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Exact non-Markovian master equation for the spin-boson and Jaynes-Cummings models

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We provide the exact non-Markovian master equation for a two-level system interacting with a thermal bosonic bath, and we write the solution of such a master equation in terms of the Bloch vector. We show that previous approximated results are particular limits of our exact master equation. We generalize these results to more complex systems involving an arbitrary number of two-level systems coupled to different thermal baths, providing the exact master equations also for these systems. As an example of this general case we derive the master equation for the Jaynes-Cummings model.

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Understanding the dynamics of a two-level system (TLS) coupled to an external environment is a ubiquitous problem in physics, chemistry, and biology: Quantum optics, charge-transfer processes, tunneling phenomena, and light harvesting in photosynthetic systems are only a few fields where the dissipative TLS covers a crucial role [1,2]. The spin-boson model, i.e., a TLS interacting with a bosonic bath, is the paradigm for the description of these open systems [3,4]. The first step in the understanding of the spin-boson model is given by the master equation in the Markovian approximation. The validity of this master equation is restricted to those systems for which the environment can be assumed as static (i.e., the environment time scale is much shorter than that of the TLS). However, there are many processes where a Markov description is not sufficient [5]. In order to describe these systems one needs to consider a nonstatic bath, i.e., a bath that keeps track of the interaction with the TLS. Accordingly, some memory effects build up, and the dynamics is non-Markovian. Several tentatives have been made to provide a non-Markovian master equation for the spin-boson model, exploiting, e.g., the noninteracting-blip approximation [3], the time convolutionless technique (TCL) [1,6], or the stochastic approach [7]. However, only approximated results were obtained. The lack of an exact analytical description leads to investigate the problem by means of numerical techniques, among which we mention hierarchical equations of motion [8], a quasiadiabatic path integral [9], effective modes [10], real-time renormalization group (RG) in frequency space [11,12], real-time functional RG [12,13], time-dependent density-matrix RG [14], and time-dependent numerical RG [15,16].

Another paradigmatic model is the (multimode) Jaynes-Cummings model [17], which differs from the spin-boson model only for the type of coupling between the TLS and the environment. This model is widely used in quantum optics and cavity QED [18]. Also the derivation of a non-Markovian master equation for the Jaynes-Cummings model proved very difficult: An exact result has been obtained only for a bath in the ground state [1,19], whereas for a general thermal bath only approximated master equations are known.

In this Rapid Communication, we provide the solution for this long-standing problem by deriving the exact (analytical) non-Markovian master equation for the spin-boson model and by solving it in terms of the Bloch vector. Moreover, we provide the non-Markovian master equation for the Jaynes-Cummings model, and we extend our results to more complicated systems, such as the Tavis-Cummings model [20] and the Jaynes-Cummings-Hubbard model [21].

The Hamiltonian of the spin-boson model can be written as follows [3]: $\hat{H} = \hat{H}_0 + \hat{H}_I + \hat{H}_B$, where $\hat{H}_B$ is the Hamiltonian of the bath-independent bosons and $\hat{H}_0$ and $\hat{H}_I$, respectively, are system and interaction Hamiltonians:

$$\hat{H}_0 = -\frac{1}{2}\Delta\hbar\,\hat{\sigma}^x + \frac{1}{2}\epsilon\hat{\sigma}^z, \tag{1}$$

$$\hat{H}_I = \frac{k_0}{2}\hat{\sigma}^z c^i \hat{q}_i, \tag{2}$$

where $\hat{\sigma}$'s are Pauli matrices, $\hat{q}_i$'s are the positions of the bath oscillators, and k_0, c^i are arbitrary real coupling constants. Δ and ϵ, respectively, are the detuning and dephasing constants (our result still holds if these are time-dependent functions). For this reason, when $\Delta = 0$ the model is called "pure dephasing," and when $\epsilon = 0$ is said to be "pure detuning." The Einstein sum rule is understood. We assume the initial state of the open system to be factorized and the bosonic bath to be in a thermal state at temperature T. This can fully be characterized either by the environment spectral density $J(\omega)$ or by its Hermitian two-point correlation function $D(t,s)$, which are linked by well-known expressions [1],

$$D^{\mathrm{Re}}(t,s) = \hbar\int_0^\infty d\omega\, J(\omega)\coth\left(\frac{\hbar\omega}{2k_B T}\right)\cos\omega(t-s), \tag{3}$$

