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Contextuality of identical particles

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There exist quantum phenomena that cannot be explained by noncontextual hidden-variable theories, yet the majority of them requires measurements that are performed on a single quantum system at a time. This fact constrains the phenomenon of contextuality to the microscopic domain. It is therefore natural to ask if quantum contextuality can be observed in measurements on collections of particles. Since particles in nature are identical, one can expect that such contextuality would be linked to bosonic and fermionic properties. Analysis of quantum contextuality in such scenarios would broaden our understanding of nonclassical effects in composite systems and perhaps would give us a hint on how to observe quantum phenomena in the macroscopic world. In this work I propose a generalization of quantum contextuality to the case of many identical particles. I show that a type of contextuality exhibited by a collection of particles (state dependent, state independent, or noncontextual) depends on their type and their number. I also discuss further properties of this generalization and identify major open questions.

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I. INTRODUCTION

Typical tests of hidden-variable (HV) models either are derived for a single indivisible system [1] or assume that subsystems are distinguishable and can be addressed individually [2]. On the other hand, the majority of common physical systems consists of many indistinguishable parts. Therefore, derivation of HV tests for collections of indistinguishable objects is necessary to understand nonclassical phenomena in realistic scenarios.

Although the HV problem has been extensively studied in various physical setups, the case of identical particles occupying a family of local modes is still barely explored. Previous research on this topic concentrated on dimension of the system rather than on its particular bosonic and fermionic properties [3], investigated the problem of outcome assignment and exclusivity [4–6], or focused on measurements emulating spin-$1/2$ observables, which were not conserving the number of particles [7].

The goal of this work is to provide a method to test a HV description of simple measurements done on a collection of bosons or fermions which occupy some number of modes. In particular, the aim is to generalize the Kochen-Specker (KS) theorem [1] and the concept of quantum contextuality to scenarios in which HVs correspond to an assignment of particle occupation numbers. The main motivation is to find a link between contextuality and indistinguishability, which are two different fundamental properties of quantum systems. An additional goal is to pioneer a development of a tool to study truly nonclassical effects in mesoscopic and macroscopic systems consisting of many indistinguishable particles.

The standard way of proving the KS theorem [1], which due to historical reasons is also known as the Bell-Kochen-Specker theorem [8], is based on an assignment of logical values to a finite set of projectors $\{\mathcal{P}_i\}_{i\in K}$ in some finite-dimensional Hilbert space. For each subset of mutually orthogonal projectors $\{\mathcal{P}_j\}_{j\in S_{\mathcal{A}}\subset K}$ having the resolution of identity one assigns the value $v(\mathcal{P}_k)=1$ to exactly one projector and the value $v(\mathcal{P}_{l\neq k})=0$ to all the remaining projectors from this subset. In this way one constructs a classical-like description of a quantum measurement $\mathcal{A}$ in which an observed event corresponds to the projector that was assigned 1. In addition, the noncontextuality assumption states that the value assigned to a projector is the same in every subset. One also requires that each projector belongs to more than one mutually orthogonal subset. The goal is to arrive at a contradiction, i.e., to show that in some subset all projectors will be assigned zero, or that more than one projector is assigned 1.

The above contradiction occurs for any state and therefore it is an example of the state-independent contextuality (SIC). There are two ways one can arrive at SIC. The first one, discussed above, relies on finding a set of projectors that cannot be assigned values 0 and 1 in a way not leading to a contradiction. Such sets are known in the literature as the KS sets [9]. The other approach relies on an inequality for a set of projectors for which 0-1 assignments may exist. These assignments are used to derive a noncontextuality bound. However, due to the properties of projectors in the set, the bound is violated by any quantum state. Such sets of observables are known as the SIC sets [10–12]. In general, KS sets are SIC sets but not all SIC sets are KS sets.

