}{2} e^{-\gamma\tau/2} \sin^2(\omega\tau/2). \tag{16}$$

In the case of unitary dynamics ($\gamma = 0$), the quantum witness (16) reaches its maximum value of $\mathcal{W}_q^{\max} = 1 - 1/d = 1/2$ at times $\omega\tau = n\pi$ with $n \in \mathbb{N}$ [64]. We note that the maximum value of $\mathcal{W}_q^{\max}$ is equal to $C_{l_1}(\tau)/2$ (see Fig. 4). Moreover, at times $\omega\tau = n\pi$, the two-level system is in a superposition of eigenstates of σ_y and is therefore maximally disturbed by a measurement of σ_x (see Appendix C). For nonunitary dynamics ($\gamma \neq 0$), the maxima of the quantum witness decay exponentially in time with a characteristic time again equal to the coherence half-life $\tau_c = 2 \ln 2/\gamma$ of the coherence monotone C_{l_1} Eq. (7). We further observe that $\mathcal{W}_q(\tau) \leqslant C_{\ell_1}(\tau)/2$ and that the latter upper bound corresponds exactly to the envelop of the oscillatory quantum witness (see Fig. 5).

We finally remark that the quantum witness is here directly related to the expectation value of the σ_y operator,

$$\mathcal{W}_q(\tau) = \tfrac{1}{2} \langle \sigma_y(\tau/2) \rangle^2. \tag{17}$$

This is an interesting result that may simplify the experimental detection of the nonclassicality of a damped two-level system: Instead of two measurements in the σ_x basis required to realize the quantum witness, single measurements of σ_y after a suitable state preparation along σ_x should be sufficient.

VI. SUMMARY

We have presented a comprehensive examination of the quantum signatures of a damped two-level system. We have derived explicit expressions for the l_1 norm of coherence C_{l_1}, the Leggett-Garg functions K_+ and K_-, and the quantum witness $\mathcal{W}_q$ based on the no-signaling in time condition. We have shown that all three quantum indicators allow for identifying a clear boundary between quantum and classical behaviors, defined by a unique characteristic time given by the coherence half-life τ_c. This clarifies in a quantitative manner how violations of the Leggett-Garg inequalities and nonzero values of the quantum witness are linked to the existence of coherence in the system. There exists, however, a crucial qualitative difference among the three quantifiers of nonclassicality. The coherence half-life characterizes the exponential temporal decay of both the l_1 norm of coherence and the quantum witness; it thus corresponds to a soft border between quantum and classical properties. By contrast, the coherence half-life defines a sharp transition for the Leggett-Garg inequalities beyond which quantum features abruptly disappear, akin to so-called sudden-death behaviors [65]. We finally mention that each of the three tests of quantumness faces different experimental challenges: full state tomography for the l_1 norm of coherence, the noninvasive measurement of two-time correlation functions for the Leggett-Garg inequalities, and ideal state preparation for the quantum witness. The noted direct connection (17) of the quantum witness to the expectation value of the σ_y Pauli operator may simplify such an experimental test.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE CORRELATION FUNCTION

In this appendix, we compute the two-point correlation function $C(\tau)$ Eq. (3). According to the quantum regression theorem, the equation of motion for the two-time correlation function is the same as that for the corresponding one-time function in the limit of weak coupling [57,61]. We thus have in general for operators O and A,

$$\frac{d}{d\tau}\langle \hat{O}(t)\vec{A}(t+\tau)\rangle = \mathcal{L}\langle \hat{O}(t)\vec{A}(t+\tau)\rangle. \tag{A1}$$

Considering only the upper left submatrix of the superoperator $\mathcal{L}$ Eq. (2), we find

$$\frac{d}{d\tau}\vec{C} = \begin{pmatrix} -\frac{\gamma}{2} & -\omega \\ \omega & -\frac{\gamma}{2} \end{pmatrix}\vec{C}(\tau), \tag{A2}$$

where the correlation vector $\vec{C}(\tau)$ is defined as

$$\vec{C}(\tau) = \begin{pmatrix} \langle \sigma_x(t)\sigma_x(t+\tau)\rangle \\ \langle \sigma_x(t)\sigma_y(t+\tau)\rangle \end{pmatrix}. \tag{A3}$$

The solution to Eq. (A2) is given by

$$\vec{C}(\tau) = \begin{pmatrix} e^{-(\gamma\tau/2)}\cos(\omega\tau) & -e^{-(\gamma t/2)}\sin(\omega\tau) \\ e^{-(\gamma\tau/2)}\sin(\omega\tau) & e^{-(\gamma t/2)}\cos(\omega\tau) \end{pmatrix}\vec{C}(0), \tag{A4}$$

with the initial condition,

$$\vec{C}(0) = \begin{pmatrix} \langle \sigma_x\sigma_x\rangle(t) \\ \langle \sigma_x\sigma_y\rangle(t) \end{pmatrix} = \begin{pmatrix} 1 \\ i\langle \sigma_z\rangle(t) \end{pmatrix}. \tag{A5}$$

