

Exact master equation for a spin interacting with a spin bath: Non-Markovianity and negative entropy production rate

Samyadeb Bhattacharya,^{*} Avijit Misra, Chiranjib Mukhopadhyay, and Arun Kumar Pati[†]

Harish-Chandra Research Institute, Allahabad 211019, India

and Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400 085, India

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An exact canonical master equation of the Lindblad form is derived for a central spin interacting uniformly with a sea of completely unpolarized spins. The Kraus operators for the dynamical map are also derived. The non-Markovianity of the dynamics in terms of the divisibility breaking of the dynamical map and the increase of the trace distance fidelity between quantum states is shown. Moreover, it is observed that the irreversible entropy production rate is always negative (for a fixed initial state) whenever the dynamics exhibits non-Markovian behavior. In continuation with the study of witnessing non-Markovianity, it is shown that the positive rate of change of the purity of the central qubit is a faithful indicator of the non-Markovian information backflow. Given the experimental feasibility of measuring the purity of a quantum state, a possibility of experimental demonstration of non-Markovianity and the negative irreversible entropy production rate is addressed. This gives the present work considerable practical importance for detecting the non-Markovianity and the negative irreversible entropy production rate.

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I. INTRODUCTION

In many-body problems the dynamics of microscopic (e.g., spin systems) or mesoscopic [e.g., Superconducting Quantum Interference Devices (SQUIDs)] systems always gets complicated owing to its interaction with a background environment. To obtain the reduced dynamics of the quantum system that we are interested in, it is a general custom to model the environment as a collection of oscillators or spin- $\frac{1}{2}$ particles [1] which is often abbreviated as bath. They constitute two different universal classes of quantum environment [2]. In the oscillator bath model, the environment is described as a set of uncoupled harmonic oscillators. Paradigmatic examples of this kind of bath are the spin-boson model [3,4] and the Caldeira-Leggett model [4,5] originating from a scheme proposed by Feynman and Vernon [6]. These oscillator models have been widely studied in the context of various physical phenomena under Markovian approximation [1,7,8]. On the other hand, the spin bath models remain relatively less explored. However, the spin bath models play a pivotal role in the quantum theory of magnetism [9], in quantum spin glasses [10], and in the theory of conductors and superconductors [3]. To get the exact dynamics of a quantum system under this spin bath model is of paramount importance yet is a difficult task. Indeed, in most of the cases the dynamics cannot be described exactly and several approximation techniques, both local and nonlocal in time, have been employed [1,11–16].

In this work, we focus on the dynamical behavior of a central spin interacting uniformly with a spin bath and derive an exact time-local master equation of the Lindblad type. Moreover, the Kraus representation of the dynamical map is also derived. Reduced dynamics of this particular spin bath model has been considered before [15,16] where a correlated projection operator technique has been used to approximate the

master equation of the central spin. However, the given master equation is time nonlocal and not of the standard canonical form. In contrast, we start from the exact reduced state of the central spin at an arbitrary given time [15] to derive the canonical master equation without considering any approximations. The thrust of our result is not only that the master equation is exact but also that the method used here allows us to unravel the less explored but far-reaching consequences of the strong coupling regimes which can be instrumental in performing information theoretic tasks, quantum thermodynamic tasks, and several other quantum technological tasks. Moreover, the relaxation rates in the canonical master equation are insightful to understand several physical processes, such as dissipation, absorption, and dephasing, and thus the nature of decoherence.

One of the characteristics of the spin bath models is to exhibit the non-Markovian features [17,18]. The non-Markovianity has been identified as a key resource in information theoretic [19–21], thermodynamic [22–24], and precision measurement protocols [25–27]. We study the non-Markovian features of the reduced dynamics and it is shown that the non-Markovianity increases with the interaction strength.

An irreversible increase of entropy due to dissipation of energy and work into the environment is inevitable for systems out of equilibrium. The analysis of irreversible or nonequilibrium entropy production and its rate have been instrumental to understanding nonequilibrium phenomena in different branches of physics [28–33]. According to Spohn's theorem [34], the irreversible entropy production rate is always non-negative under the Markovian dynamics. Whereas non-Markovianity of the dynamics allows a negative irreversible entropy production rate and thereby this partial reversibility of the work and entropy influences the performance of quantum heat engines, refrigerators, and memory devices. As our study enables us to probe the strong coupling regime, it can be far reaching to unravel the hitherto unexplored consequences of the non-Markovian dynamics in the strong coupling regime for more efficient thermodynamic protocols. Here, we investigate the entropy production rate and show that

^{*}samyadebbhattacharya@hri.res.in

[†]akpati@hri.res.in

the non-Markovianity of the dynamics is always associated with a negative entropy production rate of the central spin for a certain initial state. We also investigate the non-Markovianity in terms of the rate of change of the purity of the central qubit and it is observed that the rate of change of the purity of the qubit is positive for the same aforesaid initial state, whenever the dynamics is non-Markovian. Experimental detection of the non-Markovianity and the entropy production rates for quantum systems is of paramount interest in current research. As purity can be measured in the laboratory, the study of this article can pave novel avenues to experimentally demonstrate non-Markovian features and negative entropy production rate in spin bath models.

The organization of the paper is as follows. In Sec. II, we derive the proposed canonical master equation of Lindblad type. The non-Markovian features of the dynamics of the central qubit are demonstrated, explaining the indivisibility of the dynamical map and non-monotonicity of the trace distance fidelity. In this section, we also derive the Kraus operators for the dynamical evolution. The nonequilibrium entropy production rate and the dynamics of purity of the qubit are studied in Sec. III. Finally we conclude in Sec. IV.

II. CENTRAL SPIN MODEL AND ITS REDUCED DYNAMICS

In this section we first describe the central spin bath model. Then we derive the exact canonical master equation of the Lindblad type. From the master equation of the Lindblad form we show that the reduced dynamics of the central spin exhibits non-Markovian features throughout. We also derive the Kraus operators for the dynamical map.