There is also another type of contextuality that occurs for specific states (state-dependent contextuality). In this case possible assignments provide a noncontextual model for some subset of states. The goal is to look for a set of projectors minimizing the subset of noncontextual states and for states lying outside of this subset [13,14].

II. THE GENERAL IDEA

Here I study the problem of a noncontextual hidden-variable model for N identical particles. The main idea is to establish a correspondence between the set of M propositions needed to prove contextuality for a single qudit and a system of N identical particles populating M modes.

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Let me first discuss the case $N = 1$. One starts with a finite set of modes $\{\mathcal{M}_i\}_{i\in K}$ from which one chooses a subset $\{\mathcal{M}_j\}_{j\in S_{\mathcal{A}}\subset K}$ consisting of d mutually orthogonal ones. A measurement $\mathcal{A}$, corresponding to this subset, reveals that a particle can be found in exactly one mode; hence one can consider a classical-like description in which one assigns $v(\mathcal{M}_k) = 1$ to the occupied mode and $v(\mathcal{M}_{l\neq k}) = 0$ to unoccupied modes. Moreover, if one also assumes noncontextuality, then the value assigned to a mode is the same in every subset in which this mode occurs. The goal is to show that such an assignment is not possible.

The single-particle example is basically the same as the standard KS scenario. Perhaps the only difference is in the interpretation of the contradiction at which one arrives. In the original case one has a logical contradiction; i.e., one cannot assign logical values to propositions. In the particle-mode example the contradiction has a more physical basis and corresponds to a lack of the particle number conservation.

The situation becomes more interesting when $N > 1$ and particles are indistinguishable. The case of many distinguishable particles can be reduced to many single-particle cases, because mode occupation can be considered for each type of particle separately; hence it does not differ from the standard scenario. However, the problem reveals new complex features when the particles are bosons or fermions. In the case of bosons the values assigned to modes are $v_B(\mathcal{M}_i) = 0,1,2,\ldots,$ whereas for fermions they are $v_F(\mathcal{M}_i) = 0,1$. In addition, for each subset of mutually orthogonal modes having the resolution of identity the following holds:

$$\sum_{i\in S_{\mathcal{A}}} v_B(\mathcal{M}_i) = \sum_{i\in S_{\mathcal{A}}} v_F(\mathcal{M}_i) = N. \tag{1}$$

Just like in the standard KS scenario, the problem can be formulated for a system having $d \geqslant 3$ distinguishable modes. For bosons N is arbitrary, but for fermions $N \leqslant d$.

In the rest of this work I focus on the 18-projector KS set [15,16] and on its properties in $N > 1$ scenarios; however, the model developed here allows one to consider any set of modes. The reason to study the 18-projector set is to show how the familiar model changes when more particles are considered. For this set $d = 4$ (see Fig. 1). Following the notation in Ref. [16], the modes are labeled v_{ij}, where i and j denote the measurement contexts to which the mode belongs. For $N = 1$ one arrives at a contradiction [15]. In the following sections I discuss the case $N > 1$ for fermions and bosons.

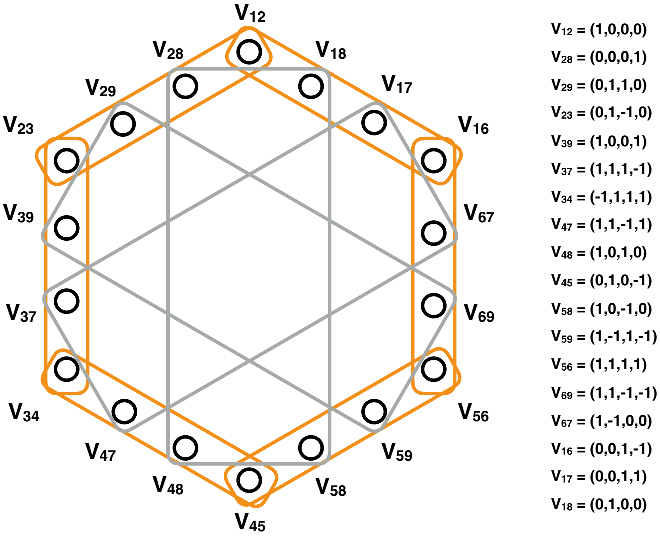

FIG. 1. A hypergraph from Ref. [16] representing measurements in the 18-projector KS scenario in dimension $d = 4$ [15]. Hyperedges (grey and orange rectangular areas) correspond to measurement contexts, i.e., sets of mutually orthogonal projectors (modes). The projectors in the right-hand column are unnormalized.