$$D^{\mathrm{Im}}(t,s) = -\hbar\int_0^\infty d\omega\, J(\omega)\sin\omega(t-s), \tag{4}$$

where D^{Re} and D^{Im}, respectively, are real symmetric and imaginary antisymmetric parts of D and k_B is the Boltzmann constant. We introduce the left-right (LR) formalism [22,23], denoting by a subscript L (R) the operators acting on $\hat{\rho}$ from the left (right), e.g., $\hat{A}_L\hat{B}_R\hat{\rho} = \hat{A}\hat{\rho}\hat{B}$. In a recent paper [23] the most general trace preserving completely positive non-Markovian map $\mathcal{M}_t$ has been derived such that $\hat{\rho}_t = \mathcal{M}_t\hat{\rho}_0$.

For a bilinear system-bath interaction of the type $\hat{H}_I = \hat{A}^i\hat{\phi}_i$ (with $\hat{A}^i$ as the Hermitian system operators and $\hat{\phi}_i$ as the Hermitian linear combinations of the bath modes), in the

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interaction picture such a map reads

$$\mathcal{M}_t = T\ \exp\left\{\int_0^t d\tau \int_0^t ds\, D_{jk}(\tau,s) \times \left[\hat{A}_L^k(s)\hat{A}_R^j(\tau) - \theta_{\tau s}\hat{A}_L^j(\tau)\hat{A}_L^k(s) - \theta_{s\tau}\hat{A}_R^k(s)\hat{A}_R^j(\tau)\right]\right\}, \tag{5}$$

where $\theta_{\tau s}$ denotes the step function that is 1 for $\tau > s$ and the two-point correlation function is $D_{ij} = \mathrm{Tr}_B[\hat{\phi}_i\hat{\phi}_j\hat{\rho}_B]$. In the spin-boson interaction Hamiltonian (2), the TLS is coupled to the environment via $\frac{1}{2}k_0\hat{\sigma}^z$. Hence, one just needs to define $\hat{\phi}_z = c^i\hat{q}_i$ and perform the substitution $\hat{A} \to \frac{1}{2}k_0\hat{\sigma}^z$ (there is only one $\hat{A}$) to obtain the correct map. After some manipulation, one finds that the completely positive map describing the spin-boson model reads

$$\mathcal{M}_t = T\ \exp\left\{-\int_0^t d\tau[\hat{\sigma}_L(\tau) - \hat{\sigma}_R(\tau)] \times \int_0^\tau ds\, D(\tau,s)\hat{\sigma}_L(s) - D^*(\tau,s)\hat{\sigma}_R(s)\right\}, \tag{6}$$

where the asterisk denotes complex conjugation. In order to simplify the notation, we have dropped the index z, and we have absorbed the factor $\frac{1}{2}k_0$ in $\hat{\sigma}$. We observe that, by choosing a local correlation function $D(\tau,s) = D(\tau)\delta(\tau - s)$, one obtains the Markovian map [1],

$$\mathcal{M}_t = T\exp\left\{\int_0^t d\tau\, D(\tau)[\hat{\sigma}_L(\tau)\hat{\sigma}_R(\tau) - \hat{I}]\right\}, \tag{7}$$

where $\hat{I}$ denotes the identity operator. Differentiation of Eq. (7) provides the well-known Lindbald equation. In order to obtain the non-Markovian master equation we need to differentiate the general $\mathcal{M}_t$ of Eq. (6) and express $\dot{\mathcal{M}}_t$ in terms $\mathcal{M}_t$. This goal is hard to achieve because the double integral in the exponent of $\mathcal{M}_t$ is such that $\dot{\mathcal{M}}_t$ displays the time ordering of nonlocal arguments. This problem is overcome by exploiting Wick's theorem [24]. We expand the map $\mathcal{M}_t$ (6) in the Dyson series,

$$\mathcal{M}_t = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}M_t^n, \tag{8}$$

where $M_t^n = T[\prod_{i=1}^n \diamond_i]$ and

$$\diamond_i = \int_0^t dt_i[\hat{\sigma}_L(t_i) - \hat{\sigma}_R(t_i)] \times \int_0^{t_i} ds_i[D(t_i,s_i)\hat{\sigma}_L(s_i) - D^*(t_i,s_i)\hat{\sigma}_R(s_i)]. \tag{9}$$

By differentiating M_t^n one finds

$$\dot{M}_t^n = n[\hat{\sigma}_L(t) - \hat{\sigma}_R(t)] \times T\left[\int_0^t ds_1[D(t,s_1)\hat{\sigma}_L(s_1) - D^*(t,s_1)\hat{\sigma}_R(s_1)]\prod_{i=2}^n \diamond_i\right]. \tag{10}$$

