

Quantum relative Lorenz curvesFrancesco Buscemi^{1,*} and Gilad Gour^{2,†}¹*Department of Computer Science and Mathematical Informatics, Nagoya University, Chikusa-ku, Nagoya 464-8601, Japan*²*Institute for Quantum Science and Technology and Department of Mathematics and Statistics, University of Calgary, 2500 University Drive NW, Calgary, Alberta, Canada T2N 1N4*

(Received 19 July 2016; published 9 January 2017)

The theory of majorization and its variants, including thermomajorization, have been found to play a central role in the formulation of many physical resource theories, ranging from entanglement theory to quantum thermodynamics. Here we formulate the framework of quantum relative Lorenz curves, and show how it is able to unify majorization, thermomajorization, and their noncommutative analogs. In doing so, we define the family of Hilbert α divergences and show how it relates with other divergences used in quantum information theory. We then apply these tools to the problem of deciding the existence of a suitable transformation from an initial pair of quantum states to a final one, focusing in particular on applications to the resource theory of athermality, a precursor of quantum thermodynamics.

DOI: [10.1103/PhysRevA.95.012110](https://doi.org/10.1103/PhysRevA.95.012110)**I. INTRODUCTION**

Lorenz curves, originally introduced to give a quantitative and pictorially clear representation of the inequality of the wealth distribution in a country [1], have since then been used also in other contexts in order to effectively compare different distributions (see, for example, [2], and references therein). In their typical formulation, Lorenz curves fully capture the notion of “nonuniformity” [3] of a distribution, in the sense that comparing the Lorenz curves associated to two given distributions (say, p and q) induces an ordering equivalent to the relation of *majorization*, which, in turn, is well known to be equivalent to the existence of a random permutation (i.e., a bistochastic channel) transforming p into q [2].

More recently, some variants of the original definition were proposed in order to capture other aspects of a given distribution, besides its mixedness. In particular, *thermomajorization* was introduced in [4] to characterize state transitions under thermal operations or Gibbs preserving operations [5]. Here, the corresponding Lorenz curve characterizes a partial ordering relative to the Gibbs distribution, rather than the uniform one.

This suggests that Lorenz curves are best understood not as properties of one given distribution, but rather of a given *pair* of distributions, one being the “state” at hand and the other being the “reference.” For example, the original Lorenz curve contains information about a given distribution p with respect to the uniform one: it is in this precise sense, then, that the Lorenz curve characterizes the degree of nonuniformity of p —exactly because the reference distribution is chosen to be the uniform one. In the same way, thermomajorization measures the degree of “athermality” because, in this case, the reference distribution is chosen to be the thermal (Gibbs) distribution.

A lot of attention has been devoted recently to the generalization of the above ideas to the case in which, rather than comparing distributions, one wants to compare

quantum states, namely, density operators defined on a Hilbert space. This is one of the topics lying at the core of theories like quantum thermodynamics and, more generally, quantum resource theories [6–8]. However, a general theory of quantum Lorenz curves would be interesting in its own right, providing insights on the rich analogies existing between quantum theory and classical probability theory, despite their differences.

In this paper we develop such a theory by introducing the notions of quantum testing region, quantum relative Lorenz curves, and quantum relative majorization in much analogy with their classical counterparts. We find equivalent conditions for quantum relative majorization in terms of a family of divergences that we call Hilbert α divergences, with $\alpha \in (1, \infty)$, and show that in the limits $\alpha \rightarrow 1$ and $\alpha \rightarrow \infty$ the Hilbert α divergences are equivalent to the trace distance and the max-relative entropy, respectively. As an application to quantum thermodynamics, we show that only the min- and max-relative entropies are needed to determine whether it is possible to convert one qubit athermality resource to another by Gibbs preserving operations. Finally, we show that in higher dimensions, quantum relative Lorenz curves can be used to determine the existence of a test-and-prepare channel converting one pair of states to another.

II. QUANTUM RELATIVE LORENZ CURVES

Consider the task of distinguishing which, among two possible distributions, is the one that originated a set of observed sample data. This scenario, central in statistics, is usually treated within the framework known as *hypothesis testing* [9]: the two distributions are called the *null hypothesis* and the *alternative hypothesis*, respectively, and the task of the statistician is to minimize the so-called *type II error* (i.e., the probability of wrongly accepting the null hypothesis, namely, the probability of false negatives) given that the *type I error* (i.e., the probability of wrongly rejecting the null hypothesis, namely, the probability of false positives) falls below a certain threshold. The whole hypothesis testing problem is hence “encoded” in the shape of the region of the x, y plane containing all achievable points $(x, y) = (\text{type I}, \text{type II})$. Such a region is, by construction, convex, always contains the points $(0, 0)$ and

*buscemi@is.nagoya-u.ac.jp

†gour@ucalgary.ca

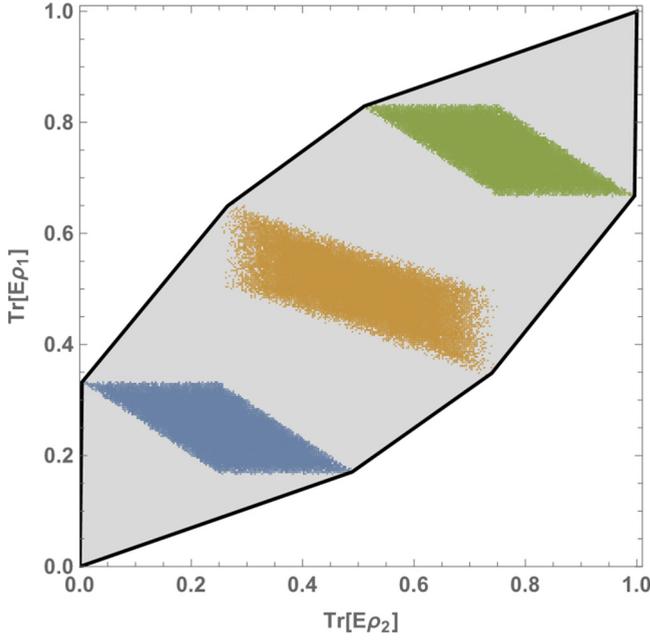


FIG. 1. Example of classical testing region in dimension $n = 4$ with $\vec{p}_1 = (1/2, 1/4, 1/4, 0)^T$ and $\vec{p}_2 = (1/6, 1/6, 1/3, 1/3)^T$. The Lorenz curve (i.e., the upper boundary) is determined by the vertices at which the Lorenz curve changes slope.

$(1, 1)$, and is symmetric, in the sense that (x, y) belongs to the region if and only if $(1 - x, 1 - y)$ does, as this corresponds to exchanging the roles of null and alternative hypotheses (see Fig. 1). Hence, the hypothesis testing is fully characterized by the upper boundary of the region. In particular, as noticed by Renes [10], when testing p against the uniform distribution, such boundary coincides with the usual Lorenz curve; when testing p against the Gibbs distribution, it coincides with the thermomajorization curve.

The observations in [10] exhibit a fundamental connection between the theory of (thermo)majorization and hypothesis testing. It is then extremely natural for us here to introduce the definition of Lorenz curves for a pair of quantum states, leveraging on the fact that hypothesis testing is well understood in the quantum case too [11–15]:

Definition 1. Given two density matrices ρ_1 and ρ_2 on \mathbb{C}^n , the associated *testing region* $\mathcal{T}(\rho_1, \rho_2) \subset \mathbb{R}^2$ is defined as the set of achievable points

$$(x, y) = (\text{Tr}[E\rho_2], \text{Tr}[E\rho_1]),$$

with $0 \leq E \leq \mathbb{1}_n$. The *quantum Lorenz curve* of ρ_1 relative to ρ_2 is defined as the upper boundary of $\mathcal{T}(\rho_1, \rho_2)$ (see Fig. 2).

Closely related to the testing region, the *hypothesis testing relative entropy* (see, e.g., [16–18], and references therein) is defined, for $0 \leq \epsilon \leq 1$, as follows:

$$D_H^\epsilon(\rho_1 \parallel \rho_2) \triangleq -\log Q^\epsilon(\rho_1 \parallel \rho_2),$$

$$Q^\epsilon(\rho_1 \parallel \rho_2) \triangleq \min_{\substack{0 \leq E \leq \mathbb{1}_n \\ \text{Tr}[\rho_1 E] \geq 1 - \epsilon}} \text{Tr}[\rho_2 E]. \quad (1)$$

As noted in Ref. [17], the computation of $Q^\epsilon(\rho_1 \parallel \rho_2)$ can be solved efficiently by semidefinite linear programming (SDP). In fact, in what follows [see Eqs. (27) and (28) in Sec. IV]

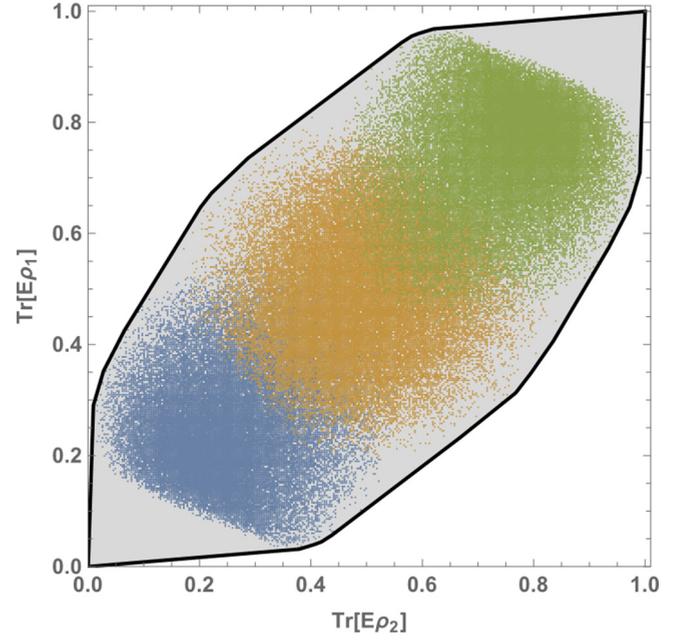


FIG. 2. Numerical example of the quantum testing region for two random four-dimensional density matrices. Notice that the curve is only roughly approximated as the sampled measurements are not enough to determine it neatly. Quantum Lorenz curves are efficiently obtained by semidefinite linear programming, e.g., using Eq. (2) in the main text.

we show that, using the strong duality relation of SDP, it is possible to write $Q^\epsilon(\rho_1 \parallel \rho_2)$, for any fixed ϵ , as the maximum of a simple function of one real variable, namely, $Q^\epsilon(\rho_1 \parallel \rho_2) = \max_{r \geq 0} f_\epsilon(r)$, where

$$f_\epsilon(r) \triangleq (1 - \epsilon)r - \text{Tr}(r\rho_1 - \rho_2)_+$$

$$= \frac{1}{2}[1 + (1 - 2\epsilon)r - \|\text{Tr}[\rho_1 - \rho_2]\|_1]. \quad (2)$$

This observation will play an important role in what follows, by considerably simplifying our analysis.

