Majorization of quantum polarization distributions

Alfredo Luis and Gonzalo Donoso
Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain
(Received 2 October 2016; published 28 December 2016)

Majorization provides a rather powerful partial-order classification of probability distributions depending only on the spread of the statistics, and not on the actual numerical values of the variable being described. We propose to apply majorization as a metameasure of quantum polarization fluctuations, this is to say of the degree of polarization. We compare the polarization fluctuations of the most relevant classes of quantum and classical-like states. In particular we test Lieb's conjecture regarding classical-like states as the most polarized and a complementary conjecture that the most unpolarized pure states are the most nonclassical.

DOI: 10.1103/PhysRevA.94.063858

I. INTRODUCTION

Light fluctuations are relevant both from fundamental as well as practical perspectives. On the one hand field statistics are the key feature distinguishing classical from quantum light [1]. On the other hand, fluctuations and uncertainty usually limit the performance of optical applications. In this regard is worth noting that polarization and two-beam linear interferometry share the same fundamental SU(2) symmetry, so we may say that they are isomorphic. Deep down, this equivalence holds because interference and polarization are the two main manifestations of coherence.

Both in classical and quantum optics, polarization uncertainty is assessed via the degree of polarization [2–4]. The classic definition in terms of the Stokes parameters involves just second-order statistics of the field complex amplitudes. This cannot reflect statistical properties involving higher-order moments, in particular polarization fluctuations, which are crucial in quantum optics [4]. For example, there are states with vanishing degree of polarization that nevertheless cannot be regarded as being unpolarized, which is usually referred to as hidden polarization [4,5].

These and similar reasonings have motivated the introduction of other measures of polarization fluctuations, actually plenty of them [3,4,6,7]. In this work we go beyond particular definitions of the degree of polarization by applying the mathematical idea of majorization to quantum polarization distributions. Majorization provides a rather powerful partial-order classification of probability distributions depending only on the spread of the statistics, and not on the actual numerical values of the variable being described [8]. This ordering is respected by the entropic measures. So majorization actually becomes a kind of metameasure of uncertainty. In our case this means to go beyond all measures of the degree of polarization introduced so far.

As a suitable polarization distribution in quantum optics we focus on the SU(2) Q function because of its good properties, especially SU(2) invariance [6,9]. We apply this technique to the most relevant classical and nonclassical polarization states. In particular we test Lieb's conjecture regarding SU(2) coherent states as the most polarized states in quantum optics [10]. Since SU(2) coherent states are also regarded as the most classical states [11,12], this suggests the ensuing complementary conjecture: that the most quantum states should be the most unpolarized pure states [13]. This conjecture can readily be tested also via majorization.

In Sec. II we present the main ingredients such as the polarization SU(2) Q function, majorization, and the most relevant classes of states to be compared. This includes the SU(2) coherent states as the most classical-like, as well as nonclassical examples such as squeezed states, the so-called NOON states, the phase states, and finally the most nonclassical states according to the Hilbert-Schmidt distance. In Sec. III the polarization distributions of these states are compared via majorization. Since in principle polarization and intensity are independent degrees of freedom, we focus mainly on states with definite total number of photons. Nevertheless, we consider also more practical and experimentally generable states with nondefinite total number of photons.

II. PROCEDURE

A. Polarization distribution

A suitable polarization distribution can be introduced via the SU(2) Q function $Q(\Omega)$ defined by projection of the density matrix ρ on the SU(2) coherent states as [6,9]

$$Q(\Omega) = \sum_{n=0}^{\infty} \frac{n+1}{4\pi} \langle n,\Omega|\rho|n,\Omega\rangle, \tag{1}$$

where $|n,\Omega\rangle$ are the SU(2) coherent states [12]

$$|n,\Omega\rangle = \sum_{m=0}^{n} \binom{n}{m}^{1/2} \left(\sin\frac{\theta}{2}\right)^{n-m} \left(\cos\frac{\theta}{2}\right)^{m} \times e^{-im\phi}|m,n-m\rangle, \tag{2}$$

and $|n_1,n_2\rangle = |n_1\rangle_1|n_2\rangle_2$ denote the product of photon number states in the corresponding two field modes sustaining the polarization degree of freedom. The variables $\Omega = (\theta,\phi)$ represent points on a unit sphere, the Poincaré sphere, with polar angle θ, azimuthal angle ϕ, and surface element $d\Omega = \sin\theta d\theta d\phi$.