A. The model

Let us first describe the central spin bath model. We consider a spin- $\frac{1}{2}$ particle that interacts uniformly with N other spin- $\frac{1}{2}$ particles constituting the bath. The spins of the bath do not interact with each other. The Hamiltonian for this spin bath model is given by

$$H = H_S + H_{SB} \\ = \frac{\hbar}{2}\omega_0\sigma_z^0 + \frac{\hbar}{2}\sum_{i=1}^N\alpha(\sigma_x^0\sigma_x^i + \sigma_y^0\sigma_y^i + \sigma_z^0\sigma_z^i), \quad (1)$$

where σ_k^i ($k = x, y, z$) are the Pauli matrices of the i th spin of the bath, σ_k^0 ($k = x, y, z$) are the Pauli matrices for the central spin, and α is the interaction strength. Here H_S and H_{SB} are the system and interaction Hamiltonian, respectively. Initially the system and reservoir are uncorrelated and the reservoir is in a thermal state at infinite temperature, i.e., a completely unpolarized state [15]. The composite state of the system and bath evolves unitarily under the total Hamiltonian $H + H_B$, starting from the factorized initial state, $\rho_{SB}(0) = \rho_S(0) \otimes \frac{1}{2^N}\mathbb{I}_B$, where \mathbb{I}_B is an N -qubit identity matrix and H_B is the bath Hamiltonian. Note that as we are only concerned with the reduced dynamics of the central spin and the bath is completely unpolarized at $t = 0$, there is no loss of generality to drop the bath Hamiltonian H_B from the effective Hamiltonian H to get the reduced dynamics of the

spin. Therefore, the reduced quantum state $\rho_S(t)$ of the central spin at time t can be obtained by tracing out the bath degrees of freedom as

$$\rho_S(t) = \text{Tr}_B \left[e^{-i(H+H_B)t/\hbar} \left\{ \rho_S(0) \otimes \frac{1}{2^N}\mathbb{I}_B \right\} e^{i(H+H_B)t/\hbar} \right]. \quad (2)$$

Hereafter, we drop the subscript ‘‘S’’ for brevity to denote central spin as we will only deal with it. The total angular momentum of the bath is given by $\mathbf{J} = \frac{1}{2}\sum_i\boldsymbol{\sigma}^i$. The basis $|j, m\rangle$ is defined as the simultaneous eigenbases of both \mathbf{J}^2 and J_z . For even N , j takes the values $j = 0, 1, 2, \dots, N/2$, and for odd N , we have $j = 1/2, 3/2, \dots, N/2$. m goes from $-j$ to j . It can be shown that [15] the z component of the total angular momentum $\frac{1}{2}\sigma_z^0 + \frac{1}{2}\sum_i\sigma_z^i$ and \mathbf{J}^2 are conserved quantities. There are now two-dimensional subspaces spanned by $|+\rangle \otimes |j, m\rangle$ and $|-\rangle \otimes |j, m+1\rangle$ which are invariant under time evolution. Now the task of finding the analytical solution to the reduced dynamics of the central spin is broken down into solving the equations of motion in each subspace. Solving the equation of motion exactly [15], the initial reduced state of the central spin, $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$, can be shown to evolve as

$$\rho_{11}(t) = A(t)\rho_{11}(0) + B(t)\rho_{22}(0), \\ \rho_{12}(t) = C(t)\rho_{12}(0), \quad (3)$$

where

$$A(t) = \sum_{j,m} \frac{N_j}{2^N} \left\{ \cos^2[\mu_+(j,m)t] + \frac{\Omega_+^2(m)}{4\mu_+^2(j,m)} \right. \\ \left. \times \sin^2[\mu_+(j,m)t] \right\}, \\ B(t) = \sum_{j,m} \frac{N_j}{2^N} \frac{\alpha^2 b^2(j,m)}{4\mu_+^2(j,m)} \sin^2[\mu_+(j,m)t], \\ C(t) = e^{i\omega_0 t} \sum_{j,m} \frac{N_j}{2^N} \left\{ \cos[\mu_+(j,m)t] - \frac{i\Omega_+(m)}{2\mu_+(j,m)} \right. \\ \left. \times \sin[\mu_+(j,m)t] \right\} \\ \times \left\{ \cos[\mu_-(j,m)t] + \frac{i\Omega_-(m)}{2\mu_-(j,m)} \sin[\mu_-(j,m)t] \right\},$$

and

$$N_j = \binom{N}{\frac{N}{2} + j} - \binom{N}{\frac{N}{2} + j + 1}, \\ \Omega_{\pm} = \pm\omega_0 + \alpha(\pm m + 1/2), \\ \mu_{\pm} = \frac{1}{2}\sqrt{\Omega_{\pm}^2 + \alpha^2 b_{\pm}^2}, \\ b_{\pm} = \sqrt{j(j+1) - m(m \pm 1)}.$$

It follows from the above expressions that $A(t) + B(t) = 1$, which implies the dynamical map is unital. The unitality of the dynamics has to be satisfied as the environment and the system start from a product state while the environment is in the maximally mixed state. We are now in position to derive the canonical master equation.

B. Canonical master equation

Derivation of the master equation is basically finding the generator of the evolution, which is one of the fundamental problems in the theory of open quantum systems. Moreover, the Lindblad-type master equation can lead to understanding of various physical processes like dissipation, absorption, and dephasing and hence to the nature of decoherence, in a much more convincing way. Considering the importance of the spin bath to model the environmental interactions in various domains, such as magnetism, superconductors, spin glasses, etc., it is important and illustrative to have the master equation for spin bath models. Additionally, theoretical as well as experimental study of quantum thermodynamic devices (QTDs) has attracted a great deal of interest in recent times. Establishing master equations for open quantum systems is of paramount significance in the context of QTDs [24], where a single or few quantum systems are coupled with their heat baths in general. For example, in recently proposed quantum absorption refrigerators [35], three qubits interact among themselves while they are coupled to their respective baths. The Lindblad operators for the qubits under the corresponding heat baths become crucial to the study of the performance of the refrigerators in both steady and transient regimes [36–41]. Recently introduced quantum thermal transistors [42] are also worth mentioning in this context. Therefore, the canonical Lindblad-type master equation in the spin bath models can provide a way to study the QTDs in hitherto less explored strong coupling and non-Markovian regimes which might have far-reaching impacts to enhance the performance of QTDs.