III. FERMIONS

For $d = 4$ the total number of fermions can be $N = 0,1,\ldots,4$. The case $N = 0$ ($N = 4$) is obviously noncontextual since one assigns 0 (1) to each mode. As mentioned above, $N = 1$ corresponds to a standard KS scenario and the 18 modes form a KS set. The case $N = 2$ is the first nontrivial extension of the KS scenario to more than one particle and is going to be discussed in a moment. The case $N = 3$ has an interesting interpretation because it is the standard KS scenario in which zeros are swapped with 1's. Physically, this scenario can be interpreted as a search for a noncontextual model for a single hole. The symmetry between particles and holes implies that for $N = 3$ the 18 modes form a KS set, because they form a KS set for $N = 1$. In general, in the fermionic case one can speak of N particles or alternatively of $d - N$ holes. Therefore, the generalization of contextuality to many fermions is nontrivial for $d > 3$.

For $N = 2$ fermions one can find an assignment to the 18-projector set fulfilling the criteria of noncontextuality and particle number conservation (Fig. 2). Therefore, for $N = 2$ the 18 modes do not form a KS set. They do not form a simple SIC set either. The SIC set constitutes an operator C_{SIC} which in its simplest form is a sum of projectors onto all modes in the set. The corresponding SIC inequality bounds the expectation value of the SIC operator by the sum of occupation numbers over all modes in a noncontextual model. Because of the symmetry between particles and holes, any noncontextual

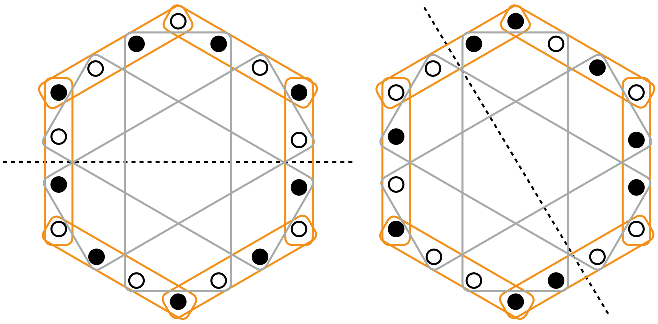

FIG. 2. Possible mode assignments of two fermions in the 18-projector KS set. Solid nodes correspond to particles and empty nodes correspond to holes. These assignments exhibit a symmetry between particles and holes; i.e., if one exchanges particles with holes one obtains the same assignment (up to the reflection denoted by the dashed lines).

model assigns exactly 9 particles and 9 holes to all 18 modes (see Fig. 2). Therefore, the SIC operator is bounded by 9.

In order to evaluate the average value of the expectation of the SIC operator, note that 18 modes constitute 9 contexts, each having a resolution of identity, and each mode appears in exactly two contexts. Therefore, $C_{\text{SIC}} = \frac{9}{2}\hat{N}$, where $\hat{N}$ is an operator of the total number of particles. As a result, for any state of two fermions (or bosons) $|\psi\rangle$ one has $\langle\psi|C_{\text{SIC}}|\psi\rangle = 9$ and the corresponding SIC inequality cannot be violated. However, the possibility of violation of some other SIC inequality based on an operator which is not a simple sum of projectors onto all modes was not excluded here and remains an open problem.