The SU(2) symmetry reflects the fact that all points on the sphere are equivalent. This is conveniently respected by the SU(2) Q function since the $Q(\Omega)$ function for the transformed state has the same form of the original one, but simply centered at another point of the Poincaré sphere. SU(2) transformations are quite simply implemented in practice via phase plates or beam splitters.

To simplify the comparison between distributions via majorization we shall discretize the polarization distribution by dividing the Poincaré sphere into N surface elements, say pixels. The key point to maintain the natural SU(2) invariance is that all pixels should be of the same area. Taking into account that $d\Omega = \sin\theta d\theta d\phi = |d\cos\theta| d\phi$, we accomplish this by dividing the ranges of variation of $\cos\theta$ and ϕ into intervals of the same length, that is,

$$\theta_\ell = \arccos\left(\frac{2\ell-1}{N_\theta} - 1\right), \quad \ell = 1,\ldots,N_\theta,$$
$$\phi_k = \frac{2\pi}{N_\phi}k - \pi, \quad k = 1,\ldots,N_\phi. \tag{3}$$

Thus the discretized version of $Q(\Omega)$ is

$$p_j = Q(\Omega_j)d\Omega, \quad \Omega_j = (\theta_\ell,\phi_k),$$
$$j = N_\phi(\ell-1)+k = 1,\ldots,N, \tag{4}$$

where $N = N_\theta N_\phi$ and $d\Omega = 4\pi/N$. More rigorously we should integrate the $Q(\Omega)$ distribution to each pixel, but this approximate form is rather simple and good enough for our purposes if the sampling is accurate. In the limit of accurate sampling neither the area nor the shape of the pixels matters.

B. Majorization

Since polarization lives on an sphere, this is a good place to apply statistical evaluations of uncertainty and fluctuations beyond variance and standard first-order moments, for example confidence intervals or entropylike measures. In this regard, both lead us to majorization as a kind of metameasure of fluctuations. Let us show this in more detail: we first present two equivalent formal definitions of majorization and then we provide some physical intuition about it.

Denoting by $\tilde{p}$ and p two given probability distributions, we say that p majorizes $\tilde{p}$, which is expressed as $\tilde{p} \prec p$, when the following relation between the ordered partial sums, or Lorenz curves, is satisfied for all k (see Fig. 1):

$$S_k(\tilde{p}^\downarrow) = \sum_{j=1}^{k} \tilde{p}_j^{\,\downarrow} \leqslant \sum_{j=1}^{k} p_j^\downarrow = S_k(p^\downarrow), \tag{5}$$

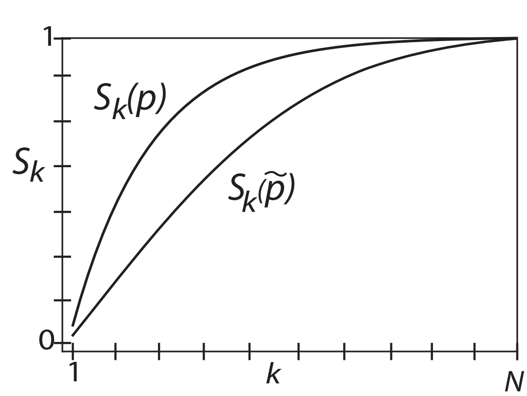

FIG. 1. Relation between partial ordered sums S_k as functions of k when the majorization $\tilde{p} \prec p$ holds. Although k is discrete, in all plots the points have been joined by continuous lines as an aid to the eye.

where $k = 1,2,\ldots,N$ represents the number of pixels the probabilities of which are added in the corresponding ordered partial sum S_k, always with $S_N = 1$. The superscript ↓ denotes that the p_j values are arranged in decreasing order: $p_1^\downarrow \geqslant p_2^\downarrow \geqslant \cdots \geqslant p_N^\downarrow$. We will say that two distributions are comparable if one majorizes the other. Moreover, $\tilde{p} \prec p$ is equivalent to say that there exist N-dimensional permutation matrices Π_j and a probability distribution $\{\pi_j\}$ such that

$$\tilde{p} = \sum_j \pi_j \Pi_j p. \tag{6}$$

That is, $\tilde{p}$ is majorized by p when $\tilde{p}$ can be obtained from p by randomly permuting its components, and then averaging over the permutations.

Majorization is a partial ordering relation, so that not every two distributions can be compared. Thus we can find distributions that neither $\tilde{p} \prec p$ nor $p \prec \tilde{p}$. This situation will be represented as $p \bowtie \tilde{p}$. In such a case the Lorenz curves S_k will intersect.