The main difference between M_t^n and $\dot{M}_t^n$ is that the former are the time-ordered products (T products) of an even number of $\hat{\sigma}$, whereas the latter display odd-T products. Here is where Wick's theorem enters the calculations, allowing us to rewrite each $\dot{M}_t^n$ as a sum of even-T products that are eventually rewritten in terms of M_t^n. We note that different $\hat{\sigma}$'s acting on the same side of ρ ($_{LL,RR}$) anticommute with each other, whereas mixed contributions ($_{LR}$) commute

$$\{\hat{\sigma}_L,\hat{\sigma}_L\} = \{\hat{\sigma}_R,\hat{\sigma}_R\} = 0, \quad [\hat{\sigma}_L,\hat{\sigma}_R] = 0. \tag{11}$$

Accordingly, a Wick contraction is defined as follows [24]:

$$\overline{\hat{\sigma}_L(s_1)\hat{\sigma}_L(s_2)} = \overline{\hat{\sigma}_R(s_1)\hat{\sigma}_R(s_2)} = -\{\hat{\sigma}(s_1),\hat{\sigma}(s_2)\}\theta_{s_2,s_1}, \tag{12}$$

$$\overline{\hat{\sigma}_L(s_1)\hat{\sigma}_R(s_2)} = 0. \tag{13}$$

Since $\hat{H}_0$ of Eq. (1) gives linear Heisenberg equations for $\hat{\sigma}^z$, these contractions are c functions. This is a crucial feature because it implies that contractions commute with the T ordering. Moreover, according to Eqs. (12) and (13) the contraction of two $\hat{\sigma}$'s separated by a product of $n\hat{\sigma}$ between them is

$$\overline{\hat{\sigma}_L(s_1)(\cdots)\hat{\sigma}_L(s_2)} = (-1)^m\,\overline{\hat{\sigma}_L(s_1)\hat{\sigma}_L(s_2)}(\cdots), \tag{14}$$

where $m \leqslant n$ is the number of $\hat{\sigma}_L$'s contained in $(\cdots)$ (similarly for R contractions). These prescriptions allow us to rearrange the odd-T product of Eq. (10) exploiting Wick's theorem. Precisely, this is decomposed in an even-T product (that can be linked to M_t^n) plus another odd-T product of lower order with the same structure as the second line of Eq. (10). This procedure provides us with a rule that we can apply recursively to $\dot{M}_t^n$, allowing us to decompose it in terms of even-T products. The calculations are rather involved and require some delicate manipulation. We report the details of the derivation in the Supplemental Material [25]. The final result is the following integral master equation (the full notation has been restored):

$$\dot{\hat{\rho}}_t = -\frac{k_0^2}{4}\left[\hat{\sigma}_L^z(t) - \hat{\sigma}_R^z(t)\right] \times \left[\int_0^t ds\, \mathbb{D}_{zz}(t,s)\hat{\sigma}_L^z(s) - \mathbb{D}_{zz}^*(t,s)\hat{\sigma}_R^z(s)\right]\hat{\rho}_t, \tag{15}$$

where

$$\mathbb{D}_{zz} = \sum_{n=1}^{\infty}(-1)^{n-1}D_{zz(n)}. \tag{16}$$

The explicit expressions of the $D_{zz(n)}$'s are reported in Ref. [25]. The last step of our derivation is to provide a master equation that displays only operators at time t. We do so by solving the Heisenberg equations for $\hat{H}_0$: Since these are linear we can write

$$\hat{\sigma}^i(s) = b_j^i(s-t)\hat{\sigma}^j(t), \tag{17}$$

where the indices i,j run over the components x,y,z of the TLS and b is a real matrix (for explicit expressions of its entries see Ref. [25]). Substituting this expression in Eq. (15) one obtains

$$\dot{\hat{\rho}}_t = -\left[\hat{\sigma}_L^z(t) - \hat{\sigma}_R^z(t)\right]\left[B_{zi}(t)\hat{\sigma}_L^i(t) - B_{zi}^*(t)\hat{\sigma}_R^i(t)\right]\hat{\rho}_t, \tag{18}$$

with

$$B_{zi}(t) = \frac{k_0^2}{4}\int_0^t ds\, \mathbb{D}_{zz}(t,s) b_i^z(s-t). \tag{19}$$