The assignments discussed above exhibit a symmetry between particles and holes, which suggests that in order to observe contextuality one has to find a state which breaks this symmetry. Therefore, one has to consider a state-dependent version of contextuality. Indeed, for $N = 2$ one can find a Hardy-like proof [17] of quantum contextuality [18,19].

Proof of Hardy-like contextuality for two fermions. Let $f_{ij}^\dagger$ be an operator creating a fermion in the mode v_{ij} (see Fig. 1). These operators obey standard fermionic anticommutation rules $f_{ij}^\dagger f_{kl}^\dagger + f_{kl}^\dagger f_{ij}^\dagger = 0$. Next, consider the state

$$|\psi\rangle = f_{67}^\dagger f_{69}^\dagger |0\rangle. \tag{2}$$

This state has a well-defined fermionic occupation in the measurement context 6:

$$C_6 = \{v_{16} = 0, v_{67} = 1, v_{69} = 1, v_{56} = 0\}. \tag{3}$$

Then, consider the same state in the measurement context 3:

$$|\psi\rangle = \left(\frac{f_{39}^\dagger f_{23}^\dagger}{2\sqrt{2}} + \frac{f_{37}^\dagger f_{23}^\dagger}{4} - \frac{f_{37}^\dagger f_{39}^\dagger}{4} - \frac{3 f_{34}^\dagger f_{23}^\dagger}{4} + \frac{f_{34}^\dagger f_{39}^\dagger}{4} - \frac{f_{34}^\dagger f_{37}^\dagger}{2\sqrt{2}}\right)|0\rangle. \tag{4}$$

According to Eq. (4) there is a probability of $\frac{1}{16}$ that one finds particles in modes v_{37} and v_{39}. If this happens, context 3 has the following assignment:

$$C_3 = \{v_{34} = 0, v_{37} = 1, v_{39} = 1, v_{23} = 0\}. \tag{5}$$

Next, C_3 and C_6 imply

$$C_7 = \{v_{17} = 0, v_{67} = 1, v_{47} = 0, v_{37} = 1\}, \tag{6}$$

$$C_9 = \{v_{69} = 1, v_{59} = 0, v_{39} = 1, v_{29} = 0\}. \tag{7}$$

Because v_{17} and v_{16} have been already assigned zero, one gets

$$C_1 = \{v_{12} = 1, v_{18} = 1, v_{17} = 0, v_{16} = 0\}. \tag{8}$$

Similarly, $v_{56} = 0$ and $v_{59} = 0$ imply

$$C_5 = \{v_{56} = 0, v_{59} = 0, v_{58} = 1, v_{45} = 1\}. \tag{9}$$

One can follow the chain of implications to obtain

$$C_8 = \{v_{18} = 1, v_{58} = 1, v_{48} = 0, v_{28} = 0\}. \tag{10}$$

At this point all the modes have been already assigned an occupation number. However, one obtains a contradiction (see Fig. 3) because there are two contexts that are assigned only a single particle:

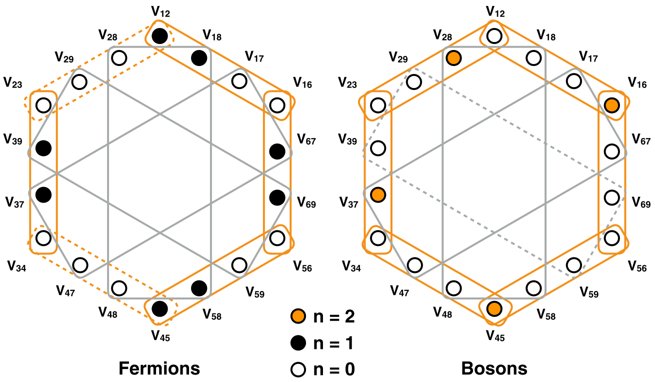

FIG. 3. Mode assignments and contradictions in the Hardy-like proofs of contextuality for two identical particles. The contexts with contradictory assignments are denoted by dashed hyperedges.