The last equality is a result of the algebraic properties of the Pauli operators. The time-symmetrized correlation function $C(\tau) = \langle\{\sigma_x(t),\sigma_x(t+\tau)\}\rangle/2$ is equal to the real part of the above correlation function and reads

$$C(\tau) = \exp\left(-\frac{\gamma}{2}\tau\right)\cos(\omega\tau). \tag{A6}$$

APPENDIX B: CALCULATION OF THE QUANTUM WITNESS

In this appendix, we evaluate the quantum witness $\mathcal{W}_q$ Eq. (16). In order to first compute the propagator Ω, it is convenient to express the Liouville superoperator $\mathcal{L}$ in a basis consisting of the projectors $\Pi_\pm$ onto the σ_x eigenstates $|\pm\rangle$ and the Pauli operators σ_x and σ_y (instead of the basis σ_x, σ_y, σ_z, and I used previously). In that basis, the master equation (1) takes the form

$$\frac{d}{dt}\begin{pmatrix} \Pi_+ \\ \Pi_- \\ \sigma_y \\ \sigma_z \end{pmatrix} = \mathcal{L}\begin{pmatrix} \Pi_+ \\ \Pi_- \\ \sigma_y \\ \sigma_z \end{pmatrix} = \begin{pmatrix} -\frac{\gamma}{4} & \frac{\gamma}{4} & -\frac{\omega}{2} & 0 \\ \frac{\gamma}{4} & -\frac{\gamma}{4} & \frac{\omega}{2} & 0 \\ \omega & -\omega & -\frac{\gamma}{2} & 0 \\ -\gamma_0 & -\gamma_0 & 0 & -\gamma \end{pmatrix}\begin{pmatrix} \Pi_+ \\ \Pi_- \\ \sigma_y \\ \sigma_z \end{pmatrix}. \tag{B1}$$

The formal solution of Eq. (B1) is

$$e^{\mathcal{L}t} = \begin{pmatrix} \frac{1}{2}[1+e^{-(\gamma t/2)}\cos(\omega t)] & \frac{1}{2}[1-e^{-(\gamma t/2)}\cos(\omega t)] & -\frac{1}{2}e^{-(\gamma t/2)}\sin(\omega t) & 0 \\ \frac{1}{2}[1-e^{-(\gamma t/2)}\cos(\omega t)] & \frac{1}{2}[1+e^{-(\gamma t/2)}\cos(\omega t)] & \frac{1}{2}e^{-(\gamma t/2)}\sin(\omega t) & 0 \\ e^{-(\gamma t/2)}\sin(\omega t) & -e^{-(\gamma t/2)}\sin(\omega t) & e^{-(\gamma t/2)}\cos(\omega t) & 0 \\ \frac{(-1+e^{-\gamma t})\gamma_0}{\gamma} & \frac{(-1+e^{-\gamma t})\gamma_0}{\gamma} & 0 & e^{-\gamma t} \end{pmatrix}. \tag{B2}$$

Assuming that the system is initially at $t=0$ in state $|+\rangle$, the time evolution of the Π_+ operator follows as:

$$\Pi_+(t) = \tfrac{1}{2}[1+e^{-(\gamma t/2)}\cos(\omega t)]. \tag{B3}$$

The (quantum) probability $p_+(\tau)$ is then simply the expectation $\langle \Pi_+(\tau)\rangle$. On the other hand, the propagator $\Omega_{mn}(\tau,\tau/2)$ with $(m,n)=(+,-)$ is described by the upper left 2×2 matrix of the full propagator Eq. (B2),

$$\Omega\left(\tau,\frac{\tau}{2}\right) = \frac{1}{2}\begin{pmatrix} 1+e^{-(\gamma\tau/4)}\cos\left(\omega\frac{\tau}{2}\right) & 1-e^{-(\gamma\tau/4)}\cos\left(\omega\frac{\tau}{2}\right) \\ 1-e^{-(\gamma\tau/4)}\cos\left(\omega\frac{\tau}{2}\right) & 1+e^{-(\gamma\tau/4)}\cos\left(\omega\frac{\tau}{2}\right) \end{pmatrix}. \tag{B4}$$

The (classical) probabilities to find the two-level system in states $|\pm\rangle$ is at $t = \tau/2$ are further given by

$$\begin{pmatrix} p'_+ \\ p'_- \end{pmatrix}\left(\frac{\tau}{2}\right) = \Omega\left(\frac{\tau}{2},0\right)\begin{pmatrix} p'_+ \\ p'_- \end{pmatrix}(0). \tag{B5}$$

Combining these two expressions, the classical probability to find the system in state $|+\rangle$ at $t = \tau$ is

$$p'_+(\tau) = \frac{1}{2}\left[1 + e^{-(\gamma t/2)}\cos^2\left(\frac{\omega t}{2}\right)\right]. \tag{B6}$$

Equations (B3) and (B6) finally lead to the witness,

$$\mathcal{W}_q = |\langle\Pi_+(\tau)\rangle - p'_+(\tau)| = \frac{1}{2}e^{(-\gamma/2)\tau}\sin^2\left(\frac{\omega}{2}\tau\right). \tag{B7}$$