Roughly speaking, if p majorizes $\tilde{p}$ we may say that p presents less dispersion or less uncertainty than $\tilde{p}$ regarding the underlying physical property. This is because the partial sums (5) indicate that more probability is concentrated in a lesser number of pixels. This idea that $\tilde{p}$ is more random that p is also clearly expressed by the randomization procedure in Eq. (6).

This interpretation can be further illustrated if we consider the two extremes situations. If there were no uncertainty, all the distribution should be concentrated in a single pixel, $p_1^\downarrow = 1$, $p_{j\neq 1}^\downarrow = 0$, and $S_k = 1$ for all k. This clearly majorizes any other distribution. On the other hand, the uniform distribution $p_j^\downarrow = 1/N$ is majorized by any other distribution [14].

This intuition is further confirmed by the deep relation between majorization and other measures of uncertainty. Let us present two clear examples: confidence intervals $K(\alpha)$ and entropies $R_q(p)$.

Confidence intervals $K(\alpha)$ are defined as the minimum number of pixels K such that the partial sum up to $p_K^\downarrow$ comprises a given fraction α of the probability [15], that is,

$$\sum_{j=1}^{K} p_j^\downarrow \geqslant \alpha \longleftrightarrow K \geqslant K(\alpha). \tag{7}$$

When two distributions are comparable, $\tilde{p} \prec p$ is equivalent to saying that all confidence intervals of $\tilde{p}$ are larger than or equal to those of p (see Fig. 2):

$$\tilde{p} \prec p \longleftrightarrow \tilde{K}(\alpha) \geqslant K(\alpha) \ \forall\alpha. \tag{8}$$

Otherwise, if the distributions are incomparable $p \bowtie \tilde{p}$ we will have $\tilde{K}(\alpha) > K(\alpha)$ and $\tilde{K}(\beta) < K(\beta)$ for different α and β.

Regarding the relation between entropylike measures and majorization we may consider, for example, the Rényi entropies [16]

$$R_q(p) = \frac{1}{1-q} \ln\left(\sum_{j=1}^{N} p_j^q\right), \tag{9}$$

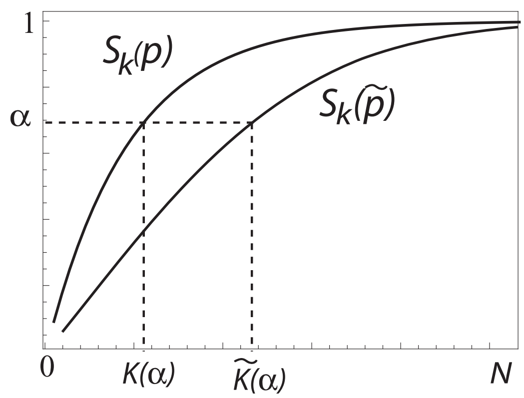

FIG. 2. Relation between partial ordered sums S_k and confidence intervals when the majorization $\tilde{p} \prec p$ holds.

where $q > 0$ is an index labeling different entropies, so we have that if $\tilde{p} \prec p$ then $R_q(\tilde{p}) > R_q(p)$ for all q. The limiting case $q \to 1$ is the Shannon entropy $R_1 = -\sum_{j=1}^{N} p_j \ln p_j$ while $q = 2$ is essentially the degree of polarization introduced in Ref. [6]. If the distributions are incomparable $p \bowtie \tilde{p}$ different entropies may provide contradictory conclusions: $R_q(\tilde{p}) > R_q(p)$ while $R_r(\tilde{p}) < R_r(p)$ for some $r \neq q$.

We think this reveals the powerfulness of majorization as a kind of metameasure. When majorization holds there is unanimity of confidence intervals and entropies regarding which distribution is more ordered and has less uncertainty. When there is no majorization the unanimity is lost.

C. Distributions for relevant field states

Let us recall the classes of classical-like and nonclassical field states the polarization distributions of which will be compared. We will focus mainly on field states defined within the subspaces $\mathcal{H}_n$ of fixed total photon number n. These subspaces have dimension $n+1$ being spanned by the product of number states $|m,n-m\rangle, m = 0,\ldots,n$. We consider pure states to focus exclusively on uncertainty with quantum origin.

1. SU(2) coherent states

These are considered as the most classical polarization states [11]. According to Lieb's conjecture they should majorize any other one within $\mathcal{H}_n$ [10,11]. After Eq. (6) this is particularly clear for classical-like states of the form $\rho = \int d\Omega P(\Omega)|n,\Omega\rangle\langle n,\Omega|$, with a *bona fide* classical probability distribution $P(\Omega)$. This is because all the SU(2) coherent states are connected by an SU(2) transformation, so the corresponding discretized $Q(\Omega)$ are just connected by pixel permutations. Maybe, the surprising result is that this extends to nonclassical light with highly singular $P(\Omega)$ distributions.