$$C_2 = \{v_{23} = 0, v_{29} = 0, v_{28} = 0, v_{12} = 1\}, \tag{11}$$

$$C_4 = \{v_{45} = 1, v_{48} = 0, v_{47} = 0, v_{34} = 0\}. \tag{12}$$

Finally, note that the above contradiction can be obtained for any state of the form $f_m^\dagger f_n^\dagger |0\rangle$, where the modes m and n are orthogonal. One simply needs to find an arbitrary unitary operation which transforms $f_{67}^\dagger \to f_m^\dagger$ and $f_{69}^\dagger \to f_n^\dagger$ and apply it to all 18 modes. Next, one needs to follow similar steps as above. ■

IV. BOSONS

The bosonic case for $N = 0,1,2$ resembles the fermionic one. The 18-mode set for $N = 0$ bosons is trivially noncontextual and $N = 1$ corresponds to the standard KS scenario. The two-fermion assignments from Fig. 2 can also describe a system of bosons. Therefore, the case of $N = 2$ bosons does not provide a typical KS contradiction and the 18 modes do not form a bosonic KS set. Due to the reasons discussed in the previous sections, the 18 modes do not form a simple SIC set either. However, one can find a Hardy-like proof by exploiting the bosonic properties of the two particles.

Proof of Hardy-like contextuality for two bosons. Let $b_{ij}^\dagger$ be an operator creating a boson in the mode v_{ij}. The bosonic creation operators obey standard commutation rules $b_{ij}^\dagger b_{kl}^\dagger - b_{kl}^\dagger b_{ij}^\dagger = 0$. Next, consider the state

$$|\psi\rangle = \frac{b_{16}^{\dagger 2}}{\sqrt{2}}|0\rangle, \tag{13}$$

which has a well-defined bosonic occupation in contexts 1 and 6. The operator $b_{16}^\dagger$ has the following representation in the basis of operators from context 4:

$$b_{16}^\dagger = \frac{b_{45}^\dagger}{2} + \frac{b_{48}^\dagger}{2} - \frac{b_{47}^\dagger}{\sqrt{2}}. \tag{14}$$

This implies that the state $|\psi\rangle$ has an equivalent form

$$|\psi\rangle = \left(\frac{b_{45}^{\dagger 2}}{4\sqrt{2}} + \frac{b_{48}^{\dagger 2}}{4\sqrt{2}} - \frac{b_{47}^{\dagger 2}}{2\sqrt{2}} + \frac{b_{45}^{\dagger}b_{48}^{\dagger}}{4\sqrt{2}} - \frac{b_{45}^{\dagger}b_{47}^{\dagger}}{4} - \frac{b_{47}^{\dagger}b_{48}^{\dagger}}{4}\right)|0\rangle \tag{15}$$

and that there is a probability $\frac{1}{16}$ to measure two particles in mode v_{45}. If this happens one has the following assignments:

$$C_4 = \{v_{45} = 2, v_{48} = 0, v_{47} = 0, v_{34} = 0\}, \tag{16}$$

$$C_5 = \{v_{56} = 0, v_{59} = 0, v_{58} = 0, v_{45} = 2\}. \tag{17}$$

In addition, the initial assumption about the state (13) implies

$$C_1 = \{v_{12} = 0, v_{18} = 0, v_{17} = 0, v_{16} = 2\}, \tag{18}$$

$$C_6 = \{v_{16} = 2, v_{67} = 0, v_{69} = 0, v_{56} = 0\}. \tag{19}$$

From the above four assignments one finds that since $v_{17} = v_{67} = v_{47} = 0$ and $v_{18} = v_{58} = v_{48} = 0$ then

