Tensor power of dynamical maps and positive versus completely positive divisibility

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The are several nonequivalent notions of Markovian quantum evolution. In this paper we show that the one based on the so-called CP divisibility of the corresponding dynamical map enjoys the following stability property: the dynamical map Λ_t is CP divisible if and only if the second tensor power $\Lambda_t \otimes \Lambda_t$ is CP divisible as well. Moreover, the P divisibility of the map $\Lambda_t \otimes \Lambda_t$ is equivalent to the CP divisibility of the map Λ_t. Interestingly, the latter property is no longer true if we replace the P divisibility of $\Lambda_t \otimes \Lambda_t$ by simple positivity and the CP divisibility of Λ_t by complete positivity. That is, unlike when Λ_t has a time-independent generator, positivity of $\Lambda_t \otimes \Lambda_t$ does not imply complete positivity of Λ_t.

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I. INTRODUCTION

Usually, the nonunitary, dissipative time evolution of an open quantum system S, that we take as a finite-level system, for the sake of simplicity, is approximated by dynamical maps Λ_t that are constrained to be completely positive [1,2]. Namely, if the system S is initially statistically coupled to an inert, nonevolving copy of it, the dynamics $\Lambda_t \otimes \mathrm{id}$ of $S+S$ must be positive, that is, it must map all possible initial states of $S+S$ into density matrices, thus guaranteeing the positivity of their spectrum at all times. Otherwise, there surely exist entangled states of $S+S$ whose spectrum acquires negative eigenvalues that cannot then be interpreted as probabilities [1]. However, in view of the generic and uncontrollable character of the ancilla, such a motivation for the necessity of complete positivity is scarcely physically palatable, above all because of the ensuing constraints which, in the case of a Markovian dynamical semigroup, $\Lambda_t = \exp(tL)$, are embodied by the celebrated Gorini-Kossakowski-Sudarshan-Lindblad form of the generator L [3,4],

$$L[\rho] = -i[H,\rho] + \frac{1}{2}\sum_\alpha \gamma_\alpha([V_\alpha,\rho V_\alpha^\dagger] + [V_\alpha\rho, V_\alpha^\dagger]) \quad (1)$$

with positive decoherence and dissipation rates $\gamma_\alpha > 0$.

In [5] a more physical point of view was presented, whereby both systems are embedded into a same environment thus undergoing the same dissipative Markovian time evolution, Λ_t. In this case, the ancilla is not out of practical control and not inert, the compound system $S+S$ dynamics being described by $\Lambda_t \otimes \Lambda_t$. Physical consistency then demands that the latter map be positive so to exclude the appearance of negative probabilities in the spectrum of time-evolving states of $S+S$. It turns out that the complete positivity of $\Lambda_t = \exp(tL)$ is equivalent to the positivity of the tensor product $\Lambda_t \otimes \Lambda_t$. Hence positivity of $e^{tL} \otimes e^{tL}$ implies the generator L to be of the Lindblad form (1).

Consider now the time-local master equation

$$\frac{d}{dt}\Lambda_t = L_t\Lambda_t, \quad \Lambda_{t=0} = \mathbb{I}, \quad (2)$$

governed by time-local generator L_t. One may wonder wether a similar result holds for the solution Λ_t; that is, is it true that positivity of $\Lambda_t \otimes \Lambda_t$ implies complete positivity of Λ_t? In this paper we show that this result is no longer true for general L_t (in the next section we provide a concrete counterexample of random unitary qubit evolution). However, it holds for Markovian dynamical maps. Hence, violation of the above implication may be considered as another witness of non-Markovianity of Λ_t. Non-Markovian quantum evolutions have recently been extensively analyzed (see [6–8] for recent reviews and the collection of papers in [9]). There are several nonequivalent approaches to (non)Markovian evolution [6–8]. The most popular ones are based on distinguishability of quantum states [10], divisibility of the corresponding dynamical map [11], quantum mutual information [12], quantum Fisher information [13], capacity of quantum channels [14], volume of accessible states [15], and quantum interferometric power [16] (this list is by no means exhaustive). In this paper we adopt the one based on the concept of divisibility. Recall that Λ_t is divisible if $\Lambda_t = \Lambda_{t,s}\Lambda_s$ for all $t \geqslant s \geqslant 0$. Moreover, a divisible map Λ_t is

(1) CP divisible if $\Lambda_{t,s}$ is completely positive;

(2) P divisible if $\Lambda_{t,s}$ is positive.