Using the SU(2) symmetry we will consider the Q function for the SU(2) coherent state C which is just the product of a number state with n photons and the vacuum state:

$$|n,C\rangle = |n,0\rangle. \tag{10}$$

The corresponding Q function is concentrated at the north pole of the Poincaré sphere. The other SU(2) coherent states $|n,\Omega\rangle$ are just SU(2) orbits of this state.

2. Phase states

These are complementary to the number states [17]:

$$|n,\phi\rangle = \frac{1}{\sqrt{n+1}} \sum_{m=0}^{n} e^{-im\phi}|m,n-m\rangle. \tag{11}$$

Using the SU(2) symmetry we will consider the Q function for the phase state P with $\phi = 0$, that is, $|n,P\rangle = |n,\phi = 0\rangle$.

3. Squeezed states

These are quite distinguished states regarding quantum applications, including metrology as a relevant example. There are no simple criteria translating the simple quadrature squeezing into SU(2) squeezing [18]. For our purposes we can focus on the most squeezed states S regarding metrological applications, exemplified by the twin-number states

$$|n,S\rangle = |n/2,n/2\rangle \tag{12}$$

for even n [19], and the closets analog for n odd [20]:

$$|n,S\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{n+1}{2},\frac{n-1}{2}\right\rangle + \left|\frac{n-1}{2},\frac{n+1}{2}\right\rangle\right). \tag{13}$$

4. NOON states

Further states with interesting practical applications relying on their strong quantum properties are the NOON or Schrödinger's cat states [21], which we shall refer to as N:

$$|n,N\rangle = \frac{1}{\sqrt{2}}(|n,0\rangle + |0,n\rangle). \tag{14}$$

5. Most nonclassical states via Hilbert-Schmidt distance

These are the most nonclassical states according to the Hilbert-Schmidt distance to the convex set of classical-like states defined as the incoherent mixture of SU(2) coherent states [22]. They have no simple general expression and we will consider just the examples with lower number of photons, say

$$|n = 4,H\rangle = \frac{1}{\sqrt{3}}(|0,4\rangle + \sqrt{2}|3,1\rangle), \tag{15}$$

and

$$|n = 5,H\rangle = \frac{1}{\sqrt{2}}(|1,4\rangle + |4,1\rangle), \tag{16}$$

while for $n = 2,3$ they coincide with the NOON states. It is worth noting that these states coincide with the so-called anticoherent states, defined as those with mean value and variance of Stokes-operators vector invariant under SU(2) transformations, and some other approaches [13].

III. RESULTS

In this section we present the results obtained for the lowest photon numbers, that nevertheless clearly illustrate the situation regarding the mutual relationship between classical-like and nonclassical states.

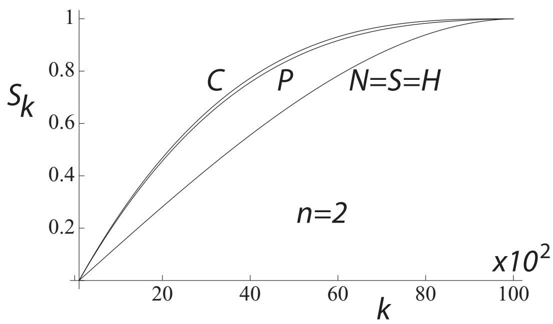

FIG. 3. Ordered partial sums S_k as functions of k for coherent C, squeezed S, NOON N, most quantum H, and phase states P for two-photon states $n = 2$.

A. One-photon states $n = 1$

The case of a single photon is trivial since all pure states are SU(2) coherent states, so all pure states have the same polarization distribution, modulus SU(2) transformations.

B. Two-photon states $n = 2$

In this case after SU(2) symmetry all the above classes of states reduce to the comparison of the SU(2) coherent state $|2,C\rangle = |2,0\rangle$, the phase state $|2,P\rangle$, and the product of one-photon states $|1,1\rangle$ that is simultaneously NOON, squeezed, and the most nonclassical state $|2,N\rangle = |2,S\rangle = |2,H\rangle = |1,1\rangle$. Their ordered partial sums S_k are plotted in Fig. 3 as functions of k, where it can be appreciated that the following sequence holds:

$$N = S = H \prec P \prec C, \tag{17}$$

so that the most polarized is the most classical and the most unpolarized is the most nonclassical. Note also that coherent and phase states are so close that they can be hardly distinguished.