In what follows, we derive the exact canonical master equation of the Lindblad type for the central spin starting from the dynamical map given in Eq. (3). The dynamical map described in Eq. (3) can be notationally represented as

$$\rho(t) = \Phi[\rho(0)]. \quad (4)$$

$$L(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ [\dot{A}(t) + \dot{B}(t)] + \left(\frac{\dot{A}(t) - \dot{B}(t)}{A(t) - B(t)}\right) \{1 - [A(t) + B(t)]\} \end{pmatrix}$$

where $C_R(t)$ and $C_I(t)$ are the real and imaginary part of $C(t)$, respectively. Now from Eq. (8), we get the equation of motion as given by

$$\begin{aligned} \dot{\rho}_{11}(t) &= \frac{L_{z0} + L_{zz}}{2} \rho_{11}(t) + \frac{L_{z0} - L_{zz}}{2} \rho_{22}(t), \\ \dot{\rho}_{12}(t) &= (L_{xx} + iL_{xy}) \rho_{12}(t). \end{aligned} \quad (11)$$

Equation (11) gives the time rate of change of the density matrix. However, one needs to have the Lindblad-type master equation to understand various processes like dissipation, absorption, and dephasing in a more convincing way. Moreover, it is of prime importance to have the master equation to study the non-Markovian behavior of the reduced dynamics as we will see later. Therefore, our immediate aim is to derive

The equation of motion of the reduced density matrix of the form

$$\dot{\rho}(t) = \Lambda[\rho(t)] \quad (5)$$

can be obtained from Eq. (3), which is characterized by the time-dependent generator $\Lambda[.]$. By following the method [43] given below, we find the master equation and thus the generator of the specific reduced dynamics. Consider a convenient orthonormal basis set $\{G_a\}$ with the properties $G_a^\dagger = G_a$ and $\text{Tr}[G_a G_b] = \delta_{ab}$. The map given in Eq. (4) can now be represented as

$$\Phi[\rho(0)] = \sum_{k,l} \text{Tr}[G_k \Phi[G_l]] \text{Tr}[G_l \rho(0)] G_k = [F(t)r(0)]G^T, \quad (6)$$

where $F_{kl} = \text{Tr}[G_k \Phi[G_l]]$ and $r_l = \text{Tr}[G_l \rho(0)]$. Differentiating Eq. (6), we get

$$\dot{\rho}(t) = [\dot{F}(t)r(0)]G^T. \quad (7)$$

Let us consider a matrix L , with elements $L_{kl} = \text{Tr}[G_k \Lambda[G_l]]$. We can now represent Eq. (5) as

$$\dot{\rho}(t) = \sum_{k,l} \text{Tr}[G_k \Lambda[G_l]] \text{Tr}[G_l \rho(t)] G_k = [L(t)r(t)]G^T. \quad (8)$$

By comparing Eq. (7) and (8), we find

$$\dot{F}(t) = L(t)F(t) \Rightarrow L(t) = \dot{F}(t)F(t)^{-1}. \quad (9)$$

We can arrive at Eq. (9) given the inverse of $F(t)$ does exist and $F(0) = \mathbb{I}$. Considering the specific map of the central spin in Eq. (3), and taking the orthonormal basis set $\{G_a\}$ as $\{\frac{\mathbb{I}_2}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}\}$, we find the $L(t)$ matrix to be

$$L(t) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{C_R(t)\dot{C}_R(t) + C_I(t)\dot{C}_I(t)}{C_R(t)^2 + C_I(t)^2} & -\frac{C_I(t)\dot{C}_R(t) - C_R(t)\dot{C}_I(t)}{C_R(t)^2 + C_I(t)^2} & 0 \\ \frac{C_I(t)\dot{C}_R(t) - C_R(t)\dot{C}_I(t)}{C_R(t)^2 + C_I(t)^2} & \frac{C_R(t)\dot{C}_R(t) + C_I(t)\dot{C}_I(t)}{C_R(t)^2 + C_I(t)^2} & 0 \\ 0 & 0 & \left(\frac{\dot{A}(t) - \dot{B}(t)}{A(t) - B(t)}\right) \end{pmatrix}, \quad (10)$$

the Lindblad-type master equation starting from Eq. (11). Equation (5) can be written in the form [44]

$$\dot{\rho}(t) = \Lambda[\rho(t)] = \sum_k X_k(t)\rho(t)Y_k(t)^\dagger, \quad (12)$$

where $X_k(t) = \sum_i G_i x_{ik}(t)$, $Y_k(t) = \sum_i G_i y_{ik}(t)$, and $\{G_a\}$ are the basis operators as defined before. Using this decomposition of $X(t)$ and $Y(t)$, Eq. (12) can be rewritten as

$$\dot{\rho}(t) = \sum_{i,j=\{0,x,y,z\}} z_{ij}(t) G_i \rho(t) G_j, \quad (13)$$

where $z_{ij}(t) = \sum_k x_{ik}(t)y_{jk}(t)^*$ are the elements of a Hermitian matrix. Using a new set of operators [44], $\mathcal{F}(t) = (z_{00}(t)/8)\mathbb{I}_2 + \sum_i (z_{i0}/2)G_i$ and $H(t) = \frac{i}{2}\hbar[\mathcal{F}(t) - \mathcal{F}^\dagger(t)]$,

after some algebra, Eq. (13) can be written as

$$\dot{\rho}(t) = \frac{i}{\hbar}[\rho(t), H(t)] + \sum_{i,j=\{x,y,z\}} z_{ij}(t) \left[G_i \rho G_j - \frac{1}{2} \{G_j G_i, \rho(t)\} \right], \quad (14)$$

where the curly braces stand for anticommutator. Hence, the canonical master equation of the Lindblad form reads as

$$\begin{aligned} \dot{\rho}(t) = & \frac{i}{\hbar} U(t)[\rho(t), \sigma_z] + \Gamma_{\text{deph}}(t)[\sigma_z \rho(t) \sigma_z - \rho(t)] \\ & + \Gamma_{\text{dis}}(t) \left[\sigma_- \rho(t) \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, \rho(t) \} \right] \\ & + \Gamma_{\text{abs}}(t) \left[\sigma_+ \rho(t) \sigma_- - \frac{1}{2} \{ \sigma_- \sigma_+, \rho(t) \} \right], \end{aligned} \quad (15)$$

where $\sigma_{\pm} = \frac{\sigma_x \pm i \sigma_y}{2}$, and $\Gamma_{\text{dis}}(t), \Gamma_{\text{abs}}(t), \Gamma_{\text{deph}}(t)$ are the rates of dissipation, absorption, and dephasing processes, respectively, and $U(t)$ corresponds to the unitary evolution. $A(t) + B(t) = 1$, for this specific system, is used to derive the master equation. The rates of dissipation, absorption, and dephasing and the unitary evolution are, respectively, given as

$$\begin{aligned} \Gamma_{\text{dis}}(t) &= -\frac{L_{z0} + L_{zz}}{2} = \frac{d}{dt} \left[\ln \left(\frac{1}{\sqrt{A(t) - B(t)}} \right) \right], \\ \Gamma_{\text{abs}}(t) &= -\frac{L_{zz} - L_{z0}}{2} = \frac{d}{dt} \left[\ln \left(\frac{1}{\sqrt{A(t) - B(t)}} \right) \right], \\ \Gamma_{\text{deph}}(t) &= -\frac{2L_{xx} - L_{zz}}{4} = \frac{1}{4} \frac{d}{dt} \left[\ln \left(\frac{A(t) - B(t)}{|C(t)|^2} \right) \right], \\ U(t) &= -\frac{L_{xy}}{2} = -\frac{1}{2} \frac{d}{dt} \left[\ln \left(1 + \left(\frac{C_R(t)}{C_I(t)} \right)^2 \right) \right]. \end{aligned} \quad (16)$$