Notice that, if Λ_t is CP divisible, then $\Lambda_t = \Lambda_{t,s=0}$ is completely positive for all $t \geqslant 0$. Analogously, if Λ_t is P divisible, then Λ_t is at least positive for all $t \geqslant 0$. We call the quantum evolution Markovian if and only if the corresponding dynamical map Λ_t is CP divisible [11,17–19]. In the following, we show that the P divisibility of $\Lambda_t \otimes \Lambda_t$ implies that Λ_t is CP divisible, hence corresponding to a Markovian evolution. Clearly, the CP divisibility of Λ_t implies the CP divisibility of $\Lambda_t \otimes \Lambda_t$; then, on the level of the tensor product $\Lambda_t \otimes \Lambda_t$ P and CP divisibility are equivalent. This proves that the notion of Markovianity based on the concept of CP divisibility is stable with respect to replacing Λ_t with the tensor product $\Lambda_t \otimes \Lambda_t$.

II. POSITIVE, NOT COMPLETELY POSITIVE QUBIT DYNAMICS

In this section we construct a positive (but not completely positive) map Λ_t such that $\Lambda_t \otimes \Lambda_t$ is positive. Consider the following qubit time-local generator

$$L_t[\rho] = \frac{\alpha}{2}\sum_{k=1}^{3}\gamma_k(t)(\sigma_k\rho\sigma_k - \rho), \tag{3}$$

where σ_j, $j = 1,2,3$ are the Pauli matrices, $\gamma_1(t) = \gamma_2(t) = 1$, and $\gamma_3(t) = -\tanh(t)$. The parameter $\alpha > 0$ controls the property of the corresponding map Λ_t. For $\alpha = 1$ this generator was already considered in [20] as an example of so-called *eternal* non-Markovian evolution (see also [21]).

Proposition 1. The corresponding map Λ_t is

(i) positive for all $\alpha > 0$;

(ii) completely positive if and only if $\alpha \geqslant 1$.

Proof. Let us represent a qubit density matrix by a Bloch vector $\mathbf{r} = (r_1,r_2,r_3)$ such that

$$\varrho = \frac{1}{2}\left(\mathbb{I} + \sum_{j=1}^{3} r_j\sigma_j\right), \quad r_j \in \mathbb{R}, \quad \sum_{j=1}^{3} r_j^2 \leqslant 1. \tag{4}$$

When complemented with the identity matrix $\sigma_0 = \mathbb{I}$, the matrices σ_μ, $\mu = 0,1,2,3$, are eigenvectors of L_t,

$$L_t[\sigma_\mu] = \lambda_\mu(t)\sigma_\mu, \tag{5}$$

with eigenvalues

$$\lambda_0(t) = 0, \quad \lambda_1(t) = \lambda_2(t) = \alpha[\tanh(t) - 1], \quad \lambda_3(t) = -2\alpha.$$

One then readily gets the following time-evolution equations for the Bloch vector components of $r_j(t)$ of $\varrho_t = \Lambda_t[\varrho]$,

$$\frac{dr_{1,2}(t)}{dt} = \alpha\frac{\tanh(t) - 1}{2} r_{1,2}(t)\ , \quad \frac{dr_3(t)}{dt} = -2\alpha r_3(t),$$

so that a straightforward integration yields

$$\Lambda_t[\varrho] = \tfrac{1}{2}[\mathbb{I} + e^{-\alpha t}\cosh^\alpha(t)(r_1\sigma_1 + r_2\sigma_3) + e^{-2\alpha t} r_3\sigma_3]. \tag{6}$$