The above definition of relative Lorenz curve generalizes the classical Lorenz curve to the quantum case. In particular, if ρ_1 and ρ_2 commute, they can be simultaneously diagonalized, and the testing region in this case becomes the collection of points

$$\mathcal{T}_{\text{cl}}(\vec{p}_1, \vec{p}_2) \triangleq \{(\vec{t} \cdot \vec{p}_1, \vec{t} \cdot \vec{p}_2) : \vec{t} \in \mathbb{R}_+^n, \vec{t} \leq (1, 1, \dots, 1)^T\},$$

where \vec{p}_1 and \vec{p}_2 are the diagonals of ρ_1 and ρ_2 written in a vector form. In this case, Blackwell proved a very strong relation [19,20]: given two pairs of distributions (\vec{p}_1, \vec{p}_2) and (\vec{q}_1, \vec{q}_2) , the inclusion $\mathcal{T}_{\text{cl}}(\vec{q}_1, \vec{q}_2) \subseteq \mathcal{T}_{\text{cl}}(\vec{p}_1, \vec{p}_2)$ holds if and only if there exists a column stochastic matrix M such that $\vec{q}_1 = M\vec{p}_1$ and $\vec{q}_2 = M\vec{p}_2$. Known results about classical (thermo) majorization are therefore special cases of Blackwell's theorem, even though Blackwell's work actually predates some of them (see the discussion in Refs. [10,21]).

In the rest of the paper we explore the extent to which statements similar to Blackwell's theorem can be proved in the quantum case. However, our interest here does not lie as much in the general case, for which we know that many classical results cease to hold [21–26], but rather in restricted

scenarios of practical relevance, especially for the growing field of quantum resource theories.

III. HILBERT α DIVERGENCES

In analogy with the notation used for majorization, we write

$$(\rho_1, \rho_2) \succ (\rho'_1, \rho'_2)$$

and say that (ρ_1, ρ_2) *relatively majorizes* (ρ'_1, ρ'_2) , whenever the quantum Lorenz curve of ρ_1 relative to ρ_2 lies everywhere above the quantum Lorenz curve of ρ'_1 relative to ρ'_2 , that is, $\mathcal{T}(\rho_1, \rho_2) \supseteq \mathcal{T}(\rho'_1, \rho'_2)$.

As a tool to characterize quantum relative majorization, we introduce here a family of divergences as follows: given two density matrices ρ and σ on \mathbb{C}^n , we define, for all $\alpha \geq 1$, the following quantity:

$$\sup_\alpha(\rho/\sigma) \triangleq \sup_{\alpha^{-1}\mathbb{1}_n \leq E \leq \mathbb{1}_n} \frac{\text{Tr}[E\rho]}{\text{Tr}[E\sigma]}, \quad (3)$$

and the corresponding divergence:

$$H_\alpha(\rho\|\sigma) \triangleq \frac{\alpha}{\alpha-1} \log_2 \sup_\alpha(\rho/\sigma). \quad (4)$$

The notation used in Eq. (3) is adapted from Refs. [27–29]: there the quantity

$$\begin{aligned} \sup(\rho/\sigma) &\triangleq \inf\{\lambda : \lambda\sigma - \rho \geq 0\} \\ &= \lim_{\alpha \rightarrow \infty} \sup_\alpha(\rho/\sigma) \end{aligned}$$

is used to define the Hilbert projective metric

$$\mathfrak{h}(\rho, \sigma) \triangleq \ln[\sup(\rho/\sigma) \sup(\sigma/\rho)].$$

We note that, in Ref. [29], the quantity $\inf(\rho/\sigma)$ is also introduced as $\sup\{\lambda : \rho - \lambda\sigma \geq 0\}$; in our notation it coincides with $\inf_{0 \leq E \leq \mathbb{1}_n} \{\text{Tr}[E\rho]/\text{Tr}[E\sigma]\} = 1/\sup(\sigma/\rho)$. Due to the relation with the Hilbert’s metric, we refer to the divergences in Eq. (4) as *Hilbert α divergences*. Their main properties are summarized in the following theorem:

Theorem 1. Let ρ and σ be two density matrices on \mathbb{C}^n . Then

(i) for all $\alpha \geq 1$, $H_\alpha(\rho\|\sigma) \geq 0$, with equality if and only if $\rho = \sigma$;

(ii) for all $\alpha \geq 1$, the data-processing inequality holds: for any (not necessarily completely) positive trace-preserving map Φ , $H_\alpha[\Phi(\rho)\|\Phi(\sigma)] \leq H_\alpha(\rho\|\sigma)$;

(iii) $H_\infty(\rho\|\sigma) \triangleq \lim_{\alpha \rightarrow \infty} H_\alpha(\rho\|\sigma) = D_{\max}(\rho\|\sigma)$, namely, the max-relative entropy of Ref. [30];

(iv) $H_1(\rho\|\sigma) \triangleq \lim_{\alpha \rightarrow 1} H_\alpha(\rho\|\sigma) = \frac{1}{2\ln 2} \|\rho - \sigma\|_1$.

Remark 1. H_α is thus a family of divergences connecting the trace distance (when $\alpha \rightarrow 1$) with D_{\max} (when $\alpha \rightarrow \infty$). In passing by, we also notice that, while in point (ii) above the data-processing inequality is stated to hold for any positive trace-preserving map, Hilbert α divergences are in fact monotonically decreasing for an even larger set of transformations, called *2-statistical morphisms*: while this point is outside the scope of the present work, we refer the interested reader to Refs. [23–25,31].

Proof. Properties (ii) and (iii) are direct consequences of the definition of $\sup_\alpha(\rho/\sigma)$.

In order to prove property (iv), we start by taking $\alpha > 1$ and defining the following two quantities: $\epsilon \triangleq \alpha - 1$ and $\delta_\epsilon \triangleq \sup_{\alpha^{-1}\mathbb{1}_n \leq E \leq \mathbb{1}_n} (\rho/\sigma) - 1$. With these notations, from the definition of $\sup_\alpha(\rho/\sigma)$ we obtain

$$\delta_\epsilon \geq \frac{\text{Tr}[E\rho]}{\text{Tr}[E\sigma]} - 1 = \frac{\text{Tr}[E(\rho - \sigma)]}{\text{Tr}[E\sigma]}, \quad (5)$$

for all $\frac{1}{1+\epsilon}\mathbb{1} \leq E \leq \mathbb{1}$. Introducing the operator

$$\Delta \triangleq \frac{1}{\epsilon} [(1 + \epsilon)E - \mathbb{1}], \quad (6)$$

we get that Eq. (5) is equivalent to

$$\delta_\epsilon \geq \frac{\epsilon \text{Tr}[\Delta(\rho - \sigma)]}{1 + \epsilon \text{Tr}[\Delta\sigma]}, \quad (7)$$

for all $0 \leq \Delta \leq \mathbb{1}$. Hence, $\lim_{\epsilon \rightarrow 0} \delta_\epsilon = 0$. We therefore have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} H_{1+\epsilon}(\rho\|\sigma) &= \frac{1}{\ln(2)} \lim_{\epsilon \rightarrow 0} \frac{1 + \epsilon}{\epsilon} \ln(1 + \delta_\epsilon) \\ &= \frac{1}{\ln(2)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \delta_\epsilon \\ &\geq \frac{1}{\ln(2)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \frac{\epsilon \text{Tr}[\Delta(\rho - \sigma)]}{1 + \epsilon \text{Tr}[\Delta\sigma]} \\ &= \frac{1}{\ln(2)} \text{Tr}[\Delta(\rho - \sigma)], \end{aligned} \quad (8)$$

for all $0 \leq \Delta \leq \mathbb{1}$. We therefore conclude that

$$H_1(\rho\|\sigma) \geq \frac{1}{\ln(2)} \text{Tr}[(\rho - \sigma)_+] = \frac{1}{2\ln(2)} \|\rho - \sigma\|_1, \quad (9)$$

where we chose Δ to be the projection to the positive part of $\rho - \sigma$. To see that $H_1(\rho\|\sigma) = \frac{1}{2\ln 2} \|\rho - \sigma\|_1$ note that, in fact, by definition

$$\begin{aligned} \delta_\epsilon &= \max_{\frac{1}{1+\epsilon}\mathbb{1} \leq E \leq \mathbb{1}} \frac{\text{Tr}[E(\rho - \sigma)]}{\text{Tr}[E\sigma]} \\ &= \max_{0 \leq \Delta \leq \mathbb{1}} \frac{\epsilon \text{Tr}[\Delta(\rho - \sigma)]}{1 + \epsilon \text{Tr}[\Delta\sigma]} \\ &= \epsilon \max_{0 \leq \Delta \leq \mathbb{1}} \text{Tr}[\Delta(\rho - \sigma)] + O(\epsilon^2). \end{aligned}$$

Hence, in the limit $\epsilon \rightarrow 0$ we get $\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \delta_\epsilon = \text{Tr}[(\rho - \sigma)_+] = \frac{1}{2} \|\rho - \sigma\|_1$.

Finally, property (i) is proved as follows. Since $\sup_\alpha(\rho\|\sigma) \geq 1$, we always have $H_\alpha(\rho\|\sigma) \geq 0$. For $\alpha > 1$ if $H_\alpha(\rho\|\sigma) = 0$, then $\sup_\alpha(\rho\|\sigma) = 1$. Hence, $\text{Tr}[E\rho] \leq \text{Tr}[E\sigma]$ for all $\alpha^{-1}\mathbb{1} \leq E \leq \mathbb{1}$. Introducing

$$\Delta \triangleq \frac{\alpha}{\alpha-1} \left(E - \frac{1}{\alpha} \mathbb{1} \right),$$

we get $\text{Tr}[\Delta\rho] \leq \text{Tr}[\Delta\sigma]$, namely, $\text{Tr}[\Delta(\rho - \sigma)] \leq 0$, for all $0 \leq \Delta \leq \mathbb{1}$. We therefore must have $\rho = \sigma$. The case $\alpha = 1$ follows from property (iv). ■

IV. RELATIVE MAJORIZATION AS SETS OF INEQUALITIES

We are now in a position to provide a set of alternative conditions, reformulating the relative majorization ordering $(\rho_1, \rho_2) \succ (\rho'_1, \rho'_2)$ as sets of inequalities.

Theorem 2. Consider two pairs of density matrices (ρ_1, ρ_2) on \mathbb{C}^n and (ρ'_1, ρ'_2) on \mathbb{C}^m . The following are equivalent:

- (i) $(\rho_1, \rho_2) \succ (\rho'_1, \rho'_2)$;
- (ii) for all $t \geq 0$, $\|\rho_1 - t\rho_2\|_1 \geq \|\rho'_1 - t\rho'_2\|_1$;
- (iii) for all $\alpha \geq 1$,

$$H_\alpha(\rho_1 \|\rho_2) \geq H_\alpha(\rho'_1 \|\rho'_2),$$

$$H_\alpha(\rho_2 \|\rho_1) \geq H_\alpha(\rho'_2 \|\rho'_1);$$

- (iv) for all $0 \leq \epsilon \leq 1$, $D_H^\epsilon(\rho_1 \|\rho_2) \geq D_H^\epsilon(\rho'_1 \|\rho'_2)$.

We split the proof into several lemmas.

Lemma 1. Given two pairs of density operators (ρ_1, ρ_2) and (ρ'_1, ρ'_2) on \mathbb{C}^n and \mathbb{C}^m , respectively, the following are equivalent:

- (i) $(\rho_1, \rho_2) \succ (\rho'_1, \rho'_2)$;
- (ii) $\mathcal{T}(\rho_1, \rho_2) \supseteq \mathcal{T}(\rho'_1, \rho'_2)$;
- (iii) $\|t_1\rho_1 + t_2\rho_2\|_1 \geq \|t_1\rho'_1 + t_2\rho'_2\|_1$ for all $t_1, t_2 \in \mathbb{R}$;
- (iv) $\|\rho_1 - t\rho_2\|_1 \geq \|\rho'_1 - t\rho'_2\|_1$, for all $t \geq 0$.