Note that the system environment interaction generates a time-dependent driving Hamiltonian evolution in the form of $U(t)$. Since the coefficients of dissipation and absorption are equal, the master equation (15) can also be rewritten as

$$\begin{aligned} \dot{\rho}(t) = & \frac{i}{\hbar} U(t)[\rho(t), \sigma_z] + \frac{\Gamma_{\text{dis}}(t)}{2} [\sigma_x \rho(t) \sigma_x - \rho(t)] \\ & + \frac{\Gamma_{\text{dis}}(t)}{2} [\sigma_y \rho(t) \sigma_y - \rho(t)] + \Gamma_{\text{deph}}(t) [\sigma_z \rho(t) \sigma_z - \rho(t)]. \end{aligned} \quad (17)$$

The above equation implies that $\frac{\mathbb{I}}{2}$ is a fixed point of the reduced dynamics and, hence, it confirms the unitality of the dynamical map. As the bath is in a thermal state at infinite temperature, the probabilities of losing energy to the bath modes and absorbing from it become equal which causes the dissipation and absorption rates to be the same. This is quite similar to the bosonic thermal baths, as it follows from the Kubo-Martin-Schwinger condition [1] that given the baths having canonical equilibrium distribution the rates of the absorption and dissipation processes are balanced by the equation $\Gamma(-\omega) = \Gamma(\omega) \exp(-\beta\omega)$. Here β is the inverse temperature of the bath and it implies that $\Gamma(-\omega) = \Gamma(\omega)$, iff $\beta = 0$.

One of the important properties of a quantum dynamical map is complete positivity [17,18,45–52]. The notion ‘‘complete’’ comes with the argument that, for any valid quantum

dynamical map, the positivity must be preserved if the map is acting on a system which is correlated to an ancilla of any possible dimension. For a Lindblad-type canonical master equation with time-dependent coefficients, as in Eq. (15), the complete positivity is guaranteed by the following condition $\int_0^t \Gamma_i(s) ds \geq 0$ [53], which can be easily verified for the specific decay rates given in Eq. (16). It is worth mentioning that, since the dynamical map for this specific spin bath model is derived starting from an initial product system plus environment state, it is always guaranteed to be completely positive [54,55]. However, the complete positivity of the dynamical map for the reduced system can break down in the presence of a system-environment initial correlation [55].

C. Operator sum representation

Another important aspect of general quantum evolution is the Kraus operator sum representation, given as $\rho(t) = \sum_i K_i(t) \rho(0) K_i^\dagger(t)$. The Kraus operators can be constructed [56] from the eigenvalues and eigenvectors of the corresponding Choi-Jamiolkowski state [57]. The Choi-Jamiolkowski state for a dynamical map $\Phi[\rho]$ acting on a d -dimensional system is given by $(\mathbb{I}_d \otimes \Phi)[\Phi_+]$, with $\Phi_+ = |\Phi_+\rangle\langle\Phi_+|$ being the maximally entangled state in $d \times d$ dimension. For the particular evolution considered here, we find the Choi state to be

$$\begin{pmatrix} A(t)/2 & 0 & 0 & C(t)/2 \\ 0 & B(t)/2 & 0 & 0 \\ 0 & 0 & B(t)/2 & 0 \\ C^*(t)/2 & 0 & 0 & A(t)/2 \end{pmatrix}. \quad (18)$$

The positive semidefiniteness of the above density matrix demands $B(t) \geq 0; A(t) \geq |C(t)|$. From the eigensystem of the Choi state given in Eq. (18), we derive the Kraus operators as

$$\begin{aligned} K_1(t) &= \sqrt{B(t)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ K_2(t) &= \sqrt{B(t)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ K_3(t) &= \sqrt{\frac{A(t) - |C(t)|}{2}} \begin{pmatrix} -e^{i\theta(t)} & 0 \\ 0 & 1 \end{pmatrix}, \\ K_4(t) &= \sqrt{\frac{A(t) + |C(t)|}{2}} \begin{pmatrix} e^{i\theta(t)} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (19)$$

where $\theta(t) = \arctan[C_I(t)/C_R(t)]$. It is straight forward to verify that the Kraus operators satisfy the unitality property $\sum_i K_i(t) K_i^\dagger(t) = \mathbb{I}$.

D. Non-Markovianity

The characterization and quantification of the non-Markovianity is a fundamental aspect of open quantum dynamics. There are several proposed measures based on divisibility [50,58] and non-Markovianity witness [48,49,59–64]. A well-accepted characterization and quantification of non-Markovianity based on the composition law of the dynamical map has been introduced by Rivas-Huelga-Plenio [50], commonly known as the RHP measure of non-Markovianity.

In this approach, the non-Markovian behavior is attributed to the deviation from divisibility and the quantification of non-Markovianity is done based on the amount of the deviation. A complete positive and trace-preserving (CPTP) dynamical map $\Phi(t,0)$ is divisible, when for all intermediate time τ , it follows that

$$\Phi(t + \tau, 0) = \Phi(t + \tau, t)\Phi(t, 0). \quad (20)$$

In Ref. [50], it has been shown that the dynamical map Φ is divisible or indivisible if the (right) time derivative,

$$q(t) = \lim_{\epsilon \rightarrow 0^+} \frac{||[\mathbb{I}_d \otimes \Phi(t + \epsilon, t)]\Phi_+|| - 1}{\epsilon}, \quad (21)$$

is zero or greater than zero, respectively. Here, d is the dimension of the Hilbert space and $||\cdot||$ denotes the trace norm, and $\Phi_+ = |\Phi_+\rangle\langle\Phi_+|$ is the maximally entangled state in $d \times d$ dimension. To illustrate this measure, we consider the dynamical equation (5). In the limit $\epsilon \rightarrow 0^+$, the solution formally reads $\Phi(t + \epsilon, t) \rightarrow e^{\epsilon\Lambda}$. To the first-order expansion, the parameter $q(t)$ is given as

$$q(t) = \lim_{\epsilon \rightarrow 0^+} \frac{||[\mathbb{I}_{d \times d} + \mathbb{I}_d \otimes \Lambda(t + \epsilon, t)]\Phi_+|| - 1}{\epsilon}. \quad (22)$$