The map $\varrho \to \Lambda_t[\varrho]$ is positive since $e^{-\alpha t}[\cosh(t)]^\alpha \leqslant 1$ for $t \geqslant 0$. In order to analyze the complete positivity of Λ_t, let us observe that its action can be recast in the form

$$\Lambda_t = \sum_{\mu=0}^{3} p_\mu(t) S_\mu, \tag{7}$$

where $S_\mu[\varrho] = \sigma_\mu\varrho\sigma_\mu$ and the parameters $p_\mu(t)$ are [22,23]

$$\begin{aligned} p_0(t) &= \tfrac{1}{4}[1 + 2e^{-\alpha t}\cosh^\alpha(t) + e^{-2\alpha t}],\\ p_1(t) &= p_2(t) = \tfrac{1}{4}(1 - e^{-2\alpha t}),\\ p_3(t) &= \tfrac{1}{4}[1 - 2e^{-\alpha t}\cosh^\alpha(t) + e^{-2\alpha t}]. \end{aligned} \tag{8}$$

Clearly, Λ_t is CP if and only if (7) corresponds to a Kraus representation, that is, if and only if $p_\mu(t) \geqslant 0$ for all $t \geqslant 0$. The only nontrivial condition $p_3(t) \geqslant 0$ is equivalent to

$$\cosh(\alpha t) \geqslant \cosh^\alpha(t), \tag{9}$$

which is satisfied if and only if $\alpha \geqslant 1$. Indeed, $f(t) = \ln\cosh t$ has a positive second derivative and is thus convex. Hence, since $f(0) = 0$, for any $0 \leqslant \alpha \leqslant 1$ one has

$$f(\alpha t + (1-\alpha)\times 0) \leqslant \alpha f(t) + (1-\alpha) f(0)\ ,$$

so that (9) is violated. On the other hand, if $\alpha \geqslant 1$,

$$f(t) = f\left(\frac{1}{\alpha}(\alpha t) + \left(1 - \frac{1}{\alpha}\right)\times 0\right) \leqslant \frac{1}{\alpha} f(\alpha t),$$

whence (9) follows. ■

As briefly outlined in the Introduction, when Λ_t has a time-independent generator, the lack of complete positivity of Λ_t and thus of positivity of $\Lambda_t \otimes \mathrm{id}_2$, is often not regarded as a compelling argument in favor of complete positivity. This is so because envisioning possible initial quantum correlations of the system of interest with an ancilla, another generic qubit in the present case, otherwise completely independent and inert, looks more like a mathematical request than a necessary physical constraint. Moreover, the consequences of such an abstract motivation are nonetheless physically quite relevant. Indeed, despite the fact that it is perfectly well behaved on single qubit states, a time evolution as Λ_t in (6) is ruled out as physically inconsistent because it is $\Lambda_t \otimes \mathrm{id}_2$, which is physically ill defined: indeed, it cannot keep positive all possible initially entangled states.

However, if instead of a generic, uncontrollable ancilla, one considers another system under the same physical conditions of an open system as the previous one and noninteracting with the former, then the dynamics of the compound system becomes $\Lambda_t \otimes \Lambda_t$. Unlike $\Lambda_t \otimes \mathrm{id}_2$, $\Lambda_t \otimes \Lambda_t$ is physically more tenable and physical consistency demands it to be positive. In [5] it was proved that, in the Markovian case when the time-local generator L_t is in fact time independent, $L_t = L$, $\Lambda_t \otimes \Lambda_t$ is positive if and only if Λ_t is completely positive.

We now show, by means of a counterexample, that this conclusion does not hold in the more general setting represented by the master equation (2). The main technical tool is the following result proved in Proposition 4 of [24].

Proposition 2. If Λ_t is a linear map on the algebra $M_2(\mathbb{C})$ of 2×2 matrices and Λ_t^2 is completely positive, then $\Lambda_t \otimes \Lambda_t$ is positive.

Proposition 3. The maps Λ_t in (6) satisfy the following property: $\Lambda_t \otimes \Lambda_t$ is positive for all $\alpha \geqslant \frac{1}{2}$.