Proof. The first equivalence holds by definition. Denoting by (p, \bar{p}) and (q, \bar{q}) the generic element of $\mathcal{T}(\rho_1, \rho_2)$ and $\mathcal{T}(\rho'_1, \rho'_2)$, respectively, the separation theorem for convex sets, applied to $\mathcal{T}(\rho_1, \rho_2)$ and $\mathcal{T}(\rho'_1, \rho'_2)$, states that $\mathcal{T}(\rho_1, \rho_2) \supseteq \mathcal{T}(\rho'_1, \rho'_2)$ if and only if, for any $v = (a, b) \in \mathbb{R}^2$,

$$\max_{(p, \bar{p}) \in \mathcal{T}(\rho_1, \rho_2)} [ap + b\bar{p}] \geq \max_{(q, \bar{q}) \in \mathcal{T}(\rho'_1, \rho'_2)} [aq + b\bar{q}]. \quad (10)$$

The next step is to show that

$$\max_{(p, \bar{p}) \in \mathcal{T}(\rho_1, \rho_2)} [ap + b\bar{p}] = \frac{a + b + \|a\rho_1 - b\rho_2\|_1}{2}, \quad (11)$$

and, analogously, for (ρ'_1, ρ'_2) . This is done by the following simple passages:

$$\begin{aligned} \max_{(p, \bar{p}) \in \mathcal{T}(\rho_1, \rho_2)} [ap + b\bar{p}] &= \max_{0 \leq E \leq \mathbb{1}} \{a \operatorname{Tr}[\rho_1 E] + b \operatorname{Tr}[\rho_2 E]\} \\ &= \max_{0 \leq E \leq \mathbb{1}} \operatorname{Tr}[(a\rho_1 + b\rho_2)E] \\ &= \operatorname{Tr}(a\rho_1 + b\rho_2)_+, \end{aligned} \quad (12)$$

where the last expression denotes the positive part of the self-adjoint operator $a\rho_1 + b\rho_2$. Then, since $2\operatorname{Tr}(A)_+ = \|A\|_1 + \operatorname{Tr}[A]$ for any self-adjoint operator, we have that

$$2 \max_{(p, \bar{p}) \in \mathcal{T}(\rho_1, \rho_2)} [ap + b\bar{p}] = a + b + \|a\rho_1 + b\rho_2\|_1. \quad (13)$$

This proves that Eq. (10) is satisfied if and only if $\|a\rho_1 + b\rho_2\|_1 \geq \|a\rho'_1 + b\rho'_2\|_1$, for all $a, b \in \mathbb{R}$.

We are left to prove that (iii) is equivalent to (iv). However, since (iv) is a special case of (iii), we only need to prove that (iv) implies (iii). To this end, we notice that, whenever $t_1, t_2 \geq 0$ or $t_1, t_2 \leq 0$, $\|t_1\rho_1 + t_2\rho_2\|_1 = \|t_1\rho'_1 + t_2\rho'_2\|_1$ always, simply due to the positivity of $\rho_1, \rho_2, \rho'_1, \rho'_2$. We can hence consider only the cases $t_1 > 0 > t_2$ or $t_2 > 0 > t_1$. However, since $\|X\|_1 = \|-X\|_1$, for any matrix X , we can further restrict the parameters t_1 and t_2 to the case $t_2 < 0 < t_1$. The statement is finally obtained by rescaling both t_1 and t_2 by the (positive) factor $1/t_1$. ■

Lemma 1 above shows that statements (i) and (ii) of Theorem 2 are indeed equivalent. We now move on to proving the equivalence of the point (iii). We begin with the following lemma.

Lemma 2. For any choice of density operators ρ and σ ,

$$\sup_\alpha(\rho/\sigma) = \inf \left\{ \lambda \geq 1 : \frac{\|\lambda\sigma - \rho\|_1}{\lambda - 1} \leq \frac{\alpha + 1}{\alpha - 1} \right\}.$$

Proof. Note first that

$$\begin{aligned} \sup_\alpha(\rho/\sigma) &\triangleq \sup_{\alpha^{-1}\mathbb{1} \leq E \leq \mathbb{1}} \{\operatorname{Tr}[E\rho]/\operatorname{Tr}[E\sigma]\} \\ &= \inf\{\lambda : \lambda \geq \operatorname{Tr}[E\rho]/\operatorname{Tr}[E\sigma] \text{ for all } \alpha^{-1}\mathbb{1} \leq E \leq \mathbb{1}\} \\ &= \inf\{\lambda : \operatorname{Tr}[E(\lambda\sigma - \rho)] \geq 0 \text{ for all } \alpha^{-1}\mathbb{1} \leq E \leq \mathbb{1}\} \\ &= \inf\{\lambda \in \mathbb{R} : \alpha^{-1} \operatorname{Tr}[(\lambda\sigma - \rho)_+] \geq \operatorname{Tr}[(\lambda\sigma - \rho)_-]\}, \end{aligned}$$

where, in the last equality, we used the decomposition $A = A_+ - A_-$ for Hermitian operators and the choice $E = \alpha^{-1}\Pi_+ + \Pi_-$, being Π_\pm the projectors onto the positive and negative parts of $(\lambda\sigma - \rho)$, respectively. Indeed, this is the choice for the operator E that poses the toughest constraints compatible with the fixed value of the parameter α . (Equivalently, if $\operatorname{Tr}[E(\lambda\sigma - \rho)] \geq 0$ for such a choice of E , then it is positive for any $\alpha^{-1}\mathbb{1} \leq E \leq \mathbb{1}$.)

Then, using the relations

$$\lambda - 1 = \operatorname{Tr}[(\lambda\sigma - \rho)_+] - \operatorname{Tr}[(\lambda\sigma - \rho)_-]$$

and

$$\|\lambda\sigma - \rho\|_1 = \operatorname{Tr}[(\lambda\sigma - \rho)_+] + \operatorname{Tr}[(\lambda\sigma - \rho)_-]$$

gives

$$\sup_\alpha(\rho/\sigma) = \inf \left\{ \lambda \in \mathbb{R} : \operatorname{Tr}[(\lambda\sigma - \rho)_-] \leq \frac{\lambda - 1}{\alpha - 1} \right\}.$$

Then, since $\operatorname{Tr}[(\lambda\sigma - \rho)_-] \leq \frac{\lambda-1}{\alpha-1}$ if and only if $\|\lambda\sigma - \rho\|_1 \leq \frac{\lambda-1}{\alpha-1} + \operatorname{Tr}[(\lambda\sigma - \rho)_+] = \frac{\lambda-1}{\alpha-1} + \frac{\lambda-1 + \|\lambda\sigma - \rho\|_1}{2}$, after an easy manipulation we obtain

$$\sup_\alpha(\rho/\sigma) = \inf \left\{ \lambda \in \mathbb{R} : \frac{\|\lambda\sigma - \rho\|_1}{\lambda - 1} \leq \frac{\alpha + 1}{\alpha - 1} \right\}.$$

The statement is finally recovered by noticing that no loss of generality comes from restricting λ to values greater than or equal to 1. ■

Lemma 3. For any choice of density operators ρ and σ , the function

$$f(\lambda) = \frac{\|\lambda\sigma - \rho\|_1}{\lambda - 1}$$

is monotonically nonincreasing in the domain $\lambda \geq 1$ with $f(1) = \infty$ and $f(\infty) = 1$.

Remark 2. In particular, Lemma 2 and Lemma 3 above imply that, for any pair of density operators ρ and σ ,

$$\frac{\|\sup_\alpha(\rho/\sigma)\sigma - \rho\|_1}{\sup_\alpha(\rho/\sigma) - 1} = \frac{\alpha + 1}{\alpha - 1}.$$

Proof. Set $t = 1/(\lambda - 1)$ and define $g(t) = \|(1+t)\sigma - t\rho\|_1$. Hence, $g(t) = f(\lambda)$, and it is enough to show that $g(t)$ is monotonically nondecreasing in its domain $t \in [0, \infty)$. First note that for any $0 < p < 1$ and

$t_1, t_2 \in \mathbb{R}_+$ we have

$$\begin{aligned} &g[pt_1 + (1-p)t_2] \\ &= \|p(1+t_1)\sigma - pt_1\rho + (1-p)(1+t_2)\sigma - (1-p)t_2\rho\|_1 \\ &\leq \|p(1+t_1)\sigma - pt_1\rho\|_1 + \|(1-p)(1+t_2)\sigma - (1-p)t_2\rho\|_1 \\ &= pg(t_1) + (1-p)g(t_2). \end{aligned}$$

Hence $g(t)$ is a convex function. Moreover, note that $g(0) = 1 \leq g(t)$ for all $t \geq 0$. These two properties of $g(t)$ together imply that it is monotonically nondecreasing in t . ■

Lemma 4. Consider two pairs of states (ρ, σ) and (ρ', σ') . Then, the following are equivalent:

(i) for all $\alpha \geq 1$, $\sup_\alpha(\rho/\sigma) \geq \sup_\alpha(\rho'/\sigma')$ and $\inf_\alpha(\rho/\sigma) \leq \inf_\alpha(\rho'/\sigma')$;

(ii) $\|t\sigma - \rho\|_1 \geq \|t\sigma' - \rho'\|_1$ for all $t \geq 0$.

Proof. We only need to show that the condition $\sup_\alpha(\rho/\sigma) \geq \sup_\alpha(\rho'/\sigma')$ for all $\alpha \geq 1$ is equivalent to $\|t\sigma - \rho\|_1 \geq \|t\sigma' - \rho'\|_1$ for all $t \geq 1$. Then, if this holds, the remaining statement, namely, that $\inf_\alpha(\rho/\sigma) \leq \inf_\alpha(\rho'/\sigma')$ for all $\alpha \geq 1$ is equivalent to $\|t\sigma - \rho\|_1 \geq \|t\sigma' - \rho'\|_1$ for all $t \in [0, 1]$, simply follows from the definitions.

Set $M_\alpha \equiv \sup_\alpha(\rho/\sigma)$, $M'_\alpha \equiv \sup_\alpha(\rho'/\sigma')$, and recall the definition of f in Lemma 3. Then, as noticed in Remark 2 above, it follows that

$$f(M_\alpha) = \frac{\|M_\alpha\sigma - \rho\|_1}{M_\alpha - 1} = \frac{\alpha + 1}{\alpha - 1} = \frac{\|M'_\alpha\sigma' - \rho'\|_1}{M'_\alpha - 1}. \quad (14)$$

Since f is monotonically nonincreasing, we get that $M_\alpha \geq M'_\alpha$ implies that

$$\frac{\|M'_\alpha\sigma' - \rho'\|_1}{M'_\alpha - 1} \geq \frac{\|M_\alpha\sigma - \rho\|_1}{M_\alpha - 1}. \quad (15)$$

Combining the above two equations gives

$$\|M_\alpha\sigma - \rho\|_1 \geq \|M'_\alpha\sigma' - \rho'\|_1 \quad \forall \alpha \geq 1. \quad (16)$$

We now make the simple observation that, by definition, the function $\sup_\alpha(\rho/\sigma) = M_\alpha$ is continuous and monotonically nondecreasing in α , with $\sup_{\alpha=1}(\rho/\sigma) = 1$ and $\sup_{\alpha \rightarrow \infty}(\rho/\sigma) = \sup(\rho/\sigma)$. Hence,

$$\|t\sigma - \rho\|_1 \geq \|t\sigma' - \rho'\|_1, \quad 1 \leq \forall t \leq \sup(\rho/\sigma), \quad (17)$$

and the above is enough to conclude that the same ordering holds in fact for all $t \geq 1$.