It is straightforward to calculate $q(t)$ from Eq. (22) [50]. Hence, the RHP measure of non-Markovianity can be defined [50,58] based on the strict positivity of $q(t)$ as follows:

$$G = \frac{\eta}{\eta + 1}, \quad (23)$$

where $\eta = \int_0^\infty q(t)dt$. Note that for the Markovian evolution G is zero and the maximum non-Markovianity corresponds to $G = 1$, i.e., when $\eta \rightarrow \infty$. The positivity of the function $q(t)$ or the indivisibility of the map appears when the relaxation rates ($\{\Gamma_i(t)\}$) take negative values. We show in the following that, for the specific dynamical evolution considered in the present work, the decay rates periodically get negative and hence break the divisibility of the map, although they always maintain the complete positivity condition. For this particular evolution, we get

$$q(t) = [|\Gamma_{\text{dis}}(t)| - \Gamma_{\text{dis}}(t)] + [|\Gamma_{\text{deph}}(t)| - \Gamma_{\text{deph}}(t)] \\ = q_{\text{dis}}(t) + q_{\text{deph}}(t), \quad (24)$$

where $q_{\text{deph}}(t) = |\Gamma_{\text{deph}}(t)| - \Gamma_{\text{deph}}(t)$ is the non-Markovianity for the dephasing channel and $q_{\text{dis}}(t) = |\Gamma_{\text{dis}}(t)| - \Gamma_{\text{dis}}(t)$ is that for the thermal part of the channel including the dissipation and absorption processes.

In Figs. 1 and 2, we plot the total non-Markovianity $q(t)$ and the contribution due to the thermal channel $q_{\text{dis}}(t)$ with different values of α to show the non-Markovian behavior of the dynamics. We see that the revival of $q_{\text{dis}}(t)$ increases with the increasing interaction strength α . In Figs. 3 and 4, we plot $q(t)$ and total non-Markovianity $q_{\text{dis}}(t)$, respectively, but for different numbers of bath spins N with a fixed interaction strength.

Let us now investigate the aspect of non-Markovianity from another well-known perspective, namely the distinguishability of two quantum states [48,49]. Consider any distance measure $D(\cdot)$ between two quantum states, following the contraction

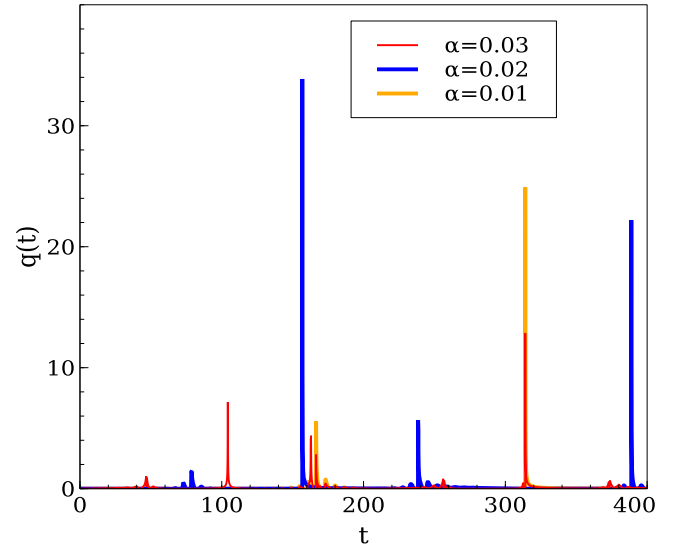


FIG. 1. Variation of $q(t)$ with time t for various interaction strengths α . The number of bath spins is kept fixed at $N = 20$. Positive $q(t)$ implies the non-Markovian nature of the dynamics according to the RHP measure.

property

$$D(\Phi[\rho^1], \Phi[\rho^2]) \leq D(\rho^1, \rho^2), \quad (25)$$

where $\Phi[\cdot]$ represents any CPTP map. Under any Markovian evolution, the time derivative of $D(\cdot)$ will always be negative, owing to this contraction property. Therefore, nonmonotonicity of these distances can be understood as a witness of the non-Markovian information feedback into the system. One such distance measure is the trace distance between quantum states [65]. Taking the trace distance between two states

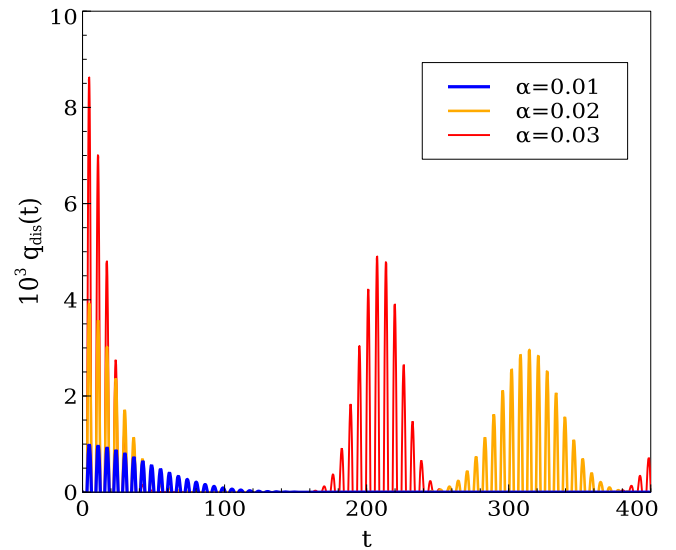


FIG. 2. Variation of $q_{\text{dis}}(t)$ with time t for various interaction strengths α . The number of bath spins is kept fixed at $N = 20$. To distinguish the effect on the thermal part of the quantum channel, we separately plot $q_{\text{dis}}(t)$. It can be seen from the plot that the non-Markovian revival for the thermal part of the channel increases with the increase of the interaction strength α for fixed N .