Proof. We show that for $\alpha \geqslant \frac{1}{2}$ the map Λ_t^2 is completely positive and hence, due to Proposition 2, the tensor product $\Lambda_t \otimes \Lambda_t$ is positive. Using the Pauli matrix algebra, one reduces the product $S_\mu S_\nu$ to the action of single S_λ and finds

$$\Lambda_t^2 = \sum_{\mu=0}^{3} q_\mu(t) S_\mu, \tag{10}$$

with parameters

$$\begin{aligned} q_0(t) &= \tfrac{1}{4}[1 + 2e^{-2\alpha t}\cosh^{2\alpha}(t) + e^{-4\alpha t}],\\ q_1(t) &= q_2(t) = \tfrac{1}{4}(1 - e^{-4\alpha t}),\\ q_3(t) &= \tfrac{1}{4}[1 - 2e^{-2\alpha t}\cosh^{2\alpha}(t) + e^{-4\alpha t}], \end{aligned}$$

which differ from (8) by an obvious replacement $\alpha \to 2\alpha$. Then, if $\alpha \geqslant \frac{1}{2}$ one has $q_\mu(t) \geqslant 0$. ■

Remark 1. Putting together Proposition 2 and Proposition 3, it follows that for $\alpha \in [\frac{1}{2},1)$ the map Λ_t is positive but not completely positive, whereas the tensor product $\Lambda_t \otimes \Lambda_t$ is positive. This way we provided a counterexample to the naive expectation that the property—Λ_t is completely positive if and only if $\Lambda_t \otimes \Lambda_t$ is positive—that holds for time-independent generators [5], might also hold for general master equations of the form (2). Thus, in general, the relations between the (complete) positivity of the maps Λ_t and the (complete) positivity of the maps $\Lambda_t \otimes \Lambda_t$ can be summarized by the following diagram:

$$\begin{tikzcd}
\Lambda_t \otimes \Lambda_t \textit{ is positive} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "/" marking] & \Lambda_t \textit{ is positive} \\
\Lambda_t \textit{ is completely positive} \arrow[ur, Rightarrow] \arrow[d, Leftrightarrow] & \\
\Lambda_t \otimes \Lambda_t \textit{ is completely positive} \arrow[uu, Rightarrow, bend left] &
\end{tikzcd}$$

Remark 2. Interestingly, Proposition 3 also provides a counterexample to another naive expectation that if L_t generates completely positive dynamical maps Λ_t, then the rescaled cL_t with $c > 0$ does the same. Noticeably, the model of random unitary evolution (7) was recently used for describing the effective dynamics of disordered quantum systems [23,25].

III. P AND CP DIVISIBILITY

While the positivity of $\Lambda_t \otimes \Lambda_t$ does not in general require Λ_t to be completely positive when the generator of Λ_t is time dependent, in this section, we shall instead show that the P divisibility of $\Lambda_t \otimes \Lambda_t$ implies the CP divisibility of Λ_t. Indeed, the following result holds whose proof is an adaptation from [5].

Theorem 1. The one-parameter family $\{\Lambda_t\}_{t\geqslant 0}$ on the state space of a d-level system is CP divisible if and only if $\{\Lambda_t \otimes \Lambda_t\}_{t\geqslant 0}$ is P divisible.

Proof.

The scheme of the proof is as follows:

$$\begin{tikzcd}
\Lambda_t \otimes \Lambda_t \text{ is P-divisible} \arrow[d, Rightarrow] \\
\Lambda_t \text{ is CP-divisible} \arrow[d, Rightarrow] \\
\Lambda_t \otimes \Lambda_t \text{ is CP-divisible} \arrow[uu, Rightarrow, bend right]
\end{tikzcd}$$

Because of linearity and trace preservation, the action of the local time-dependent generator L_t on a state ϱ can always be written in the form

$$L_t[\varrho] = -i[H_t,\varrho] + \sum_{i,j=1}^{d^2-1} C_{ij}(t)\left(F_i\varrho F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_i,\varrho\}\right),$$

with respect to an orthonormal Hilbert-Schmidt basis of d^2-1 $d\times d$ traceless matrices such that $\mathrm{Tr}(F_j^\dagger F_i) = \delta_{ij}$ complemented with $F_{d^2} = 1/\sqrt{d}$. The only consistency request on the $(d^2-1)\times(d^2-1)$ matrix $C(t) = [C_{ij}(t)]$ is that it be Hermitian.