$$C_7 = \{v_{17} = 0, v_{67} = 0, v_{47} = 0, v_{37} = 2\}, \tag{20}$$

$$C_8 = \{v_{18} = 0, v_{58} = 0, v_{48} = 0, v_{28} = 2\}. \tag{21}$$

Next, since $v_{28} = v_{37} = 2$ one gets

$$C_2 = \{v_{23} = 0, v_{29} = 0, v_{28} = 2, v_{12} = 0\}, \tag{22}$$

$$C_3 = \{v_{34} = 0, v_{37} = 2, v_{39} = 0, v_{23} = 0\}. \tag{23}$$

The occupation numbers are already assigned to all the modes; however, one obtains a contradiction (see Fig. 3) since

$$C_9 = \{v_{69} = 0, v_{59} = 0, v_{39} = 0, v_{29} = 0\}. \tag{24}$$

Just like in the case of two fermions, the proof can be adopted to any state of the form $\frac{b_j^{\dagger 2}}{\sqrt{2}}|0\rangle$. ■

V. CONTEXTUALITY OF $N > 2$ BOSONS

The above proof of Hardy-like contextuality can be easily generalized to the case of an arbitrary $N > 0$ number of bosons. One starts with a state

$$|\psi\rangle = \frac{b_{16}^{\dagger N}}{\sqrt{N!}}|0\rangle, \tag{25}$$

and measures the occupation numbers in context 4. From Eq. (14) one finds that the probability of finding N particles in mode v_{45} is $\frac{1}{4^N}$. If this happens, one follows the same steps as in the case of $N = 2$. This time in all the assignments there is N instead of 2. Finally, one arrives at Eq. (24).

VI. CONCLUSIONS

In this work I presented a generalization of quantum contextuality to multipartite scenarios in which mode occupation numbers are measured. The main result is the link between two seemingly unrelated quantum phenomena: contextuality and indistinguishability. In particular, for the system of 18 modes that can be grouped in 9 contexts of 4 mutually orthogonal ones (corresponding to a single-particle Hilbert space of dimension 4), I showed that the contextuality depends on the number and on the type of particles. The bosonic (fermionic) case $N = 0$ ($N = 0,4$) is trivially noncontextual. The case $N = 1$ ($N = 1,3$) exhibits state-independent contextuality. For the case $N = 2$ there is no state-independent contextuality (for both bosons and fermions) in the form of a KS set or a simple SIC set; however, a state-dependent Hardy-like proof of contextuality can be found. This proof also works for $N > 2$ bosons. The possibility of state-independent contextuality in multibosonic systems remains to be investigated.

In general, any macroscopic manifestation of effects discussed in this work would involve a large number of particles. In the case of bosonic systems, for example, photons in linear optical setups, one can have $N \gg 1$ and at the same time keep the number of modes relatively small. Note that the Hardy-like contextuality for photonic Fock states can be observed with present technology. However, Fock states are considered to be highly nonclassical and require a nontrivial preparation. The real challenge would be to find a method to observe contextuality on coherent states of light, which would require relaxation of the particle number conservation assumption in the discussed scenario.

On the other hand, fermionic properties can be tested in solid-state systems. In this case macroscopic effects demand $d \geqslant N \gg 1$. Such scenarios seem to be more demanding than bosonic ones because of the large number of measurements one would have to perform. However, a possible simplification may result from degenerate (non-rank-1) measurements in which each observable mode consists of many micromodes.

VII. OTHER OPEN PROBLEMS

The results reported in this work lead to a number of open questions. Apart from the ones mentioned above, the other major problems are the following: Can one find the state-independent contextuality of bosons for $N > 1$ (fermions for $d - 1 > N > 1$) and for an arbitrary $d \geqslant 3$ ($d \geqslant 4$)? For a given d and N what is the minimal number of modes that lead to the state-dependent and state-independent contextuality? The Hardy-like proof is probabilistic, but is it possible to find a Greenberger-Horne-Zeilinger-like [20] proof of contextuality for $N > 1$ and some d? Can one find some system that is contextual for $N > 1$, but is noncontextual for $N = 1$?

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