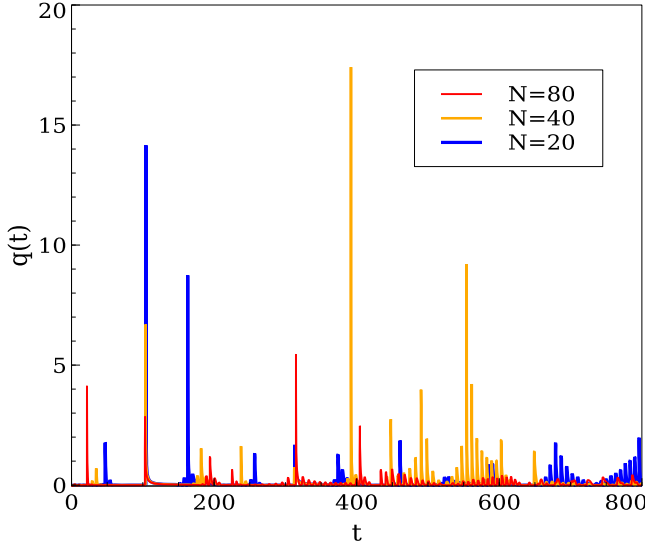


FIG. 3. Variation of $q(t)$ with time t for different numbers of bath spins N . Interaction strength $\alpha = 0.03$ is taken.

$D_T(\rho_1, \rho_2) = \frac{1}{2} \|\rho_1 - \rho_2\|$, a quantity can be defined as

$$p(t) = \frac{d}{dt} D_T(\Phi[\rho^1], \Phi[\rho^2]). \quad (26)$$

Breuer-Laine-Piilo (BLP) proposed a measure of non-Markovianity [48,49] by summing over all the positive contributions of $p(t)$ and maximizing over the input states, which is given by

$$\zeta = \max_{\rho^{1,2}} \int_{p(t)>0} p(t) dt. \quad (27)$$

It can readily be taken as a witness of non-Markovian information feedback into the system under any local decoherence

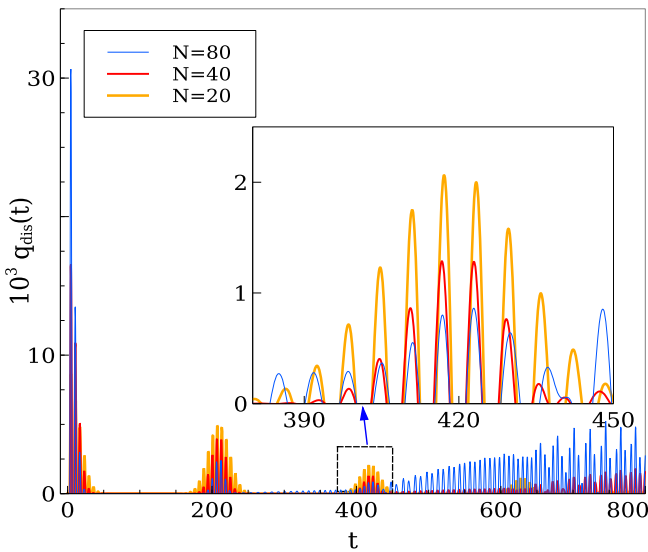


FIG. 4. Variation of $q_{\text{dis}}(t)$ with time t for different numbers of bath spins N . Interaction strength $\alpha = 0.03$ is taken. A magnified view of the rectangular region is shown in the inset. The plot depicts that the revival of $q_{\text{dis}}(t)$ increases with the increase of bath spins N .

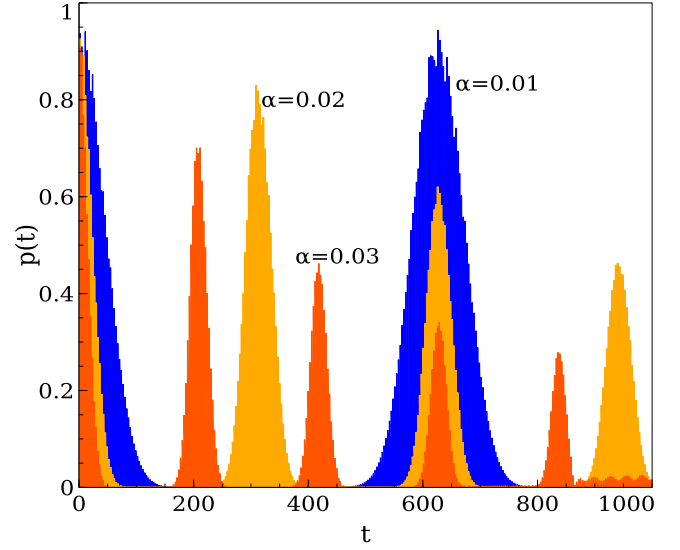


FIG. 5. Variation of $p(t)$ with time t for the two states $|\pm\rangle = \frac{1}{\sqrt{2}}|0\pm\rangle$, for different interaction strengths α (where $N = 20$). Positive $p(t)$ implies non-Markovianity according to the BLP measure.

channel. We find that for our specific quantum channel, the trace distance fidelity between two quantum states $\rho_1(t)$ and $\rho_2(t)$, at any arbitrary time after the action of the mentioned channel, can be expressed as

$$D(\Phi[\rho^1], \Phi[\rho^2]) = \sqrt{a^2[A(t) - B(t)]^2 + |b|^2|C(t)|^2}, \quad (28)$$

with $a = \rho_{11}^1(0) - \rho_{11}^2(0)$ and $b = \rho_{12}^1(0) - \rho_{12}^2(0)$. In Fig. 5, we plot the function $p(t)$ for the two states $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\pm\rangle)$. The time evolution of the function is plotted in Fig. 6, but for the case of increasing numbers of bath particles N . Note