The P divisibility of $\Lambda_t \otimes \Lambda_t$ implies that the maps

$$\Lambda_{t,s}\otimes\Lambda_{t,s} = \mathcal{T}\exp\left[\int_s^t du\,(L_u\otimes \mathrm{id} + \mathrm{id}\otimes L_u)\right],$$

with id the identity operation, are positive for all $t \geqslant s \geqslant 0$; namely, that

$$\Lambda_{t,s}\otimes\Lambda_{t,s}[|\psi\rangle\langle\psi|] \geqslant 0, \quad \forall t \geqslant s \geqslant 0, \quad \forall|\psi\rangle \in \mathbb{C}^d\otimes\mathbb{C}^d.$$

Choosing $|\phi\rangle \perp |\psi\rangle$ and expanding $\Lambda_{t,s}\otimes\Lambda_{t,s}$, one obtains, up to first order in $t-s \geqslant 0$ with $s \geqslant 0$ fixed,

$$\begin{aligned}
0 \leqslant \langle\phi|\Lambda_{t,s}\otimes\Lambda_{t,s}[|\psi\rangle\langle\psi|]|\phi\rangle \simeq \Delta_{t-s} &:= (t-s)\,(\langle\phi|L_s\otimes\mathrm{id}[|\psi\rangle\langle\psi|]|\phi\rangle + \langle\phi|\mathrm{id}\otimes L_s[|\psi\rangle\langle\psi|]|\phi\rangle)\\
&= (t-s)\sum_{i,j=1}^{d^2-1} C_{ij}(s)(\langle\phi|F_i\otimes 1|\psi\rangle\langle\psi|F_j^\dagger\otimes 1|\phi\rangle + \langle\phi|1\otimes F_i|\psi\rangle\langle\psi|1\otimes F_j^\dagger|\phi\rangle).
\end{aligned}$$

Fixing an orthonormal basis in $\mathbb{C}^d$ and regrouping the d^2 components ψ_{ab} of $|\psi\rangle$ and ϕ_{ab} of $|\phi\rangle$ into $d\times d$ matrices $\Psi = [\psi_{ab}]$ and $\Phi = [\phi_{ab}]$, one can set $u_i^* := \langle\phi|F_i\otimes 1|\psi\rangle$, $v_i^* := \langle\phi|1\otimes F_i|\psi\rangle$ and write

$$u_i^* = \sum_{a,b,c=1}^{d} \phi_{ab}^* F_i^{ac}\psi_{cb} = \mathrm{Tr}(\Psi\,\Phi^\dagger\,F_i),$$

$$v_i^* = \sum_{a,b,d=1}^{d} \phi_{ab}^* F_i^{bd}\psi_{ad} = \mathrm{Tr}[(\Phi^\dagger\,\Psi)^{tr}\,F_i],$$

so that

$$\Delta_{t-s} = (t-s)(\langle u|C(s)|u\rangle + \langle v|C(s)|v\rangle),$$

where $|u\rangle$ and $|v\rangle$ are (d^2-1)-dimensional vectors with components u_i and v_i, respectively. Choosing $\Phi^\dagger = U$, $\Psi = MU^{-1}$, $M,U \in M_{d^2-1}(\mathbb{C})$, U being the similarity matrix such that $M^{tr} = UMU^{-1}$ (such a matrix U always exists), one finds $\Psi\Phi^\dagger = M$, $\Phi^\dagger\Psi = M^{tr}$, whence $|u\rangle = |v\rangle$.