APPENDIX C: MAXIMAL MEASUREMENT DISTURBANCE

For an isolated two-level system ($\gamma = 0$), the quantum witness $\mathcal{W}_q$ Eq. (16) reaches it maximal value when $\omega\tau = n\pi$ with $n \in \mathbb{N}$ [64]. This condition corresponds to a Larmor precession of the system from the initial σ_x eigenstate to (a mixture of) eigenstates of σ_y prior to the first measurement. It is intuitively clear that measuring σ_x while the system is in a σ_y eigenstate will lead to a strong disturbance. We will here show that the disturbance is maximal by evaluating the Hilbert-Schmidt norm of the commutator of these observables. We consider two normed operators A,B of a two-level system,

$$A = \frac{1}{\sqrt{2}}\left(\alpha_0 I + \sum_{i=1}^{3}\alpha_i\sigma_i\right), \quad B = \frac{1}{\sqrt{2}}\left(\beta_0 I + \sum_{j=1}^{3}\beta_j\sigma_j\right). \tag{C1}$$

The Hilbert-Schmidt norm $\|A\|_2^2$ is then simply [3]

$$\|A\|_2^2 = \mathrm{Tr}(A^\dagger A) = \sum_{i=0}^{3}|\alpha_i|^2. \tag{C2}$$

On the other hand, the commutator of A,B reads

$$[A,B] = \sum_{i,j=1}^{3}\alpha_i\beta_j[\sigma_i,\sigma_j], \tag{C3}$$

with $[\sigma_i,\sigma_j] = \sum_k 2i\varepsilon_{ijk}\sigma_k$, where ε_{ijk} is the Levi-Civita symbol. We have here used the bilinearity of the commutator. By further introducing a new set of coefficients $\lambda_k = 2i\sum_{i,j=1}^{3}\alpha_i\beta_j\varepsilon_{ijk}$, we may write Eq. (C3) as

$$[A,B] = \sum_{k=1}^{3}\lambda_k\sigma_k. \tag{C4}$$

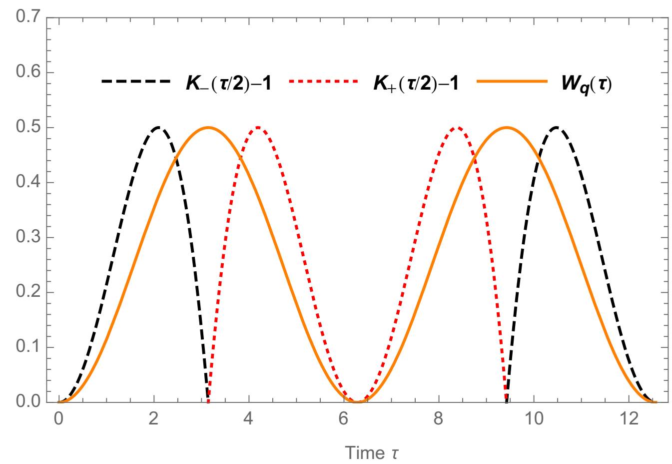

FIG. 6. Comparison among the quantum witness $\mathcal{W}_q$ (orange solid line) Eq. (14), the Leggett-Garg functions K_+ (red dotted line), and K_- (black dashed line) Eqs. (10) and (11) for an isolated two-level system ($\gamma = 0$). Maximal measurement disturbance coincides with maxima of the witness and the quantum to classical boundary $K_\pm = 1$ of the Leggett-Garg functions.

The Hilbert-Schmidt norm of the commutator follows as

$$\|[A,B]\|_2^2 = \sum_{k=1}^{3}|\lambda_k|^2. \tag{C5}$$

This expression is proportional to the squared modulus of the cross product of A,B if we interpret the operators as vectors with components $A_i = \alpha_i$, $B_i = \beta_i$. For fixed norms of the operators A,B, the maximum value is obtained by choosing orthogonal operators, corresponding to a vanishing Hilbert-Schmidt product $\langle A,B\rangle = \mathrm{Tr}(A^\dagger B) = 0$. For the normed operators $\sigma_x/\sqrt{2}$ and $\sigma_y/\sqrt{2}$, we find $\langle\sigma_x,\sigma_y\rangle/2 = \mathrm{Tr}(\sigma_x\sigma_y)/2 = 0$ and $\|[\sigma_x/\sqrt{2},\sigma_y/\sqrt{2}]\|_2^2 = 2$. The operators σ_x and σ_y are thus maximally incompatible.

We additionally observe that the maxima of the witness correspond to Leggett-Garg functions $K_\pm(\tau) = 1$, that is, to the quantum to classical boundary (see Fig. 6). The Leggett-Garg inequality indeed quantifies measurement-induced correlations, whereas the quantum witness quantifies measurement disturbance. Measurement disturbability is a prerequisite to induce correlations by a measurement. However, the maximally disturbing measurement, which corresponds to measuring σ_x in a σ_y eigenstate, does not lead to correlations between measurements. Any such measurement will yield either eigenvalue of σ_x with equally probability, independent of the state preparation or initial measurement. This point thus corresponds to the quantum to classical boundary of the Leggett-Garg functions. We finally note that the Leggett-Garg inequality and the quantum witness here identify the same nonclassical domain.

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