C. Three-photon states $n = 3$

In this case the identity between nonclassical states holds only between the NOON and the most nonclassical. Their ordered partial sums S_k are plotted in Fig. 4 showing the

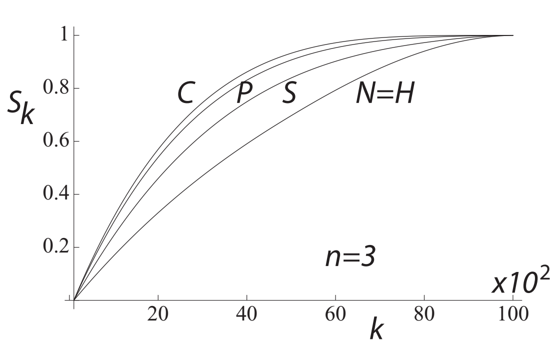

FIG. 4. Ordered partial sums S_k as functions of k for coherent C, squeezed S,NOON, and most quantum $N = H$, and phase states P for three-photon states $n = 3$.

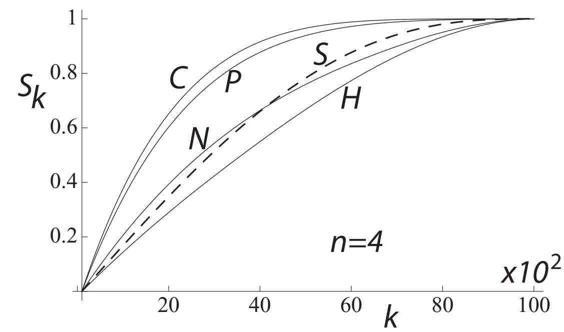

FIG. 5. Ordered partial sums S_k as functions of k for coherent C, squeezed S, NOON N, most quantum H, and phase states P for four-photon states $n = 4$. For clarity the squeezed case S is plotted with a dashed line.

following chain of majorizations:

$$N = H \prec S \prec P \prec C. \tag{18}$$

D. Four-photon states $n = 4$

In this case all the above classes of states are represented by different vectors. Their ordered partial sums S_k are plotted in Fig. 5 showing the following ordering:

$$H \prec S \bowtie N \prec P \prec C. \tag{19}$$

We get the first example of incomparability, that holds between the NOON N and squeezed S states. We have checked that exactly the same situation is repeated for six-photon states $n = 6$.

E. Five-photon states $n = 5$

For the cases we have examined with odd n there is no incomparability between squeezed and NOON states. For $n = 5$ we have the chain of majorizations

$$H \prec N \prec S \prec P \prec C, \tag{20}$$

as illustrated in Fig. 6.

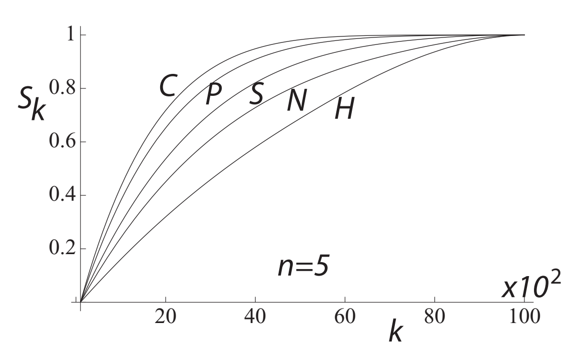

FIG. 6. Ordered partial sums S_k as functions of k for coherent C, squeezed S, NOON N, most quantum H, and phase states P for five-photon states $n = 5$.

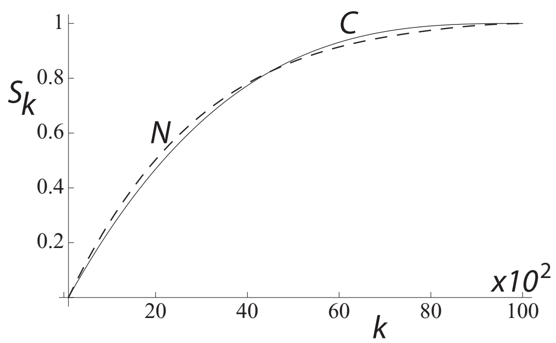

FIG. 7. Ordered partial sums S_k as functions of n for the coherent state with $n = 2$ photons C and a NOON state N with $n = 6$ photons showing that they are incomparable although very similar. For clarity the NOON case is plotted with a dashed line.