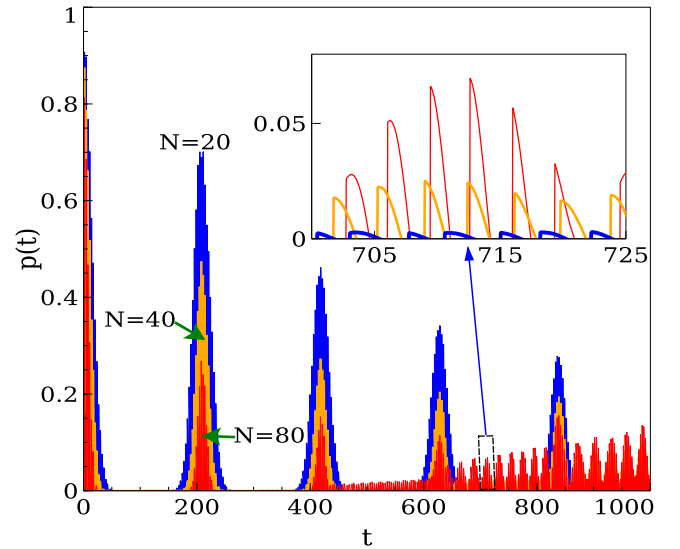


FIG. 6. Variation of $p(t)$ with time t for two density matrices $|\pm\rangle = \frac{1}{\sqrt{2}}|0\pm\rangle$, for different numbers of bath spins N and interaction strength $\alpha = 0.03$. A magnified view of the rectangular region is shown in the inset.

that calculating the maximized measure defined in Eq. (27) requires optimization over a and b , which is difficult in general. However, consideration of two specific states can demonstrate the non-Markovianity providing a lower bound of the measure. The two measures of non-Markovianity based on divisibility of the map (RHP measure, η) and distinguishability of two states under the action of the map (BLP measure, ζ) that we discuss here may not agree in general [58,66]. If a map is divisible, the evolution is Markovian and so the RHP measure of non-Markovianity η is zero. Consequently the BLP measure ζ is also zero. But the converse is generally not true; i.e., there exist some non-Markovian domains that are “bound” in terms of the BLP measure and hence not captured by it. The reason behind this is that the notion of complete positivity does not enter into the BLP measure and hence the divisibility breaking cannot be fully captured by it [58]. In this work we also consider the BLP measure of non-Markovianity to study whether the non-Markovian feature of our proposed master equation can be captured by the BLP measure also.

III. NEGATIVE ENTROPY PRODUCTION RATE

The irreversible or nonequilibrium entropy production and its rate are two fundamental concepts in the analysis of the nonequilibrium processes and the performance of thermodynamic devices [28–33]. The reduction of the nonequilibrium entropy production can significantly alter the performance of thermodynamic devices and thereby it is of utmost interest in various technological domains. The nonequilibrium entropy production rate is defined as

$$\sigma(t) = \frac{dS}{dt} + \mathcal{J}, \quad (29)$$

where S is the von Neumann entropy of the system and $\mathcal{J} = \frac{1}{KT} \frac{dQ}{dt} = \frac{1}{KT} \text{Tr}[H(t)\Lambda[\rho(t)]]$ is the entropy flux of the system. It can also be expressed as the time derivative of the relative entropy of the state ρ with respect to the thermal equilibrium state ρ_{eq} [34,54]:

$$\sigma(t) = -\frac{d}{dt} S(\rho||\rho_{\text{eq}}), \quad (30)$$

where $S(\rho||\tau) = -S(\rho) - \text{Tr}(\rho \ln \tau)$. According to Spohn’s theorem [34] the nonequilibrium entropy production rate σ is always non-negative. Spohn’s theorem is another statement of the second law of thermodynamics dictating the arrow of time. However, its validity essentially depends on the Markov approximation [48]. Under the non-Markovian dynamics σ can be negative [67,68]. Therefore, the non-Markovianity of the dynamics is a thermodynamic resource providing partial reversibility of work and entropy. In addition, negative $\sigma(t)$ is a prominent signature of the non-Markovianity and hence it can be used to detect and quantify the non-Markovianity. Since, for the specific system considered here, the absorption and the dissipation rates are equal due to the infinite temperature of the bath, the net heat flow $\frac{dQ}{dt}$ is always zero. Therefore, for this specific model, we have

$$\sigma(t) = \frac{dS}{dt}. \quad (31)$$

It is worth mentioning that under the action of the unital channel the von Neumann entropy of a system always increases

in Markovian dynamics, as it is also a doubly stochastic map. Since the given channel is unital, the negative $\frac{dS}{dt}$ also ensures the deviation from Markovianity. Note that the rate of change of entropy is given as

$$\frac{dS}{dt} = -\frac{d}{dt} \{\text{Tr}[\rho(t) \ln \rho(t)]\} = -\text{Tr}[\ln \rho(t) \Lambda[\rho(t)]]. \quad (32)$$

Here $\Lambda[\cdot]$ represents a general quantum evolution of the form

$$\Lambda[\rho(t)] = -\frac{i}{\hbar} [\rho(t), H_S(t)] + \sum_j \Gamma_j(t) \left[V_j \rho(t) V_j^\dagger - \frac{1}{2} \{V_j^\dagger V_j, \rho(t)\} \right]. \quad (33)$$

If the Lindblad operators $\{V_j\}$ are Hermitian, then Eq. (32) reads as

$$\frac{dS}{dt} = \frac{1}{2} \sum_{jkl} \Gamma_j(t) [\lambda_k(t) - \lambda_l(t)] \times [\ln \lambda_k(t) - \ln \lambda_l(t)] \langle \lambda_k(t) | V_j | \lambda_l(t) \rangle^2, \quad (34)$$

where we take the spectral decomposition of the density matrix as $\rho(t) = \sum_i \lambda_i(t) |\lambda_i(t)\rangle \langle \lambda_i(t)|$. The above equation also implies that $\frac{dS}{dt}$ is non-negative if the relaxation rates $\{\Gamma_j(t)\}$ are non-negative. However, $\frac{dS}{dt}$ can be negative if one or more of the relaxation rates $\{\Gamma_j(t)\}$ are negative, i.e., in the non-Markovian domain. For the dynamics considered here, $\sigma(t)$ can be expressed as

$$\sigma(t) = \frac{1}{2} \ln \left(\frac{1-x}{1+x} \right) \frac{dx}{dt}, \quad (35)$$

where $x = \sqrt{[\rho_{11}(t) - \rho_{22}(t)]^2 + 4|\rho_{12}(t)|^2}$. We plot the nonequilibrium entropy production rate $\sigma(t)$ starting from the pure initial state $|1\rangle$ in Fig. 7, which clearly shows that $\sigma(t)$ becomes negative whenever $\Gamma_{\text{dis}}(t)$ becomes negative. It has been shown in Ref. [67] that, for a diagonal qubit state, σ can be negative only when the non-Markovian dynamics drives the system away from its thermal equilibrium. The example considered here completely agrees with this fact.