The orthogonality of $|\psi\rangle$ and $|\phi\rangle$ amounts to asking that $\mathrm{Tr}(\Phi^\dagger\Psi) = \mathrm{Tr}(M) = 0$; then, tracelessness is the only constraint M must fulfill. Therefore, varying M one can achieve any $|u\rangle$ in $\mathbb{C}^{d^2-1}$. The positivity of $\Lambda_{t,s}\otimes\Lambda_{t,s}$ asks for

$$\Delta_{t-s} = 2(t-s)\,\langle u|C(s)\,|u\rangle \geqslant 0, \quad \forall\,|u\rangle \in \mathbb{C}^{d^2-1},$$

which in turn yields the positive semidefiniteness of the coefficient matrix $C(s) \in M_{d^2-1}(\mathbb{C})$. Such a condition is sufficient for the complete positivity of the maps $\Lambda_{t,s}$. Then, the one-parameter family $\{\Lambda_t\}_{t\geqslant 0}$ is CP divisible.

Vice-versa, if $\{\Lambda_t\}_{t\geqslant 0}$ is CP-divisible, then the maps $\Lambda_{t,s}$, $t \geqslant s \geqslant 0$, are completely positive, as well as the tensor products $\Lambda_{t,s} \otimes \Lambda_{t,s}$ so that the one-parameter family $\{\Lambda_t \otimes \Lambda_t\}_{t\geqslant 0}$ is CP- and thus P-divisible. ■

One has the following straightforward implications.

Corollary 1. $\Lambda_t \otimes \Lambda_t$ is P divisible if and only if $\Lambda_t \otimes \Lambda_t$ is CP divisible.

Corollary 2. Λ_t is Markovian if and only if $\Lambda_t \otimes \Lambda_t$ is Markovian.

The model studied in the previous section provides the following intriguing observation.

Corollary 3. For $\alpha \in [\frac{1}{2},1)$ the maps Λ_t in (6) are such that

(i) Λ_t is positive but not completely positive;

(ii) $\Lambda_{t,s}$ is positive for $t > s \geqslant 0$, which means that Λ_t is P divisible;

(iii) $\Lambda_t \otimes \Lambda_t$ is positive; and

(iv) $\Lambda_{t,s} \otimes \Lambda_{t,s}$ cannot be positive for all $t > s \geqslant 0$.

Proof. The one-parameter family of the maps Λ_t in (6) is P divisible: this follows from a result in [22] together with the fact that $\gamma_i(t) + \gamma_j(t) \geqslant 0$ for $i \neq j$. Then, the maps $\Lambda_{t,s}$ are positive for all $t \geqslant s \geqslant 0$. If the tensor product maps $\Lambda_{t,s} \otimes \Lambda_{t,s}$ were also positive for all $t \geqslant s \geqslant 0$, then the one-parameter family $\Lambda_t \otimes \Lambda_t$ would be P divisible and hence CP divisible, according to Corollary 2. Then, the maps $\Lambda_{t,s}$ would be completely positive for all $t \geqslant s \geqslant 0$ contradicting the fact that $\Lambda_t := \Lambda_{t,s=0}$ are positive, but not completely positive. ■

Remark 3. The previous corollary shows that, unlike the notion of Markovianity based on CP divisibility, the one based on the vanishing backflow of information [10] is not stable with respect to the tensor product. Let us recall that following [10] one can define the information flow by means of

$$\sigma(\varrho_1,\varrho_2;t) = \frac{d}{dt}\|\Lambda_t[\varrho_1 - \varrho_2]\|_1, \tag{11}$$

where ϱ_1 and ϱ_2 are arbitrary density operators of the system. According to [10], the evolution is defined Markovian if $\sigma(\varrho_1,\varrho_2;t) \leqslant 0$ for any ϱ_1, ϱ_2, and $t \geqslant 0$. Whenever $\sigma(\varrho_1,\varrho_2;t) > 0$, the two density matrices become more distinguishable and this fact is identified as information flowing from the environment into the system which provides a clear sign of memory effects. Now, for time evolutions generated by (3), the definition of Markovianity as the absence of backflow of information coincides with P divisibility [19,22]. Hence, Proposition 2 provides an example of dynamical maps Λ_t with vanishing backflow of information, such that their tensor product $\Lambda_t \otimes \Lambda_t$ nonetheless gives rise to nontrivial backflow of information.