Conversely, suppose the inequality above holds for all $t \geq 1$. This implies that, if

$$\frac{\|\lambda\sigma - \rho\|_1}{\lambda - 1} \leq \frac{\alpha + 1}{\alpha - 1}, \quad (18)$$

then also

$$\frac{\|\lambda\sigma' - \rho'\|_1}{\lambda - 1} \leq \frac{\alpha + 1}{\alpha - 1}. \quad (19)$$

But from Lemma 2 this implies that $\sup_\alpha(\rho'/\sigma') \leq \sup_\alpha(\rho/\sigma)$. ■

Lemma 4 above hence proves the equivalence of point (ii) and point (iii) of Theorem 2, because $\sup_\alpha(\rho_1/\rho_2) \geq \sup_\alpha(\rho'_1/\rho'_2)$ if and only if $H_\alpha(\rho_1\|\rho_2) \geq H_\alpha(\rho'_1\|\rho'_2)$, and

$\inf_\alpha(\rho_1/\rho_2) \leq \inf_\alpha(\rho'_1/\rho'_2)$ if and only if $H_\alpha(\rho_2\|\rho_1) \geq H_\alpha(\rho'_2\|\rho'_1)$.

The proof of Theorem 2 is complete if we prove the equivalence of the remaining point (iv). Also in this case, rather than proving the statement for D_H^ϵ , we will prove it for the corresponding Q^ϵ , related with D_H^ϵ as given in Eq. (1) of the main text. We recall the definition: given two density operators ρ_1 and ρ_2 on \mathbb{C}^n , for any $\epsilon \in [0, 1]$,

$$Q^\epsilon(\rho_1\|\rho_2) = \min_{\substack{0 \leq A \leq \mathbb{1}_n \\ \text{Tr}[\rho_1 A] \geq 1 - \epsilon}} \text{Tr}[\rho_2 A]. \quad (20)$$

For later convenience, we introduce the following notation: \mathbf{H}_n to denote the set of n -by- n Hermitian matrices on \mathbb{C}^n , and $\mathbf{M}_{n,+}$ to denote the set of n -by- n complex positive semidefinite matrices.

Lemma 5. Given two pairs of density operators (ρ_1, ρ_2) and (ρ'_1, ρ'_2) on \mathbb{C}^n and \mathbb{C}^m , respectively, the following are equivalent:

(i) $Q^\epsilon(\rho_1\|\rho_2) \leq Q^\epsilon(\rho'_1\|\rho'_2)$, for all $\epsilon \in [0, 1]$;

(ii) $\|\rho_2 - r\rho_1\|_1 \geq \|\rho'_2 - r\rho'_1\|_1$, for all $r \geq 0$.

Proof. Consider the following setting of linear programming. Let V_1 and V_2 be two (inner product) vector spaces with two cones $K_1 \subset V_1$ and $K_2 \subset V_2$. Consider two vectors $v_1 \in V_1$ and $v_2 \in V_2$, and a linear map $\Gamma : V_1 \rightarrow V_2$. Given a problem in its primal form:

$$\max_{\substack{x \in K_1 \\ v_2 - \Gamma(x) \in K_2}} \langle v_1, x \rangle_1, \quad (21)$$

the dual form involves the adjoint map $\Gamma^* : V_2 \rightarrow V_1$:

$$\min_{\substack{y \in K_2 \\ \Gamma^*(y) - v_1 \in K_1}} \langle v_2, y \rangle_2, \quad (22)$$

where Γ^* is defined by the relation $\langle y, \Gamma(x) \rangle = \langle \Gamma^*(y), x \rangle$, for all $x \in K_1$ and all $y \in K_2$.

In our case, denote

$$V_1 \triangleq \mathbb{R} \oplus \mathbf{H}_n = \{(r, A) | r \in \mathbb{R}; A \in \mathbf{H}_n\}$$

with inner product

$$\langle (r, A), (t, B) \rangle_1 \triangleq rt + \text{Tr}[AB].$$

Further, define $K_1 = \mathbb{R}_+ \oplus \mathbf{M}_{n,+}$ to be the positive cone in V_1 . Similarly, set $V_2 = \mathbf{H}_n$ and $K_2 = \mathbf{M}_{n,+}$. The linear map $\Gamma : V_1 \rightarrow V_2$ is given by

$$\Gamma(r, A) = r\rho_1 - A; \quad (23)$$

hence, the corresponding dual map $\Gamma^* : V_2 \rightarrow V_1$ is given by

$$\Gamma^*(B) = (\text{Tr}[\rho_1 B], -B). \quad (24)$$

Finally, set $v_1 = (1 - \epsilon, -\mathbb{1}_n)$ and $v_2 = \rho_2$.

Since, for these choices, $y \in K_2$ if and only if $y \geq 0$ and $\Gamma^*(y) - v_1 \in K_1$ if and only if $\text{Tr}[\rho_1 y] \geq 1 - \epsilon$ and $y \leq \mathbb{1}_n$, the dual form (22) becomes exactly the right-hand side of Eq. (20), namely,

$$\min_{\substack{y \in K_2 \\ \Gamma^*(y) - v_1 \in K_1}} \langle v_2, y \rangle_2 = Q^\epsilon(\rho_1\|\rho_2). \quad (25)$$

For the primal form, since $x = (r, A) \in K_1$ if and only if $r \geq 0$ and $A \geq 0$, and $v_2 - \Gamma(x) \in K_1$ if and only if $\rho_2 - r\rho_1 + A \geq 0$,

we obtain

$$\max_{\substack{x \in K_1 \\ v_2 - \Gamma(x) \in K_2}} \langle v_1, x \rangle_1 = \max_{\substack{A \geq r\rho_1 - \rho_2 \\ r, A \geq 0}} \{(1 - \epsilon)r - \text{Tr}[A]\}. \quad (26)$$

The right-hand side of the above equation can be further simplified as follows. We first fix r and optimize over A . Since the $A \geq 0$ with minimum trace such that $A \geq r\rho_1 - \rho_2$ is exactly $A = (r\rho_1 - \rho_2)_+$, we conclude that

$$\max_{\substack{x \in K_1 \\ v_2 - \Gamma(x) \in K_2}} \langle v_1, x \rangle_1 = \max_{r \geq 0} \{(1 - \epsilon)r - \text{Tr}(r\rho_1 - \rho_2)_+\},$$

that is,

$$Q^\epsilon(\rho_1 \| \rho_2) = \max_{r \geq 0} f_\epsilon(r), \quad (27)$$

where

$$f_\epsilon(r) \triangleq (1 - \epsilon)r - \text{Tr}(r\rho_1 - \rho_2)_+. \quad (28)$$

Note that

$$\text{Tr}(r\rho_1 - \rho_2)_+ = \frac{\|r\rho_1 - \rho_2\|_1 + r - 1}{2},$$

that is,

$$f_\epsilon(r) = \frac{1 + (1 - 2\epsilon)r - \|r\rho_1 - \rho_2\|_1}{2}.$$

Therefore, denoting $f'_\epsilon(r) = 2^{-1}\{1 + (1 - 2\epsilon)r - \|r\rho_1' - \rho_2'\|_1\}$, we have that

$$\|r\rho_1 - \rho_2\|_1 \geq \|r\rho_1' - \rho_2'\|_1 \Rightarrow f_\epsilon(r) \leq f'_\epsilon(r),$$

independently of r and ϵ . We thus have proved that (ii) implies (i).

To show that (i) implies (ii), suppose $Q^\epsilon(\rho_1 \| \rho_2) \leq Q^\epsilon(\rho_1' \| \rho_2')$ for all $\epsilon \in [0, 1]$. Let $r_\epsilon \geq 0$ be the minimum value of r achieving $Q^\epsilon(\rho_1' \| \rho_2')$, in formula

$$r_\epsilon \triangleq \min\{r \geq 0 : f'_\epsilon(r) = Q^\epsilon(\rho_1' \| \rho_2')\}. \quad (29)$$

In all such points r_ϵ , definition (27) together with the assumption (i) guarantee that $\|r_\epsilon\rho_1 - \rho_2\|_1 \geq \|r_\epsilon\rho_1' - \rho_2'\|_1$. This fact can be simply shown by the following chain of inequalities:

$$\begin{aligned} & \frac{1 + (1 - 2\epsilon)r_\epsilon - \|r_\epsilon\rho_1' - \rho_2'\|_1}{2} \\ &= Q^\epsilon(\rho_1' \| \rho_2') \geq Q^\epsilon(\rho_1 \| \rho_2) \\ &= \max_r \frac{1 + (1 - 2\epsilon)r - \|r\rho_1 - \rho_2\|_1}{2} \\ &\geq \frac{1 + (1 - 2\epsilon)r_\epsilon - \|r_\epsilon\rho_1 - \rho_2\|_1}{2}. \end{aligned}$$

The crucial observation now is that the points r_ϵ , representing the solutions of (29) for varying $\epsilon \in [0, 1]$, coincide with the points where the quantity $\|r\rho_1' - \rho_2'\|_1$, thought of as a function of r , changes its slope (see Fig. 3 below). For example, for $\epsilon = 0$, we have to consider the function

$$f'_0(r) = \frac{1 + r - \|r\rho_1' - \rho_2'\|_1}{2},$$

and this achieves its maximum value 1 for $r \geq r^* \equiv r_0 = \sup(\rho_2' / \rho_1')$. But then, if we know that the curve $\|r\rho_1 - \rho_2\|_1$

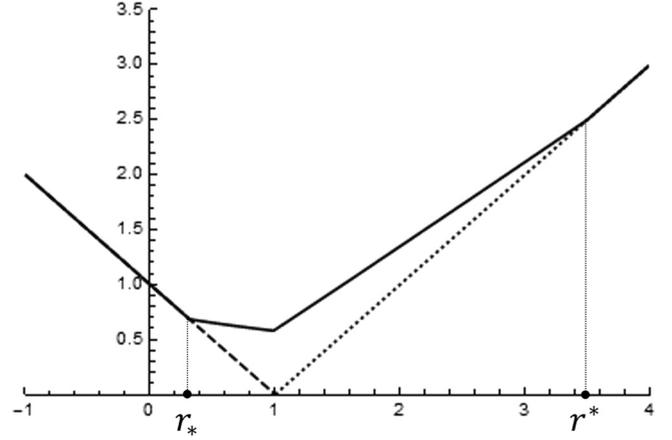


FIG. 3. Typical behavior of $\|r\rho_1 - \rho_2\|_1$, for two random density matrices ρ_1 and ρ_2 on \mathbb{C}^3 , as a function of $r \in \mathbb{R}$ (continuous line). For $r \leq r_* \equiv \inf(\rho_2/\rho_1) = \sup\{\lambda : \lambda\rho_1 - \rho_2 \leq 0\}$, the curve becomes equal to $1 - r$ (dashed line). For $r \geq r_* \equiv \sup(\rho_2/\rho_1) = \inf\{\lambda : \lambda\rho_1 - \rho_2 \geq 0\}$, the curve becomes equal to $r - 1$ (dotted line).

is not below $\|r\rho_1' - \rho_2'\|_1$ in all the points where the latter changes its slope, this is sufficient to conclude that

$$\|r\rho_1' - \rho_2'\|_1 \leq \|r\rho_1 - \rho_2\|_1 \quad \forall r \geq 0.$$

This completes the proof of the lemma. \blacksquare

The classical case

As a ‘‘consistency check’’ we separately consider the classical case here. Suppose ρ and σ are both diagonal with elements p_1, \dots, p_n and q_1, \dots, q_n , respectively. Denote $r_j \equiv q_j/p_j$ if $p_j > 0$ and otherwise $r_j = 0$. Without loss of generality suppose $r_1 \leq r_2 \leq \dots \leq r_n$. We therefore get

$$rp_j - q_j > 0 \iff r > r_j. \quad (30)$$

Hence, for $r \in (r_k, r_{k+1}]$,

$$\begin{aligned} f(r) &= (1 - \epsilon)r - \text{Tr}(r\rho - \sigma)_+ \\ &= (1 - \epsilon)r - r \sum_{j=1}^k p_j + \sum_{j=1}^k q_j. \end{aligned} \quad (31)$$

Due to the linearity in r of the expression above we conclude that in the classical case

$$\begin{aligned} Q^\epsilon(\rho \| \sigma) &= \max_{k \in \{1, \dots, n\}} f(r_k) \\ &= \max_{k \in \{1, \dots, n\}} \left\{ \left(1 - \epsilon - \sum_{j=1}^k p_j \right) r_k + \sum_{j=1}^k q_j \right\}, \end{aligned}$$

thus reconstructing the Blackwell criterion for pairs of probability distributions (including majorization and thermomajorization).