It is interesting to observe that the operators displayed by this master equation are as follows: the coupling operator ($\hat{\sigma}^z$) and the operators which are involved in the free evolution of the coupling operator [$\hat{\sigma}^i$ through Eq. (17)]. We further stress that Eq. (18) has the same structure as the bosonic case [26]: The difference between the two cases is encoded in the structure of the functions B. Moreover, in the weak-coupling limit, these functions for the TLS and the bosonic case coincide as expected [1]. Resorting to the Schrödinger picture and writing all the terms explicitly one eventually obtains

$$\begin{aligned}\dot{\hat{\rho}}_t = &-i[\hat{H}_1(t)\hat{\rho}_t - \hat{\rho}_t\hat{H}_1^\dagger(t)] - B_{zz}^{\rm Re}(t)[\hat{\sigma}^z,[\hat{\sigma}^z,\hat{\rho}_t]]\\ &+ B_{zy}(t)\hat{\sigma}^y\hat{\rho}_t\hat{\sigma}^z + B_{zy}^*(t)\hat{\sigma}^z\hat{\rho}_t\hat{\sigma}^y\\ &+ B_{zx}(t)\hat{\sigma}^x\hat{\rho}_t\hat{\sigma}^z + B_{zx}^*(t)\hat{\sigma}^z\hat{\rho}_t\hat{\sigma}^x,\end{aligned} \tag{20}$$

where $\hat{H}_1(t) = \hat{H}_0 + B_{zx}(t)\hat{\sigma}^y - B_{zy}(t)\hat{\sigma}^x$. This is the exact non-Markovian master equation for the spin-boson model. We stress that all the functions displayed by this master equation are analytical. Moreover, if one chooses time-dependent dephasing or detuning in $\hat{H}_0$, Eq. (17) still holds and so does this master equation. The first line of Eq. (20) displays a Lamb-shifted Hamiltonian and a dephasing term which changes only the nondiagonal entries of $\hat{\rho}_t$. The tunneling dynamics is driven by the second and third lines of Eq. (20): These terms modify the populations of excited and ground states of the TLS. This master equation recovers, in the appropriate limits, the results known in the literature. For the full Hamiltonian (1) the master equation for the spin-boson model is known in the weak-coupling limit [27]: In this same limit (i.e., $\mathbb{D} = D$), our exact master equation recovers that. The only exact master equation known in the literature is the one for the pure dephasing model, described by Eq. (1) with $\Delta = 0$. The master equation for this model is quite easy to derive because $\hat{H}_0$ and $\hat{H}_I$ commute. One can easily check that under this restriction $b_x^z = b_y^z = 0$, $b_z^z = 1$, and $\mathbb{D} = D$, that substituted in Eq. (20) lead to

$$\dot{\hat{\rho}}_t = -i\frac{\epsilon}{2}[\hat{\sigma}^z,\hat{\rho}_t] - \frac{k_0^2}{4}\left(\int_0^t D(t,s)ds\right)[\hat{\sigma}^z,[\hat{\sigma}^z,\hat{\rho}_t]], \tag{21}$$

which recovers the known master equation for this model [1,28]. Another interesting special case is the pure detuning model, i.e., Eq. (1) with $\epsilon = 0$. The master equation for this model is obtained simply by setting $B_{zx} = 0$ in Eq. (20). Such an exact master equation was not known, but if we restrict ourselves to the weak-coupling limit we recover previously known approximated results [29,30]. We further stress that Eq. (20) also provides the master equation for the Rabi model [1,31]: One simply needs to consider a "one-oscillator bath" by taking a δ-correlated spectral density in Eqs. (3) and (4).

In order to solve Eq. (20) it is convenient to introduce the following identity:

$$\hat{\rho}_t = \tfrac{1}{2}[\hat{I} + \langle\sigma_i(t)\rangle\hat{\sigma}^i], \tag{22}$$

where the vector with components $\langle\sigma_i\rangle$ is known as a Bloch vector. Substituting this equation in Eq. (20), after some calculation, one finds that the Bloch vector evolves according to the following equation:

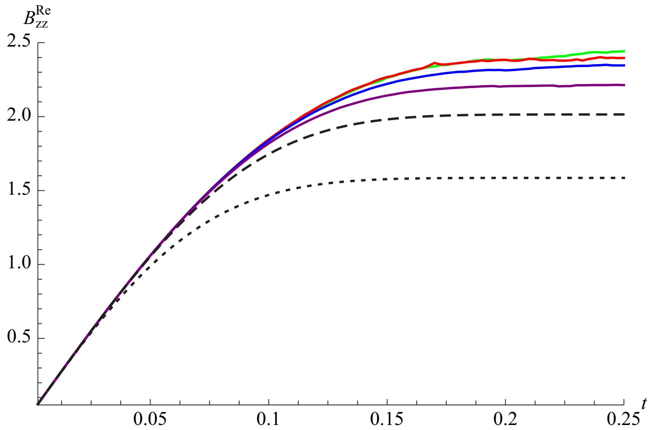

FIG. 1. Evolution of $B_{zz}^{\rm Re}$ for an increasing number of terms in the series (16) for $\mathbb{D}$. The dotted line is $n = 1$, and the dashed line is $n = 2$. The solid lines, respectively, are as follows (bottom to top): $n = 3$ (purple), $n = 4$ (blue), $n = 5$ (red), and $n = 6$ (green). Bath with ohmic spectral density and Gaussian cutoff: $J(\omega) = 2\pi\omega\, \exp[-\omega^2\Lambda^{-2}]$. Other parameters are as follows: $\epsilon = 10$, $\Delta = \epsilon$, $k_0^2 = 0.04\epsilon$, $k_BT = 0.1\epsilon$, and $\Lambda = 2\epsilon$.

$$\frac{d}{dt}\langle\sigma_i(t)\rangle = \mathcal{B}^{ij}(t)\langle\sigma_j(t)\rangle + \Sigma_i(t), \tag{23}$$

with $i,j = x,y,z$, $\Sigma = (-4B_{zy}^{\rm Im}, 4B_{zx}^{\rm Im}, 0)$, and

$$\mathcal{B} = \begin{pmatrix} -4B_{zz}^{\rm Re} & -\epsilon & 4B_{zx}^{\rm Re} \\ \epsilon & -4B_{zz}^{\rm Re} & 4B_{zy}^{\rm Re} + \hbar\Delta \\ 0 & -\hbar\Delta & 0 \end{pmatrix}. \tag{24}$$

This matrix recovers known results for the pure dephasing and pure detuning models [6]. However, unlike these special cases, the solution of the set of equations (23) with (24) is nontrivial. In general, the dynamics of $\hat{\rho}_t$ strongly depends on the bath spectral density and on the other parameters of the model. This important issue will be investigated in a dedicated forthcoming paper. Figure 1 shows the time evolution of $B_{zz}^{\rm Re}(t)$ for an increasing number of terms in the series (16) for $\mathbb{D}$ (from $n = 1$ to $n = 6$). The black lines denote previously known results: The dotted line is for the weak-coupling limit (or second-order TCL), and the dashed line is for the fourth-order TCL (known for the pure detuning model only [6]). The colored (solid) lines are the original result of this Rapid Communication: The distance between the dashed line ($n = 2$) and the green one (top solid line, $n = 6$) clearly shows how previous results are improved. Moreover, besides small numerical errors, red ($n = 5$) and green (two top solid) lines coincide, showing quite fast convergence of the series (16). The evolution of the other coefficients of the master equation (20) display a similar convergence [25].

The method we presented can be exploited to obtain more general master equations. Indeed, the map (5) provides the evolution for an interaction Hamiltonian of the type,

$$\hat{H}_I = \hat{\sigma}^i\hat{\phi}_i \tag{25}$$

[of which Eq. (2) is a special case]. The superscript i can be intended as running over different TLSs or as different components (x,y,z) of the same system (or both these options). One can then repeat the calculations previously described in the spirit of Ref. [26] and obtain the following master equation in the interaction picture:

$$\dot{\hat{\rho}}_t = -\left[\hat{\sigma}^i_L(t) - \hat{\sigma}^i_R(t)\right]\left[B_{ij}(t)\hat{\sigma}^j_L(t) - B^*_{ij}(t)\hat{\sigma}^j_R(t)\right]\hat{\rho}_t, \quad (26)$$

where we have to keep in mind that the correlation function has been promoted to a matrix D_{ij}, which implies