F. Inter-photon-number

For states of the same class we have observed the natural behavior that states with larger photon numbers majorize states with lower numbers. In this regard we consider the squeezed states with even and odd n as different classes. Naturally, the situation is richer when comparing states of different classes and different photon numbers, so that incomparability may appear. A simple example is provided in Fig. 7 showing incomparability between a coherent state with $n = 2$ and a NOON state with $n = 6$.

G. Nondefinite photon number

In all the above examples we have considered states with definite total photon number. These examples were addressed in the spirit that, in principle, intensity and polarization are independent degrees of freedom. So for simplicity we considered fixed total number. Nevertheless, such kind of states are difficult to generate in laboratories, so it would be also interesting to address the case of states that can be generated in practice without definite total number. This is the case of Glauber coherent states and thermal states, as the most classical examples, and two-mode squeezed vacuum, as a clear example of nonclassical light. For definiteness all states will be considered with the same mean total photon number $\bar{n}$.

Regarding Glauber coherent states, using SU(2) symmetry we may consider without loss of generality the product of a coherent state in the first mode and vacuum in the second mode so that

$$|C\rangle = e^{-\bar{n}/2} \sum_{n=0}^{\infty} \frac{\bar{n}^{n/2}}{\sqrt{n!}} |n,0\rangle, \tag{21}$$

with Q function

$$Q_C(\Omega) = \frac{1}{4\pi} e^{-\bar{n}\sin^2\frac{\theta}{2}} \left(1 + \bar{n}\cos^2\frac{\theta}{2}\right). \tag{22}$$

For thermal states we consider the most simple example where the second mode is also in the vacuum state:

$$\rho_T = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n,0\rangle\langle n,0|, \tag{23}$$

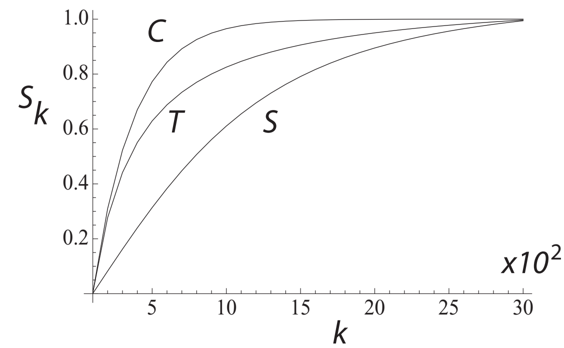

FIG. 8. Ordered partial sums S_k as functions of k for coherent C, thermal T, and two-mode squeezed vacuum S with the same total mean number $\bar{n} = 10$.

with Q function

$$Q_T(\Omega) = \frac{1+\bar{n}}{4\pi} \frac{1}{\left(1 + \bar{n}\sin^2\frac{\theta}{2}\right)^2}. \tag{24}$$

Finally, for the squeezed vacuum state

$$|S\rangle = \frac{1}{\sqrt{1+\bar{n}/2}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}/2}{1+\bar{n}/2}\right)^{n/2} |n,n\rangle, \tag{25}$$

we get the Q function

$$Q_S(\Omega) = \frac{\sqrt{2+\bar{n}}}{2\pi} \frac{1}{(2+\bar{n}\cos^2\theta)^{3/2}}. \tag{26}$$

With these explicit expressions it is simple to compute the ordered partial sums S_k as they are plotted in Fig. 8. This shows that the conclusions obtained for definite total number hold also in these most realistic cases, that is,

$$S \prec T \prec C. \tag{27}$$

IV. CONCLUSIONS

We have developed the application of majorization to quantum polarization as a metameasure of polarization fluctuations and degree of polarization. For fixed total number we have confirmed that the SU(2) coherent states are the most polarized majoring any other state. On the other hand the most nonclassical states according to the Hilbert-Schmidt distance are the most unpolarized among the pure states. We have shown that for odd dimension there is incompatibility between squeezed and NOON states. In general for states of the same class we have observed that states with larger photon numbers majorize states with lower numbers. Naturally, the situation is richer when comparing states of different classes and different photon numbers, where further cases of incomparability can be found.

ACKNOWLEDGMENTS

G.D. gratefully acknowledges a Collaboration Grant from the spanish Ministerio de Educación, Cultura y Deporte. A.L. acknowledges support from Project No. FIS2012-35583 of the Spanish Ministerio de Economía y Competitividad and from the Comunidad Autónoma de Madrid research consortium QUITEMAD+ S2013/ICE-2801.

[1] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University, Cambridge, England, 1995).

[2] J. J. Gil and R. Ossikovski, *Polarized Light and the Mueller Matrix Approach* (CRC, Boca Raton, 2016).