V. APPLICATIONS

In this section, we study how the conditions in Theorem 2 are logically related to the existence of a suitable transformation mapping (ρ_1, ρ_2) into (ρ_1', ρ_2') .

A. Coherent energy transitions with Gibbs-preserving operations

We consider here the resource theory of athermality [4,5]. In this theory, quantum systems that are not in thermal equilibrium with their environment are considered resources (e.g., work can be extracted from such systems). Hence, free systems are those prepared in the Gibbs state, i.e., $\gamma = \mathcal{Z}^{-1} \sum_{x=1}^d e^{-\beta E_x} |x\rangle\langle x|$, and permitted operations must preserve γ . Consider now two possibly noncommuting quantum states ρ and σ both with the same two-dimensional support spanned by the same two energy eigenstates, say $|x\rangle$ and $|y\rangle$. Such states form the building blocks of quantum thermodynamics as they contain the smallest units of athermality [5]. Here we find both necessary and sufficient conditions under which a Gibbs-preserving transition between ρ and σ is possible. We call such transitions *coherent energy transitions*, since not only the transitions $|x\rangle \rightarrow |y\rangle$ and $|y\rangle \rightarrow |x\rangle$ are considered, but also transitions between any linear superpositions of such energy eigenstates.

The main result about coherent energy transitions is the following:

Theorem 3. With γ , ρ , and σ as above, assuming that $\gamma > 0$ (i.e., nonzero temperature), the following are equivalent:

- (i) ρ can be transformed into σ by a γ -preserving completely positive trace-preserving (CPTP) operation (i.e., Gibbs-preserving operation);
- (ii) $(\rho, \gamma) \succ (\sigma, \gamma)$;
- (iii) it holds that

$$\begin{aligned} D_{\max}(\rho \parallel \gamma) &\geq D_{\max}(\sigma \parallel \gamma), \\ D_{\max}(\gamma \parallel \rho) &\geq D_{\max}(\gamma \parallel \sigma). \end{aligned} \tag{32}$$

In other words, in this case, we do not need to check the validity of point (iii) of Theorem 2 for all values of α , but only in the limit $\alpha \rightarrow \infty$. Moreover, in this case, we know that a CPTP map between the two pairs of states exists. In the case of zero temperature, i.e., if $\gamma \geq 0$, a third condition has to be added to the above list, namely, $D_{\min}(\gamma \parallel \rho) \geq D_{\min}(\gamma \parallel \sigma)$, where $D_{\min}(\gamma \parallel \rho) = -\log \text{Tr}[\Pi_\gamma \rho]$ denotes the *min-relative entropy* [30] and Π_γ is the projector onto the support of γ . Finally, we did not include the condition $D_{\min}(\rho \parallel \gamma) \geq D_{\min}(\sigma \parallel \gamma)$ since it is trivial, unless σ is rank one (i.e., a pure state). However, as shown below in the proof, it turns out that in this case the other conditions imply this one.

Theorem 3 generalizes an earlier work given in [32] to the generic case in which the Gibbs state is not pure. It demonstrates that three athermality monotones (given in terms of the min/max relative entropies) provide both necessary and sufficient conditions for the existence of a Gibbs preserving map connecting two nonthermal states with the same two-dimensional support. Since the set of Gibbs-preserving operations is strictly larger than the set of thermal operations [5], these three monotones, in general, will not be sufficient to determine convertibility under thermal operations [33].

The Gibbs state is given by

$$\gamma = \frac{1}{\mathcal{Z}} \sum_{x=1}^d \exp(-\beta E_x) |x\rangle\langle x|, \tag{33}$$

where $\beta = 1/kT$ is the inverse temperature, d is the dimension of the quantum system, $\{|x\rangle\}_{x=1}^d$ is the complete set of eigenstates of the Hamiltonian, E_x the eigenvalues of the Hamiltonian, and \mathcal{Z} is the partition function $\sum_{x=1}^d \exp(-\beta E_x)$. Consider now two quantum states ρ and σ both with the same two-dimensional support given by, e.g.,

$$\text{supp}(\rho) = \text{supp}(\sigma) = \text{span}\{|x\rangle\}_{x=1,2}, \tag{34}$$

but it does not matter which two energy eigenstates are chosen (with the condition $E_2 \geq E_1$). Denote further by $\gamma^{(2)}$ the Gibbs state projected onto this two-dimensional subspace:

$$\gamma^{(2)} \equiv p|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|, \tag{35}$$

where

$$p \equiv \frac{\exp(-\beta E_1)}{\exp(-\beta E_1) + \exp(-\beta E_2)} = \frac{1}{1 + \exp(-\beta \Delta E)}, \tag{36}$$

with $\Delta E = E_2 - E_1 \geq 0$ so that $p \geq 1/2$. We start with the following lemma:

Lemma 6. Let ρ and σ be as in (34), and let γ be as in (33) with $\gamma^{(2)}$ as in (35). Then, there exists a CPTP map Φ such that $\Phi(\rho) = \sigma$ and $\Phi(\gamma) = \gamma$ if and only if there exists a CPTP map \mathcal{E} such that $\mathcal{E}(\rho) = \sigma$ and $\mathcal{E}(\gamma^{(2)}) = \gamma^{(2)}$.

Proof. Suppose there exists Φ such that $\Phi(\rho) = \sigma$ and $\Phi(\gamma) = \gamma$. Then, for all $t > 0$ we have

$$\|\sigma - t\gamma\|_1 = \|\Phi(\rho) - \Phi(\gamma)\|_1 \leq \|\rho - t\gamma\|_1, \tag{37}$$

since the trace norm is contractive. Next, denoting by P the projection onto $\text{span}\{|1\rangle, |2\rangle\}$ and $r \equiv \exp(-\beta E_1) + \exp(-\beta E_2)$, we have

$$\begin{aligned} \|\rho - t\gamma\|_1 &= \|P\rho P - tP\gamma P\|_1 + t\|(I - P)\gamma(I - P)\|_1 \\ &= \left\| \rho - t \frac{r}{\mathcal{Z}} \gamma^{(2)} \right\|_1 + t \left\| \gamma - \frac{r}{\mathcal{Z}} \gamma^{(2)} \right\|_1, \end{aligned}$$

and, analogously,

$$\|\sigma - t\gamma\|_1 = \left\| \sigma - t \frac{r}{\mathcal{Z}} \gamma^{(2)} \right\|_1 + t \left\| \gamma - \frac{r}{\mathcal{Z}} \gamma^{(2)} \right\|_1.$$

Therefore, since $r/\mathcal{Z} > 0$, we can introduce the new parameter $t' \triangleq t \frac{r}{\mathcal{Z}}$ so that the inequality in Eq. (37) can be rewritten as

$$\|\sigma - t'\gamma^{(2)}\|_1 \leq \|\rho - t'\gamma^{(2)}\|_1 \quad \forall t' > 0. \tag{38}$$

From the Alberti-Uhlmann result on qubits [34] there exists \mathcal{E} as in the lemma.

Conversely, suppose there exists a CPTP map \mathcal{E} such that $\mathcal{E}(\rho) = \sigma$ and $\mathcal{E}(\gamma^{(2)}) = \gamma^{(2)}$. Then, define Φ as follows. Let $P = |1\rangle\langle 1| + |2\rangle\langle 2|$ be the projector onto the support of ρ and σ , and define

$$\Phi(\cdot) := \mathcal{E}(P(\cdot)P) + (I_d - P)(\cdot)(I_d - P). \tag{39}$$

By construction, Φ is CPTP since \mathcal{E} is CPTP, and it is easy to verify that $\Phi(\rho) = \sigma$ and $\Phi(\gamma) = \gamma$. This completes the proof. ■

Lemma 7. Let ρ and σ be two qubit density matrices and let $\gamma^{(2)}$ be the Gibbs state given in (35). Then, ρ can be converted to σ by Gibbs-preserving operations if and only if the following

three inequalities simultaneously hold:

$$\begin{aligned} D_{\max}(\rho \|\gamma^{(2)}) &\geq D_{\max}(\sigma \|\gamma^{(2)}), \\ D_{\max}(\gamma^{(2)} \|\rho) &\geq D_{\max}(\gamma^{(2)} \|\sigma), \\ D_{\min}(\gamma^{(2)} \|\rho) &\geq D_{\min}(\gamma^{(2)} \|\sigma). \end{aligned} \quad (40)$$

Before proceeding, we notice that, while in Eq. (40) above the projected Gibbs state $\gamma^{(2)}$ appears, in Eq. (32) of Theorem 3 we use the original γ . However, since $P\rho P = \rho$, $P\sigma P = \sigma$, and $P\gamma P = c\gamma^{(2)}$ (for some $c \geq 0$), and since both D_{\max} and D_{\min} in this case only depend on what there is on the support of P , the two sets of conditions are clearly equivalent.

Denote by

$$\begin{aligned} m(\rho, \gamma^{(2)}) &\triangleq \inf(\rho/\gamma^{(2)}) = \sup\{t \in \mathbb{R} : t\gamma^{(2)} - \rho \leq 0\}, \\ M(\rho, \gamma^{(2)}) &\triangleq \sup(\rho/\gamma^{(2)}) = \inf\{t \in \mathbb{R} : t\gamma^{(2)} - \rho \geq 0\}. \end{aligned} \quad (41)$$

Note that $m(\rho, \gamma^{(2)}) \leq 1 \leq M(\rho, \gamma^{(2)})$. Since we consider here the qubit case, it follows that $m(\rho, \gamma^{(2)})$ and $M(\rho, \gamma^{(2)})$ are the roots to the quadratic polynomial $\det(\rho - t\gamma^{(2)})$. A straightforward calculation gives [assuming $\det(\gamma^{(2)}) > 0$]

$$\det(\rho - t\gamma^{(2)}) = \det(\gamma^{(2)})[t - m(\rho, \gamma^{(2)})][t - M(\rho, \gamma^{(2)})] \quad (42)$$

with m and M given explicitly below after we introduce a few notations.