$$B_{ij}(t) = \int_0^t ds\, \mathbb{D}_{ik}(t,s) b^k_j(s-t). \quad (27)$$

This exact non-Markovian master equation allows describing many models of which only approximate master equations (or none) are known. Interesting examples falling in this category are the Tavis-Cummings model [20] and the Jaynes-Cummings-Hubbard model [21]. We however stress that a crucial requirement is that the free Hamiltonian $\hat{H}_0$ must provide linear Heisenberg equations, otherwise Wick's contractions would not be c functions (and the formalism would fail). Accordingly, spin chains are excluded from our treatment [25]. We exploit this general result to attain the dynamics for the Jaynes-Cummings model which covers a fundamental role in the theories of quantum optics and cavity QED [18]. The interaction Hamiltonian for this model is obtained by applying the rotating-wave approximation to Eq. (2), and it reads

$$\hat{H}_I = \hbar g\left(\hat{\sigma}^+ \sum_j \hat{a}_j + \hat{\sigma}^- \sum_j \hat{a}^\dagger_j\right), \quad (28)$$

where $\hat{\sigma}^\pm = \hat{\sigma}^x \pm i\hat{\sigma}^y$. The free Hamiltonian for this model is $\hat{H}_0 = \omega_0\hat{\sigma}^+\hat{\sigma}^-$ [our formalism also allows treating the more general $\hat{H}_0$ of Eq. (1)]. Since our formalism works with Hermitian operators, we rewrite this Hamiltonian as follows:

$$\hat{H}_I = \frac{\hbar g}{2}\left[\hat{\sigma}^x \sum_j (\hat{a}_j + \hat{a}^\dagger_j) + \hat{\sigma}^y \sum_j i(\hat{a}_j - \hat{a}^\dagger_j)\right]. \quad (29)$$

One observes that, although the Jaynes-Cummings coupling is an approximation of the standard spin-boson interaction (2), it is of the general form (25). We define $\hat{\phi}_x = \sum_j(\hat{a}_j + \hat{a}^\dagger_j)$ and $\hat{\phi}_y = i\sum_j(\hat{a}_j - \hat{a}^\dagger_j)$, and we exploit (26) to obtain

$$\begin{aligned}\dot{\hat{\rho}}_t = & -i\left(\omega_0 + B^{\mathrm{Re}}_{xy}\right)[\hat{\sigma}^+\hat{\sigma}^-,\hat{\rho}] \\ & + \left(B^{\mathrm{Re}}_{xx} - B^{\mathrm{Im}}_{xy}\right)\left(\hat{\sigma}^-\hat{\rho}\hat{\sigma}^+ - \tfrac{1}{2}\{\hat{\sigma}^+\hat{\sigma}^-,\hat{\rho}\}\right) \\ & + \left(B^{\mathrm{Re}}_{xx} + B^{\mathrm{Im}}_{xy}\right)\left(\hat{\sigma}^+\hat{\rho}\hat{\sigma}^- - \tfrac{1}{2}\{\hat{\sigma}^-\hat{\sigma}^+,\hat{\rho}\}\right). \end{aligned} \quad (30)$$

The new functions B are defined by Eq. (27), and their expressions are analytical. One can check that, if the bath is in its ground state (i.e., its temperature is zero), the following identity holds: $B^{\mathrm{Re}}_{xx} = -B^{\mathrm{Im}}_{xy}$, and Eq. (30) recovers the known master equation in this limit [1,19,32]. In Refs. [32,33] the authors provided an approximated master equation up to the fourth-order TCL for a larger class of initial bath states (namely, those commuting with the number operator). Their master equation differs from ours by a dephasing contribution of the type $\hat{\sigma}^z\hat{\rho}\hat{\sigma}^z$. Equation (30) proves that such a contribution is null for thermal states. Precisely, a coupling of the type (28) will never display a contribution, such as $\hat{\sigma}^z\hat{\rho}\hat{\sigma}^z$, because one of the two operators multiplying $\hat{\rho}$ must always be the coupling operator [as explained after Eq. (19)]. If one considers a more general $\hat{H}_0$, such as that of Eq. (1), one obtains contributions of the type $\hat{\sigma}^\pm\hat{\rho}\hat{\sigma}^z$, i.e., displaying at most one $\hat{\sigma}^z$.

In this Rapid Communication, we provided the solution of a long-standing problem, i.e., the exact non-Markovian master equation for the spin-boson model. We solved such a master equation, and we showed that our exact result recovers all known approximated results. Furthermore, we proved that the powerful formalism we developed allows for investigating more complicated systems that possibly involve more TLSs. As an example we provided the master equation for the Jaynes-Cummings model. Since the models investigated are the cornerstones for the analysis of more complicated systems, the results of this Rapid Communication will pave the way for research on such systems, both under the analytical and under the numerical points of view.

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