[3] Ch. Brosseau, *Fundamentals of Polarized Light: A Statistical Optics Approach* (Wiley, New York, 1998).

[4] A. Luis, Polarization in quantum optics, Prog. Opt. **61**, 283 (2016).

[5] V. P. Karasev and A. V. Masalov, Unpolarized light states in quantum optics, Opt. Spectrosc. **74**, 551 (1993); D. M. Klyshko, Polarization of light: Fourth-order effects and polarization-squeezed states, JETP **84**, 1065 (1997); Multiphoton interference and polarization effects, Phys. Lett. A **163**, 349 (1992); M. Y. Agre, Partially polarized light and multiphoton processes, Opt. Spectrosc. **89**, 445 (2000); P. Usachev, J. Söderholm, G. Björk, and A. Trifonov, Experimental verification of differences between classical and quantum polarization properties, Opt. Commun. **193**, 161 (2001); A. V. Burlakov and M. V. Chekhova, Polarization optics of biphotons, JETP Lett. **75**, 432 (2002).

[6] A. Luis, Degree of polarization in quantum optics, Phys. Rev. A **66**, 013806 (2002).

[7] A. P. Alodjants and S. M. Arakelian, Quantum phase measurements and nonclassical polarization states of light, J. Mod. Opt. **46**, 475 (1999); A. Picozzi, Entropy and degree of polarization for nonlinear optical waves, Opt. Lett. **29**, 1653 (2004); A. B. Klimov, L. L. Sánchez-Soto, E. C. Yustas, J. Söderholm, and G. Björk, Distance-based degrees of polarization for a quantum field, Phys. Rev. A **72**, 033813 (2005); A. Sehat, J. Söderholm, G. Björk, P. Espinoza, A. B. Klimov, and L. L. Sánchez-Soto, Quantum polarization properties of two-mode energy eigenstates, *ibid.* **71**, 033818 (2005); L. L. Sánchez-Soto, E. C. Yustas, G. Björk, and A. B. Klimov, Maximally polarized states for quantum light fields, *ibid.* **76**, 043820 (2007); J. Liñares, M. C. Nistal, D. Barral, and V. Moreno, Optical field-strength polarization of two-mode single-photon states, Eur. J. Phys. **31**, 991 (2010); A. B. Klimov, G. Björk, J. Söderholm, L. S. Madsen, M. Lassen, U. L. Andersen, J. Heersink, R. Dong, Ch. Marquardt, G. Leuchs, and L. L. Sánchez-Soto, Assessing the Polarization of a Quantum Field from Stokes Fluctuations, Phys. Rev. Lett. **105**, 153602 (2010); G. Björk, J. Söderholm, L. L. Sánchez-Soto, A. B. Klimov, I. Ghiu, P. Marian, and T. A. Marian, Quantum degrees of polarization, Opt. Commun. **283**, 4440 (2010); G. K. Gupta, A. Kumar, and R. S. Singh, Generation of hidden optical-polarization: Squeezing and non-classicality, *ibid.* **284**, 4951 (2011); R. S. Singh and H. Prakash, Degree of polarization in quantum optics through the second generalization of intensity, Phys. Rev. A **87**, 025802 (2013); P. de la Hoz, A. B. Klimov, G. Björk, Y.-H. Kim, C. Müller, Ch. Marquardt, G. Leuchs, and L. L. Sánchez-Soto, Multipolar hierarchy of efficient quantum polarization measures, *ibid.* **88**, 063803 (2013); L. L. Sánchez-Soto, A. B. Klimov, P. de la Hoz, and G. Leuchs, Quantum versus classical polarization states: When multipoles count, J. Phys. B **46**, 104011 (2013).

[8] J. Aczél and Z. Daróczy, *On Measures of Information and Their Characterizations* (Academic, New York, 1975); E. Ruch, R. Schranner, and Th. H. Seligman, The mixing distance, J. Chem. Phys. **69**, 386 (1978); A. W. Marshall and O. Olkin, *Inequalities: Theory of Majorization and Its Applications* (Academic, New York, 1979); F. Herbut and I. D. Ivanović, The principle of strictly increasing mixing character in incompatible general measurement, J. Phys. A **15**, 1775 (1982); N. Canosa, R. Rossignoli, and M. Portesi, Majorization properties of generalized thermal distributions, Physica A **368**, 435 (2006); H. Partovi, Majorization formulation of uncertainty in quantum mechanics, Phys. Rev. A **84**, 052117 (2011); R. García-Patrón, C. Navarrete-Benlloch, S. Lloyd, J. H. Shapiro, and N. J. Cerf, Majorization Theory Approach to the Gaussian Channel Minimum Entropy Conjecture, Phys. Rev. Lett. **108**, 110505 (2012); Z. Puchała, L. Rudnicki, and K. Życzkowski, Majorization entropic uncertainty relations, J. Phys. A **46**, 272002 (2013); S. Friedland, V. Gheorghiu, and G. Gour, Universal Uncertainty Relations, Phys. Rev. Lett. **111**, 230401 (2013); Universal uncertainty relations, **112**, 119905 (2014); D. G. Medranda and A. Luis, Sub-Poissonian and anti-bunching criteria via majorization of statistics, J. Phys. A **48**, 255302 (2015).