IV. CONCLUSIONS

In this paper we have discussed the consequences of asking that a one-parameter family of dynamical maps Λ_t consist of Λ_t such that $\Lambda_t \otimes \Lambda_t$ be positive. Unlike when $\Lambda_t = \exp(t\,L)$, in the case Λ_t is generated by a time-local master equation, the positivity of $\Lambda_t \otimes \Lambda_t$ does not enforce the complete positivity of Λ_t. It is, however, the P divisibility of $\Lambda_t \otimes \Lambda_t$ that implies the CP divisibility of Λ_t.

There follow interesting connections between the P divisibility, which defines classical Markovian evolutions, and the CP divisibility, which defines Markovianity in the quantum case. The crucial property of CP divisibility is stability with respect to the second tensor power of the corresponding quantum dynamical map. If one relaxes CP divisibility just to P divisibility this property is no longer true. It should be stressed, however, that the property of P divisibility is stable with respect to the second tensor power of the corresponding classical dynamical map (family of stochastic matrices).

We have also revealed an interesting phenomenon of superactivation of the back flow of information, namely, there exist dynamical maps Λ_t with vanishing flow of information from the environment into the system such that the second tensor power $\Lambda_t \otimes \Lambda_t$ nevertheless induces nonzero backflow of information.

The present paper raises an important question of stability of other non-Markovianity measures [10,12–16] with respect to tensoring of the dynamical map. The measure based on distinguishability of states [10] is not stable (Remark 3). The measure based on quantum capacity [14] is stable, since the capacity is defined as regularized coherent information. The stability of other measures is an open question.

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[1] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* (Springer, Berlin, 1987).
[2] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2007).
[3] A. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
[4] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[5] F. Benatti, R. Floreanini, and R. Romano, J. Phys. A: Math. Gen. **35**, 4955 (2002).
[6] Á. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. **77**, 094001 (2014).
[7] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys. **88**, 021002 (2016).
[8] I. de Vega and D. Alonso, arXiv:1511.06994 [Revi. Mod. Phys. (to be published)].
[9] F. Benatti, R. Floreanini, and G. Scholes, J. Phys. B **45**, 150201 (2012).
[10] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).
[11] Á. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010).
[12] S. Luo, S. Fu, and H. Song, Phys. Rev. A **86**, 044101 (2012).
[13] X.-M. Lu, X. Wang, and C. P. Sun, Phys. Rev. A **82**, 042103 (2010).

[14] B. Bylicka, D. Chruściński, and S. Maniscalco, Sci. Rep. **4**, 5720 (2014).

[15] S. Lorenzo, F. Plastina, and M. Paternostro, Phys. Rev. A **88**, 020102(R) (2013).

[16] H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. A **91**, 032115 (2015).

[17] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Phys. Rev. Lett. **101**, 150402 (2008).

[18] D. Chruściński and A. Kossakowski, J. Phys. B **45**, 154002 (2012).

[19] D. Chruściński and S. Maniscalco, Phys. Rev. Lett. **112**, 120404 (2014).

[20] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, Phys. Rev. A **89**, 042120 (2014).

[21] N. Megier, D. Chruściński, J. Piilo, and W. T. Strunz, arXiv:1608.07125.

[22] D. Chruściński and F. A. Wudarski, Phys. Lett. A **377**, 1425 (2013).

[23] D. Chruściński and F. A. Wudarski, Phys. Rev. A **91**, 012104 (2015).

[24] S. N. Filippov and K. Yu. Magadov, J. Phys. A: Math. Theor. **50**, 055301 (2017).

[25] C. M. Kropf, C. Gneiting, and A. Buchleitner, Phys. Rev. X **6**, 031023 (2016).