From Eq. (34) it is quite evident that the time rate of change of the entropy can be negative, only when the divisibility of the dynamical map breaks down. Therefore, a witness of non-Markovianity can be constructed from the negative entropy production rate for unital channels as follows:

$$\varphi = \max_{\rho_{\text{in}}} \int_{\kappa(t)>0} \kappa(t) dt, \quad (36)$$

where $\kappa(t) = -\frac{dS}{dt}$. The measure of non-Markovianity based on the entropy production rate has been considered before for unital dynamical maps [64].

A. Rate of change of purity: Detection of non-Markovianity

Let us investigate the non-Markovian behavior by the rate of change of the purity of the central qubit. If the Lindblad operators $\{V_j\}$ in Eq. (33) are Hermitian then the rate of change of the purity $P(= \text{Tr}\rho^2)$ of the central qubit can be given as

$$\frac{dP}{dt} = 2\text{Tr}[\rho(t)\Lambda[\rho(t)]] = -\sum_i \Gamma_i(t) Q_i(t), \quad (37)$$

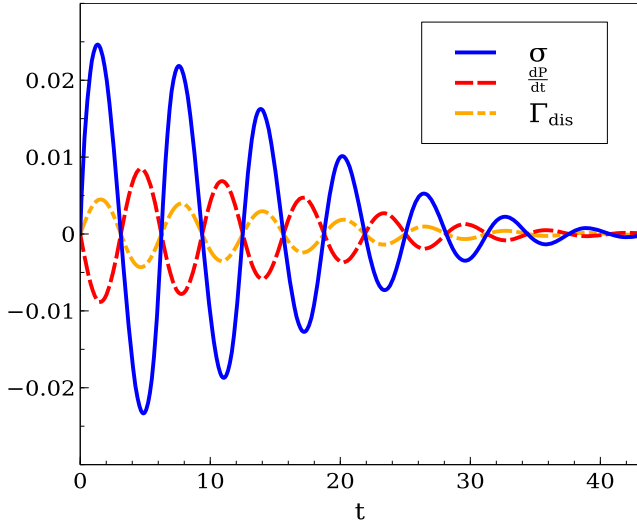


FIG. 7. Variation of the rate of change of the irreversible entropy production $\sigma(t)$, the rate of change of the purity $\frac{dP}{dt}$, and Γ_{dis} with time t for the initial state $|1\rangle$ with the interaction strength $\alpha = 0.03$ and the number of bath spins $N = 20$. It is evident that $\sigma(t)$ and $\frac{dP}{dt}$ are negative and positive, respectively, whenever Γ_{dis} is negative. This implies that the non-Markovian information backflow revives the purity of the state and causes a negative irreversible entropy production rate.

where $Q_i(t) = \|[V_i, \rho(t)]\|_{\text{HS}}^2$. The abbreviation in the subscript stands for the Hilbert-Schmidt norm $\|[X]\|_{\text{HS}} = \sqrt{\text{Tr}(X^\dagger X)}$. As $\{Q_i(t)\}$ values are always positive, the positive rate of change of purity can only occur for the negativity of one or more of $\{\Gamma_i(t)\}$ which corresponds to the divisibility breaking of the dynamical map. Note that the dynamics considered here can be expressed as a master equation with the Pauli matrices being the Lindblad operators [see Eq. (17)] and the relaxation rates given as $\Gamma_x(t) = \Gamma_y(t) = \Gamma_{\text{dis}}(t)/2, \Gamma_z(t) = \Gamma_{\text{deph}}(t)$. Since the Pauli matrices are Hermitian operators, thereby the positive rate of change of purity of the central spin clearly signifies the non-Markovianity of the dynamical map. It is also worth mentioning that when the Lindblad operators $\{V_j\}$ are Hermitian or in other words when they represent observables, then $Q_i(t) = \|[V_i, \rho(t)]\|_{\text{HS}}^2$, measures the quantumness [69,70] of the state $\rho(t)$. Therefore, Eq. (37) implies that the greater the quantumness of the state the more it is sensitive to the environment. After a little algebra, we find that the rate of change of purity for the initial central qubit state $|1\rangle$ is given as

$$\frac{dP}{dt} = [A(t) - B(t)] \frac{d}{dt} [A(t) - B(t)]. \quad (38)$$

We plot the rate change of the purity with time in Fig. 7. From Fig. 7 it can be seen that the positive rate of change of purity occurs periodically, whenever the relaxation rate $\Gamma_{\text{dis}}(t)$ is negative. Since we are taking an initial diagonal state in the

computational basis, there is no effect of the dephasing channel on the central qubit. For a qubit system, its eigenvalues have the form $\lambda = \frac{(1 \pm z)}{2}$, where $0 \leq z \leq 1$, and hence, the entropy of a qubit system is a monotonically decreasing function of the purity of the qubit. Therefore, the signs of the rates of change of purity and entropy (see Fig. 7) are opposite.

Nowadays with advanced experimental techniques, the purity of a quantum system can be directly measured [71–73]. Hence, the non-Markovian revival of purity can be experimentally verified to demonstrate the non-Markovianity and the negative nonequilibrium entropy production rate in the laboratory.

IV. CONCLUSION

To summarize, we have considered the dynamics of a central spin- $\frac{1}{2}$ particle which is interacting with a bath consisting of completely unpolarized, noninteracting spin- $\frac{1}{2}$ particles. An exact canonical Lindblad-type master equation has been derived for the central spin system. The dynamics of the system exhibits non-Markovian features which have been characterized and quantified by divisibility breaking (the RHP measure of non-Markovianity) as well as monotonicity breaking of the trace distance fidelity (the BLP measure of non-Markovianity) conditions. The Kraus operators for the dynamical evolution are also derived.

The nonequilibrium entropy production rate has been investigated. A negative entropy production rate implies the non-Markovianity of the dynamics, though the reverse does not hold true. The dynamics of the central spin considered here shows that for a specific initial state the non-Markovianity of the dynamics is always associated with a negative entropy production rate. Moreover, it has also been shown that, in this dynamics, the non-Markovianity is always accompanied by the increase of the purity of the central spin when the same initial state has been chosen. As purity is a measurable quantity, the exact canonical Lindblad-type master equation of the central spin, derived in this article, could be of paramount importance to investigate the non-Markovian features and the negative entropy production rate in the laboratory. The scheme used here to derive the canonical master equation has been proven to be fruitful to explore the strong coupling regime where the system-bath separability breaks down, which gives the present study a practical importance to unravel the far-reaching impacts of the non-Markovian dynamics in the strong coupling regime in various information theoretic and thermodynamic protocols and devices.

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