Without loss of generality we can assume that the off-diagonal terms of ρ are non-negative real numbers since $\gamma^{(2)}$ is invariant under conjugation by any 2×2 unitary matrix which is diagonal on the energy eigenbasis (i.e., commutes with $\gamma^{(2)}$). Hence, we can write

$$\rho = \begin{pmatrix} a & \varepsilon\sqrt{a(1-a)} \\ \varepsilon\sqrt{a(1-a)} & 1-a \end{pmatrix} \quad (43)$$

with $\varepsilon, a \in [0, 1]$. Taking $\gamma^{(2)}$ as in (35) we get the following explicit expressions for $m(\rho, \gamma^{(2)})$ and $M(\rho, \gamma^{(2)})$ assuming $\det(\gamma^{(2)}) > 0$ (i.e., $0 < p < 1$),

$$\begin{aligned} m(\rho, \gamma^{(2)}) &= \frac{1}{2}[r_0 + r_1 - \sqrt{(r_0 - r_1)^2 + 4r_0r_1\varepsilon^2}], \\ M(\rho, \gamma^{(2)}) &= \frac{1}{2}[r_0 + r_1 + \sqrt{(r_0 - r_1)^2 + 4r_0r_1\varepsilon^2}], \end{aligned} \quad (44)$$

where

$$r_0 \equiv \frac{a}{p} \quad \text{and} \quad r_1 \equiv \frac{1-a}{1-p}. \quad (45)$$

By definition, both m and M are monotonic in the sense that

$$\begin{aligned} M(\Phi(\rho), \Phi(\gamma^{(2)})) &\leq M(\rho, \gamma^{(2)}) \\ m(\Phi(\rho), \Phi(\gamma^{(2)})) &\geq m(\rho, \gamma^{(2)}). \end{aligned} \quad (46)$$

Note that $m(\rho, \gamma^{(2)}) = 1/M(\gamma^{(2)}, \rho)$ and M is related to the max relative entropy (in what follows, \log is a shortcut for \log_2):

$$\begin{aligned} D_{\max}(\rho \|\gamma^{(2)}) &= \log_2 M(\rho, \gamma^{(2)}), \\ D_{\max}(\gamma^{(2)} \|\rho) &= \log_2 M(\gamma^{(2)}, \rho) = -\log m(\rho, \gamma^{(2)}). \end{aligned} \quad (47)$$

A dual definition is the min-relative entropy defined by

$$D_{\min}(\gamma^{(2)} \|\rho) = -\log \text{Tr}[\rho \Pi_{\gamma^{(2)}}], \quad (48)$$

where $\Pi_{\gamma^{(2)}}$ is the projection to the support of $\gamma^{(2)}$. Clearly, if $\det(\gamma^{(2)}) > 0$ then $D_{\min}(\gamma^{(2)}, \rho) = 0$. Summarizing, we showed that the conditions in Eq. (40) are equivalent to (remember the assumption here $\gamma^{(2)} > 0$; the case of rank-one $\gamma^{(2)}$ will be considered separately below)

$$\begin{aligned} M(\rho, \gamma^{(2)}) &\geq M(\sigma, \gamma^{(2)}), \\ m(\rho, \gamma^{(2)}) &\leq m(\sigma, \gamma^{(2)}). \end{aligned}$$

To see how the above conditions can be used to prove Theorem 3, we need the following lemma from Ref. [34].

Lemma 8. (Alberti-Uhlmann) Let ρ , σ , η , and τ be qubit density matrices. Then, there exists a CPTP map Φ such that $\sigma = \Phi(\rho)$ and $\eta = \Phi(\tau)$ if and only if

$$M(\rho, \tau) \geq M(\sigma, \eta) \geq m(\sigma, \eta) \geq m(\rho, \tau), \quad (49)$$

and

$$\det(\sigma - t\eta) \geq \det(\rho - t\tau) \quad \forall m(\sigma, \eta) \leq t \leq M(\sigma, \eta). \quad (50)$$

We now apply the above lemma above to the case $\tau = \eta = \gamma^{(2)}$.

1. First case: Nonzero temperature ($\gamma^{(2)} > 0$)

We first assume $\det(\gamma^{(2)}) > 0$. The necessity of (40) follows from the fact that the min and max relative entropies both satisfy the data processing inequality. We therefore need to show that they are sufficient. With the choice $\tau = \eta = \gamma^{(2)}$ the conditions (49) are equivalent to the conditions (40) [recall the last condition of (40) is trivial since we assume for now that $\gamma^{(2)}$ is full rank]. It is therefore left to show that the conditions (50) hold automatically if Eqs. (40) hold. Indeed, recall that for $\det(\gamma^{(2)}) > 0$ we have

$$\det(\rho - t\gamma^{(2)}) = \det(\gamma^{(2)})[t - m(\rho, \gamma^{(2)})][t - M(\rho, \gamma^{(2)})] \quad (51)$$

and

$$\det(\sigma - t\gamma^{(2)}) = \det(\gamma^{(2)})[t - m(\sigma, \gamma^{(2)})][t - M(\sigma, \gamma^{(2)})]. \quad (52)$$

Hence, the inequality $\det(\sigma - t\gamma^{(2)}) \geq \det(\rho - t\gamma^{(2)})$ is equivalent to

$$\begin{aligned} t[M(\rho, \gamma^{(2)}) + m(\rho, \gamma^{(2)}) - M(\sigma, \gamma^{(2)}) - m(\sigma, \gamma^{(2)})] \\ + m(\sigma, \gamma^{(2)})M(\sigma, \gamma^{(2)}) - m(\rho, \gamma^{(2)})M(\rho, \gamma^{(2)}) \geq 0. \end{aligned} \quad (53)$$

We therefore need to show that the above inequality holds for all $m(\sigma, \gamma^{(2)}) \leq t \leq M(\sigma, \gamma^{(2)})$. It is therefore sufficient to show that it holds at the two extreme points of the interval. Indeed, for $t = m(\sigma, \gamma^{(2)})$ after some algebra the expression in (53) becomes

$$[M(\rho, \gamma^{(2)}) - m(\sigma, \gamma^{(2)})][m(\sigma, \gamma^{(2)}) - m(\rho, \gamma^{(2)})],$$

which is non-negative due to (40). Similarly, substituting $t = M(\sigma, \gamma^{(2)})$ in (53) gives

$$[M(\sigma, \gamma^{(2)}) - m(\rho, \gamma^{(2)})][M(\rho, \gamma^{(2)}) - M(\sigma, \gamma^{(2)})],$$

which is again non-negative due to (40). This completes the proof of the theorem for the case $\det(\gamma^{(2)}) > 0$.

2. Second case: Zero temperature [$\det(\gamma^{(2)}) = 0$]

In this case direct calculation gives

$$\det(\rho - t\gamma^{(2)}) = \det(\rho) - t(1 - \text{Tr}[\rho\Pi_{\gamma^{(2)}}]) \quad (54)$$

and

$$\det(\sigma - t\gamma^{(2)}) = \det(\sigma) - t(1 - \text{Tr}[\sigma\Pi_{\gamma^{(2)}}]). \quad (55)$$

Note that $\Pi_{\gamma^{(2)}}$ is either the projection $|1\rangle\langle 1|$ or $|2\rangle\langle 2|$.

Now, assuming $\rho \neq \sigma \neq \gamma^{(2)}$ (otherwise the problem becomes trivial), since $\gamma^{(2)}$ is rank one we have $M(\rho, \gamma^{(2)}) = M(\sigma, \gamma^{(2)}) = \infty$. On the other hand, in this case a simple calculation gives

$$m(\rho, \gamma^{(2)}) = \frac{\det(\rho)}{1 - \text{Tr}[\rho\Pi_{\gamma^{(2)}}]},$$

$$m(\sigma, \gamma^{(2)}) = \frac{\det(\sigma)}{1 - \text{Tr}[\sigma\Pi_{\gamma^{(2)}}]},$$

where we used $\rho \neq \gamma^{(2)}$ and $\sigma \neq \gamma^{(2)}$, so that together with $\gamma^{(2)}$ being rank one gives $\text{Tr}[\rho\Pi_{\gamma^{(2)}}] < 1$ and similarly $\text{Tr}[\sigma\Pi_{\gamma^{(2)}}] < 1$. Hence, in this case exploring the behavior of $\det(\sigma - t\gamma^{(2)}) \geq \det(\rho - t\gamma^{(2)})$ in the limit $t \rightarrow M(\sigma, \gamma^{(2)}) = \infty$ we must have

$$\text{Tr}[\sigma\Pi_{\gamma^{(2)}}] \geq \text{Tr}[\rho\Pi_{\gamma^{(2)}}],$$

which is equivalent to $D_{\min}(\gamma^{(2)}\|\rho) \geq D_{\min}(\gamma^{(2)}\|\sigma)$. At the point $t = m(\sigma, \gamma^{(2)})$, the inequality $\det(\sigma - t\gamma^{(2)}) \geq \det(\rho - t\gamma^{(2)})$ becomes

$$0 \geq \det(\rho) - m(\sigma, \gamma^{(2)})(1 - \text{Tr}[\rho\Pi_{\gamma^{(2)}}])$$

$$= (1 - \text{Tr}[\rho\Pi_{\gamma^{(2)}}]) \left(\frac{\det(\rho)}{1 - \text{Tr}[\rho\Pi_{\gamma^{(2)}}]} - m(\sigma, \gamma^{(2)}) \right)$$

$$= (1 - \text{Tr}[\rho\Pi_{\gamma^{(2)}}])[m(\rho, \gamma^{(2)}) - m(\sigma, \gamma^{(2)})],$$

which is satisfied since $m(\rho, \gamma^{(2)}) \leq m(\sigma, \gamma^{(2)})$. Hence, $\det(\sigma - t\gamma^{(2)}) \geq \det(\rho - t\gamma^{(2)})$ for all t with $m(\sigma, \gamma^{(2)}) \leq t < \infty$. This completes the proof.

B. Test-and-prepare channels

The proof of Theorem 3 above relies on a lemma proved by Alberti and Uhlmann [34], which, together with Theorem 2, implies that, if $n = m = 2$ (i.e., for qubits) then $(\rho_1, \rho_2) > (\rho'_1, \rho'_2)$ if and only if there exists a CPTP map Φ such that $\Phi(\rho_i) = \rho'_i$ ($i = 1, 2$). However, explicit counterexamples exist, showing that as soon as one leaves the qubit case, already when $n = 3$ and $m = 2$, this is not true anymore [26]. Hence, leaving aside the general case, we focus instead on a special class of CPTP maps, namely, *test-and-prepare channels* of the form

$$\mathcal{E}(\rho) \triangleq \text{Tr}[E\rho]\xi_1 + \text{Tr}[(\mathbb{1} - E)\rho]\xi_2,$$

for some effect $0 \leq E \leq \mathbb{1}$ and some density matrices ξ_1 and ξ_2 . Test-and-prepare channels are, in other words, measure-and-prepare channels for which the measurement has only two possible outcomes. Although restricted, this class seems quite natural in the framework of quantum relative Lorenz curves, which are defined only in terms of binary measurements (i.e., hypothesis tests). Indeed, a necessary and sufficient condition

for the existence of a test-and-prepare channel between two pairs of density matrices can be expressed in terms of quantum relative Lorenz curves as follows:

Theorem 4. Given two pairs of density matrices (ρ_1, ρ_2) and (ρ'_1, ρ'_2) on \mathbb{C}^n and \mathbb{C}^m , respectively, there exists a test-and-prepare channel \mathcal{E} such that $\mathcal{E}(\rho_1) = \rho'_1$ and $\mathcal{E}(\rho_2) = \rho'_2$, if and only if the quantum Lorenz curve of ρ_1 relative to ρ_2 is nowhere below the segments joining the points $(0, 0)$, $(1, 1)$ and passing through either

$$(x, y) = \left(\frac{1 - m'}{M' - m'}, \frac{M'(1 - m')}{M' - m'} \right)$$

or

$$(x, y) = \left(\frac{M' - 1}{M' - m'}, \frac{m'(M' - 1)}{M' - m'} \right),$$

whichever is higher, where

$$M' \triangleq 2^{D_{\max}(\rho'_1\|\rho'_2)} = \sup(\rho'_1/\rho'_2)$$

and

$$m' \triangleq 2^{-D_{\max}(\rho'_2\|\rho'_1)} = \inf(\rho'_1/\rho'_2).$$

Proof. Consider a test-and-prepare channel of the form

$$\mathcal{E}(\rho) = \text{Tr}[E\rho]\sigma_1 + \text{Tr}[(\mathbb{1} - E)\rho]\sigma_2, \quad (56)$$

where σ_1, σ_2 are density matrices (i.e., positive semidefinite matrices with trace 1) and $0 \leq E \leq \mathbb{1}$. If $\mathcal{E}(\rho_j) = \rho'_j$ for $j = 1, 2$, then

$$\rho'_1 = e_1\sigma_1 + (1 - e_1)\sigma_2,$$

$$\rho'_2 = e_2\sigma_1 + (1 - e_2)\sigma_2,$$

where $e_j \equiv \text{Tr}[E\rho_j]$ for $j = 1, 2$. Assuming $e_1 \neq e_2$ (otherwise, $\rho'_1 = \rho'_2$), the above equations are equivalent to

$$\sigma_1 = \frac{1}{e_1 - e_2} [(1 - e_2)\rho'_1 - (1 - e_1)\rho'_2],$$

$$\sigma_2 = \frac{1}{e_1 - e_2} [-e_2\rho'_1 + e_1\rho'_2].$$

Note that σ_1 and σ_2 have trace 1 since ρ'_1 and ρ'_2 have trace 1. Without loss of generality we can assume $e_1 > e_2$. With this choice, σ_1 and σ_2 are positive semidefinite if and only if

$$\rho'_1 - \frac{1 - e_1}{1 - e_2}\rho'_2 \geq 0 \quad \text{and} \quad \rho'_2 - \frac{e_2}{e_1}\rho'_1 \geq 0. \quad (57)$$

Note that since we assume $e_1 > e_2$ we have $1 - e_2 > 0$ and $e_1 > 0$. Denote by $m' \equiv \inf(\rho'_1/\rho'_2)$ and by $M' \equiv \sup(\rho'_1/\rho'_2)$, and note that $\inf(\rho'_2/\rho'_1) = 1/M'$. We therefore get that σ_1 and σ_2 are positive semidefinite if and only if

$$\frac{1 - e_1}{1 - e_2} \leq m' \quad \text{and} \quad \frac{e_2}{e_1} \leq \frac{1}{M'}. \quad (58)$$

The above inequalities are equivalent to

$$\text{Tr}[E(\rho_1 - m'\rho_2)] \geq 1 - m',$$

$$\text{Tr}[E(\rho_1 - M'\rho_2)] \geq 0.$$

We therefore arrive at the following lemma:

Lemma 9. There exists a channel \mathcal{E} of the form (56) such that $\mathcal{E}(\rho_1) = \rho'_1$ and $\mathcal{E}(\rho_2) = \rho'_2$, if and only if

$$W(\rho_1, \rho_2, \rho'_1, \rho'_2) \geq 0, \quad (59)$$

where the witness W is defined as

$$W(\rho_1, \rho_2, \rho'_1, \rho'_2) \triangleq m' - 1 + \max_{\substack{0 \leq E \leq \mathbb{1} \\ \text{Tr}[E(\rho_1 - M' \rho_2)] \geq 0}} \text{Tr}[E(\rho_1 - m' \rho_2)]. \quad (60)$$

The calculation of W can be simplified using the following dual formulation of linear programming, analogously to what we did in the proof of Lemma 5. Let V_1 and V_2 be two (inner product) vector spaces with two cones $K_1 \subset V_1$ and $K_2 \subset V_2$. Consider two vectors $v_1 \in V_1$ and $v_2 \in V_2$, and a linear map $\Gamma : V_1 \rightarrow V_2$. Then, the primal form is

$$\max_{\substack{x \in K_1 \\ v_2 - \Gamma(x) \in K_2}} \langle v_1, x \rangle_1. \quad (61)$$

The dual form involves the adjoint map $\Gamma^* : V_2 \rightarrow V_1$:

$$\min_{\substack{y \in K_2 \\ \Gamma^*(y) - v_1 \in K_1}} \langle v_2, y \rangle_2. \quad (62)$$

For our purposes, we take $V_1 = \mathcal{H}_n$ the space of $n \times n$ Hermitian matrices, and we take $K_1 = \mathcal{H}_{n,+}$ the cone of positive semidefinite matrices in \mathcal{H}_n . We further define the vector space

$$V_2 \equiv \mathbb{R} \oplus \mathcal{H}_n = \{(r, A) \mid r \in \mathbb{R}; A \in \mathcal{H}_n\}, \quad (63)$$

with inner product $\langle (r, A), (t, B) \rangle_1 := rt + \text{Tr}[AB]$. Further, define $K_2 = \mathbb{R}_+ \oplus \mathcal{H}_{n,+}$ to be the positive cone in V_2 . The

linear map $\Gamma : V_1 \rightarrow V_2$ is defined as follows:

$$\Gamma(A) = (-\text{Tr}[A(\rho_1 - M' \rho_2)], A). \quad (64)$$

Note that the dual map $\Gamma^* : V_2 \rightarrow V_1$ is given by

$$\Gamma^*(r, A) = A - r(\rho_1 - M' \rho_2). \quad (65)$$

Finally, set $v_1 = \rho_1 - m' \rho_2$ and $v_2 = (0, \mathbb{1}_n)$. With these choices, $v_2 - \Gamma(x) = (\text{Tr}[x(\rho_1 - M' \rho_2)], \mathbb{1}_n - x)$, so that the primal problem becomes

$$\max_{\substack{x \in K_1 \\ v_2 - \Gamma(x) \in K_2}} \langle v_1, x \rangle_1 = \max_{\substack{0 \leq E \leq \mathbb{1}_n \\ \text{Tr}[E(\rho_1 - M' \rho_2)] \geq 0}} \text{Tr}[E(\rho_1 - m' \rho_2)], \quad (66)$$

where we renamed x with E . The dual problem is given by

$$\min_{\substack{y \in K_2 \\ \Gamma^*(y) - v_1 \in K_1}} \langle v_2, y \rangle_2 = \min_{\substack{r, F \geq 0 \\ F \geq r(\rho_1 - M' \rho_2) + (\rho_1 - m' \rho_2)}} \text{Tr}[F], \quad (67)$$

where we took $y = (r, F)$. We can further simplify the above expression. First note that, for any given r , the positive semidefinite matrix F with the smallest trace that satisfies

$$\begin{aligned} F &\geq r(\rho_1 - M' \rho_2) + (\rho_1 - m' \rho_2) \\ &= (1+r)\rho_1 - (rM' + m')\rho_2 \end{aligned}$$

is of course the positive part of the left-hand side:

$$F = [(1+r)\rho_1 - (rM' + m')\rho_2]_+. \quad (68)$$

We therefore conclude that the dual problem is equivalent to

$$\min_{r \geq 0} \text{Tr}[(1+r)\rho_1 - (rM' + m')\rho_2]_+ = \min_{r \geq 0} \frac{1 - m' - r(M' - 1) + \|(1+r)\rho_1 - (rM' + m')\rho_2\|_1}{2}. \quad (69)$$

By plugging the above equation into (60), we therefore conclude that

$$\begin{aligned} W(\rho_1, \rho_2, \rho'_1, \rho'_2) &= m' - 1 + \frac{1}{2}(1 - m') + \frac{1}{2} \min_{r \geq 0} [\|(1+r)\rho_1 - (rM' + m')\rho_2\|_1 - r(M' - 1)] \\ &= \frac{1}{2}(m' - 1) + \frac{1}{2} \min_{r \geq 0} [\|(1+r)\rho_1 - (rM' + m')\rho_2\|_1 - r(M' - 1)], \end{aligned}$$

namely,

$$2W(\rho_1, \rho_2, \rho'_1, \rho'_2) = -(1 - m') + \min_{r \geq 0} (\|(1+r)\rho_1 - (rM' + m')\rho_2\|_1 - r(M' - 1)). \quad (70)$$

Introducing

$$t \triangleq \frac{rM' + m'}{1+r},$$

and noting that $t \in [m', M']$ we obtain that there exists a channel \mathcal{E} of the form (56) such that $\mathcal{E}(\rho_1) = \rho'_1$ and $\mathcal{E}(\rho_2) = \rho'_2$, if and only if

$$\begin{aligned} \|\rho_1 - t\rho_2\|_1 &\geq \frac{(M' - t)(1 - m') + (t - m')(M' - 1)}{M' - m'} \\ &= \frac{m' + M' - 2m'M' + t[m' + M' - 2]}{M' - m'}, \end{aligned} \quad (71)$$

for all $t \in [m', M']$, or equivalently,

$$\|\rho_1 - t\rho_2\|_1 \geq \|\sigma_1 - t\sigma_2\|_1, \quad \forall t \geq 0, \quad (72)$$

where

$$\begin{aligned} \sigma_1 &\equiv \frac{1}{M' - m'} \begin{pmatrix} M'(1 - m') & 0 \\ 0 & m'(M' - 1) \end{pmatrix}, \\ \sigma_2 &\equiv \frac{1}{M' - m'} \begin{pmatrix} 1 - m' & 0 \\ 0 & M' - 1 \end{pmatrix} \end{aligned} \quad (73)$$

are two diagonal 2×2 density matrices with the property that $\sup(\sigma_1/\sigma_2) = M' \equiv \sup(\rho'_1/\rho'_2)$ and $\inf(\sigma_1/\sigma_2) = m' \equiv \inf(\rho'_1/\rho'_2)$.

Finally, the statement of Theorem 4 is obtained noticing that condition (72) is equivalent, due to Theorem 2, to saying that the Lorenz curve of ρ_1 relative to ρ_2 is never below that of σ_1 relative to σ_2 . However, since the latter is the quantum Lorenz curve of two classical probability distributions, it is just made of two segments joining the points (0,0) with (1,1), passing through either $(\frac{1-m'}{M'-m'}, \frac{M'(1-m')}{M'-m'})$ or $(\frac{M'-1}{M'-m'}, \frac{m'(M'-1)}{M'-m'})$, whichever determines the steepest curve. ■

C. Probabilistic transformations

By mixing ρ'_1 with a sufficient fraction of ρ'_2 , while keeping ρ'_2 unchanged, it is always possible to decrease the gap between m' and M' , until the conditions of Theorem 4 are met. In this way, with sufficient mixing, any output pair can be obtained, but the noise due to mixing cannot be undone afterwards.

One way to overcome this problem is to relax the assumptions made on the channel, in particular, the condition of trace preservation. We hence consider *probabilistic* channels of the following form:

$$\mathcal{E}(\rho) \triangleq \text{Tr}[E\rho]\xi_1 + \text{Tr}[F\rho]\xi_2, \quad (74)$$

where $E, F \geq 0, E + F \leq \mathbb{1}$, and ξ_1, ξ_2 are two (normalized) density matrices. The above transformation constitutes a *heralded* probabilistic transformation, in the sense that we know if the protocol succeeded or not, with success probability given by $P_{\text{succ}} = \text{Tr}[(E + F)\rho]$. The main result of this section is given by the following.

Theorem 5. Consider two pairs of density matrices (ρ_1, ρ_2) and (ρ'_1, ρ'_2) on \mathbb{C}^n and \mathbb{C}^m , respectively. Then, a channel of the form (74), such that

$$\mathcal{E}(\rho_1) = p_1 \rho'_1, \quad \mathcal{E}(\rho_2) = p_2 \rho'_2, \quad (75)$$

exists if and only if

$$\frac{m}{m'} \leq \frac{p_1}{p_2} \leq \frac{M}{M'},$$

where $p_i = \text{Tr}[(E + F)\rho_i]$, and m, M, m' , and M' are as in Theorem 4.

When the protocol fails, we just prepare the state ρ'_2 independently of the input. In this way, we realize a channel that deterministically transforms ρ_2 into ρ'_2 , but is also able to transform ρ_1 into ρ'_1 , whenever the successful event is recorded.

In Ref. [29], it is shown that a probabilistic transformation from (ρ_1, ρ_2) to (ρ'_1, ρ'_2) exists if and only if $M'/m' \leq M/m$. The above theorem hence slightly extends that. For example, Theorem 5 implies that the success probability p_1 be bounded as

$$p_1 \leq M/M' \equiv e^{-\Delta F_{\text{max}}},$$

where we define the *max-free energy difference* as

$$\Delta F_{\text{max}} \triangleq D_{\text{max}}(\rho'_1 \parallel \rho'_2) - D_{\text{max}}(\rho_1 \parallel \rho_2).$$

In order to prove Theorem 5, let us consider a probabilistic test-and-prepare channel of the form

$$\Phi(\rho) = \text{Tr}[E\rho]\sigma_1 + \text{Tr}[F\rho]\sigma_2, \quad (76)$$

where σ_1, σ_2 are normalized density matrices and $E, F \geq 0$ with $E + F \leq \mathbb{1}$. If $\mathcal{E}(\rho_j) = p_j \rho'_j$ for $j = 1, 2$ with $0 <$

$p_1, p_2 \leq 1$, then

$$p_1 \rho'_1 = e_1 \sigma_1 + f_1 \sigma_2,$$

$$p_2 \rho'_2 = e_2 \sigma_1 + f_2 \sigma_2,$$

where $e_j \equiv \text{Tr}[E\rho_j]$ and $f_j \equiv \text{Tr}[F\rho_j]$ for $j = 1, 2$. Assuming $e_1 f_2 \neq e_2 f_1$ the above equations are equivalent to

$$\sigma_1 = \frac{1}{e_1 f_2 - e_2 f_1} [f_2 p_1 \rho'_1 - f_1 p_2 \rho'_2],$$

$$\sigma_2 = \frac{1}{e_1 f_2 - e_2 f_1} [-e_2 p_1 \rho'_1 + e_1 p_2 \rho'_2].$$

Note that σ_1 and σ_2 have trace 1 since ρ'_1 and ρ'_2 have trace 1. Without loss of generality, we can assume $e_1/e_2 > f_1/f_2$. With this choice, σ_1 and σ_2 are positive semidefinite if and only if

$$\rho'_1 - q^{-1} \frac{f_1}{f_2} \rho'_2 \geq 0 \quad \text{and} \quad q^{-1} \frac{e_1}{e_2} \rho'_2 - \rho'_1 \geq 0, \quad (77)$$

where $q \triangleq p_1/p_2$. Again, denote by $m' = \inf(\rho'_1/\rho'_2)$ and by $M' = \sup(\rho'_1/\rho'_2)$. We therefore get that σ_1 and σ_2 are positive semidefinite if and only if

$$\frac{f_1}{f_2} \leq qm' \quad \text{and} \quad \frac{e_1}{e_2} \geq qM'. \quad (78)$$

Recalling the definitions of e_1, e_2, f_1, f_2 , the above inequalities are equivalent to

$$\text{Tr}[F(qm' \rho_2 - \rho_1)] \geq 0, \quad \text{Tr}[E(\rho_1 - qM' \rho_2)] \geq 0. \quad (79)$$

We therefore arrive at the following lemma.

Lemma 10. There exists a CP map Φ of the form (76) such that $\Phi(\rho_1) = p_1 \rho'_1$ and $\Phi(\rho_2) = p_2 \rho'_2$, if and only if

$$\text{Tr}[F(qm' \rho_2 - \rho_1)] \geq 0 \quad (80)$$

and

$$\text{Tr}[E(\rho_1 - qM' \rho_2)] \geq 0. \quad (81)$$

Our goal is to maximize p_1 under these constraints along with the constraint $\text{Tr}[(E + F)(q\rho_2 - \rho_1)] = 0$ that defines q . Therefore, the maximum value of p_1 , with a fixed value of $q \in \mathbb{R}_+$, is given by

$$P_{\text{max}}(q) = \max_{\substack{\text{Tr}[(E + F)(q\rho_2 - \rho_1)] = 0 \\ \text{Tr}[E(\rho_1 - qM' \rho_2)] \geq 0 \\ \text{Tr}[F(qm' \rho_2 - \rho_1)] \geq 0 \\ E, F \geq 0, E + F \leq \mathbb{1}_n}} \text{Tr}[(E + F)\rho_1]. \quad (82)$$

This is an optimization problem that can be solved efficiently and algorithmically using SDP. Moreover, in the lemma below we show that if q is not in the right interval, then $P_{\text{max}}(q) = 0$.

Lemma 11. $P_{\text{max}}(q) > 0$ implies that

$$\frac{m}{m'} \leq q \leq \frac{M}{M'}.$$

Proof. Suppose $q < \frac{m}{m'}$. Then, $qm' < m$ so that $qm' \rho_2 - \rho_1 \leq 0$. Moreover,

$$\begin{aligned} \text{Tr}[F(qm' \rho_2 - \rho_1)] &= \text{Tr}[F(m\rho_2 - \rho_1)] - (m - m'q)\text{Tr}[F\rho_2] \\ &< 0, \end{aligned}$$

unless $\text{Tr}[F\rho_1] = \text{Tr}[F\rho_2] = 0$. We therefore must have $\text{Tr}[F\rho_1] = \text{Tr}[F\rho_2] = 0$. This latter condition gives

$$\begin{aligned} \text{Tr}[(E + F)(q\rho_2 - \rho_1)] &= 0 \\ \iff \text{Tr}[E(q\rho_2 - \rho_1)] &= 0 \\ \iff \text{Tr}[E\rho_1] &= q\text{Tr}[E\rho_2]. \end{aligned}$$

But this last equality gives

$$\text{Tr}[E(\rho_1 - qM'\rho_2)] = (1 - M')\text{Tr}[E\rho_1] \leq 0, \quad (83)$$

since $M' > 1$. We therefore must have $\text{Tr}[E\rho_1] = \text{Tr}[E\rho_2] = 0$. Together with $\text{Tr}[F\rho_1] = \text{Tr}[F\rho_2] = 0$, it gives $P_{\max}(q) = 0$. Following similar lines we get $P_{\max}(q) = 0$ for $q > M/M'$. ■

As a consequence of the above discussion, we obtain the following corollary, which is consistent with a result in Ref. [29], but slightly more general:

Corollary 1.

$$\max_{q \in \mathbb{R}_+} P_{\max}(q) > 0 \iff \mathfrak{h}(\rho_1, \rho_2) \geq \mathfrak{h}(\rho'_1, \rho'_2).$$

VI. DISCUSSION

In the present work we introduced quantum relative Lorenz curves and Hilbert α divergences, studied their properties, and applied them to the problem of characterizing necessary and sufficient conditions for the existence of a suitable transformation from an initial pair of states (ρ_1, ρ_2) to a final one (ρ'_1, ρ'_2) . In particular, a strong equivalence has been proved in the case of coherent energy transitions with Gibbs-preserving maps, a paradigm that has immediate applications in quantum thermodynamics and the resource theory of athermality. Finally, we also considered the cases of test-and-prepare channels and probabilistic transformations, giving necessary and sufficient conditions for both.

ACKNOWLEDGMENTS

The authors are grateful to Mark Girard for his help with Figs. 1 and 2. F.B. acknowledges financial support from the JSPS KAKENHI, Grant No. 26247016. G.G. acknowledges financial support from NSERC.

-
- [1] M. O. Lorenz, *Publ. Am. Statist. Assoc.* **9**, 209 (1905).
[2] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Springer Series in Statistics (Springer, New York, 2011).
[3] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, and N. Y. Halpern, *Phys. Rep.* **583**, 1 (2015).
[4] M. Horodecki and J. Oppenheim, *Nat. Commun.* **4**, 2059 (2013).
[5] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, *Phys. Rev. Lett.* **111**, 250404 (2013).
[6] M. Horodecki and J. Oppenheim, *Int. J. Mod. Phys. B* **27**, 1345019 (2013).
[7] B. Coecke, T. Fritz, and R. W. Spekkens, *Inf. Comput.* **250**, 59 (2016).
[8] F. G. S. L. Brandão and G. Gour, *Phys. Rev. Lett.* **115**, 070503 (2015).
[9] J. Neyman and E. S. Pearson, *Philos. Trans. R. Soc. London, Ser. A* **231**, 289 (1933).
[10] J. M. Renes, *J. Math. Phys.* **57**, 122202 (2016).
[11] K. M. R. Audenaert, J. Calsamiglia, R. Muñoz-Tapia, E. Bagan, Ll. Masanes, A. Acín, and F. Verstraete, *Phys. Rev. Lett.* **98**, 160501 (2007).
[12] M. Nussbaum and A. Szkola, *Ann. Statist.* **37**, 1040 (2009).
[13] M. Hayashi, *Phys. Rev. A* **76**, 062301 (2007).
[14] F. Hiai and D. Petz, *Commun. Math. Phys.* **143**, 99 (1991).
[15] T. Ogawa and H. Nagaoka, *IEEE Trans. Inf. Theory* **46**, 2428 (2000).
[16] L. Wang and R. Renner, *Phys. Rev. Lett.* **108**, 200501 (2012).
[17] F. Dupuis, L. Krämer, P. Faist, J. M. Renes, and R. Renner, *Generalized entropies*, in Proceedings of the XVIIIth International Congress on Mathematical Physics, 2012 (unpublished), Chap. 9, pp. 134–153.
[18] N. Datta and F. Leditzky, *J. Phys. A* **47**, 045304 (2014).
[19] D. Blackwell, *Ann. Math. Stat.* **24**, 265 (1953).
[20] E. Torgersen, *Comparison of Statistical Experiments*, Encyclopedia of Mathematics and its Applications (Cambridge University Press, Cambridge, UK, 1991).
[21] F. Buscemi, Fully quantum second-law-like statements from the theory of statistical comparisons, *arXiv:1505.00535*.
[22] E. Shmaya, *J. Phys. A* **38**, 9717 (2005).
[23] F. Buscemi, *Commun. Math. Phys.* **310**, 625 (2012).
[24] K. Matsumoto, *arXiv:1012.2650*.
[25] A. Jenčová, *Rep. Math. Phys.* **70**, 237 (2012).
[26] K. Matsumoto, *arXiv:1409.5658*.
[27] P. J. Bushell, *Arch. Ration. Mech. Anal.* **52**, 330 (1973).
[28] S. P. Eveson, *Proc. London Math. Soc.* **s3-70**, 411 (1995).
[29] D. Reeb, M. J. Kastoryano, and M. M. Wolf, *J. Math. Phys.* **52**, 082201 (2011).
[30] N. Datta, *IEEE Trans. Inf. Theory* **55**, 2816 (2009).
[31] F. Buscemi, *Prob. Inf. Trans.* **52**, 201 (2016).
[32] V. Narasimhachar and G. Gour, *Nat. Commun.* **6**, 7689 (2015).
[33] P. Cwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, *Phys. Rev. Lett.* **115**, 210403 (2015).
[34] P. M. Alberti and A. Uhlmann, *Rep. Math. Phys.* **18**, 163 (1980).