[9] G. S. Agarwal, Relation between atomic coherent-state representation, state multipoles, and generalized phase-space distributions, Phys. Rev. A **24**, 2889 (1981); M. P. Miranda, F. J. Aoiz, V. Sáez-Rábanos, and M. Brouard, Spatial distributions of angular momenta in quantum and quasiclassical stereodynamics, J. Chem. Phys. **121**, 9830 (2004); Ch. Marquardt, J. Heersink, R. Dong, M. V. Chekhova, A. B. Klimov, L. L. Sánchez-Soto, U. L. Andersen, and G. Leuchs, Quantum Reconstruction of an Intense Polarization Squeezed Optical State, Phys. Rev. Lett. **99**, 220401 (2007).

[10] E. H. Lieb and J. Ph. Solovej, Proof of an entropy conjecture for Bloch coherent spin states and its generalizations, Acta Math. **212**, 379 (2014).

[11] O. Giraud, P. Braun, and D. Braun, Classicality of spin states, Phys. Rev. A **78**, 042112 (2008).

[12] P. W. Atkins and J. C. Dobson, Angular momentum coherent states, Proc. R. Soc. A **321**, 321 (1971); F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Atomic coherent states in quantum optics, Phys. Rev. A **6**, 2211 (1972).

[13] J. Zimba, "Anticoherent" Spin states via the Majorana representation, Electron. J. Theor. Phys. **3**, 143 (2006); G. Björk, A. B. Klimov, P. de la Hoz, M. Grassl, G. Leuchs, and L. L. Sánchez-Soto, Extremal quantum states and their Majorana constellations, Phys. Rev. A **92**, 031801(R) (2015); G. Björk, M. Grassl, P. de la Hoz, G. Leuchs, and L. L. Sánchez-Soto, Stars of the quantum universe: Extremal constellations on the Poincaré sphere, Phys. Scr. **90**, 108008 (2015).

[14] I. Bengtsson and K. K. Życzkowski, *Geometry of Quantum States* (Cambridge University, Cambridge, England, 2006).

[15] A. Luis, I. Gonzalo, and M. A. Porras, Majorization applied to diffraction, Opt. Lett. **41**, 1624 (2016); A. Luis, G. M. Bosyk, and M. Portesi, Entropic measures of joint uncertainty: Effects of lack of majorization, Physica A **444**, 905 (2016).

[16] A. Rényi, On the measures of entropy and information, in *Proceedings of the Fourth Berkeley Symposium on Mathematics and Statistical Probability* (University of California, Berkeley, 1961), Vol. 1, p. 547; Ch. Beck, Generalised information and entropy measures in physics, Contemp. Phys. **50**, 495 (2009).

[17] A. Luis and L. L. Sánchez-Soto, Phase-difference operator, Phys. Rev. A **48**, 4702 (1993); Quantum phase difference, phase measurements and Stokes operators, in *Progress in Optics*, edited by E. Wolf (Elsevier, Amsterdam, 2000), Vol. 41, p. 421; T. S. Santhanam, Canonical commutation relation for

operators with bounded spectrum, Phys. Lett. A **56**, 345 (1976); I. Goldhirsch, Phase operator and phase fluctuations of spins, J. Phys. A **13**, 3479 (1980); A. Vourdas, SU(2) and SU(1, 1) phase states, Phys. Rev. A **41**, 1653 (1990); D. Ellinas, Phase operators via group contraction, J. Math. Phys. **32**, 135 (1991).

[18] C. Brif, and A. Mann, Nonclassical interferometry with intelligent light, Phys. Rev. A **54**, 4505 (1996); A. Luis and N. Korolkova, Polarization squeezing and nonclassical properties of light, *ibid.* **74**, 043817 (2006).

[19] M. J. Holland and K. Burnett, Interferometric Detection of Optical Phase Shifts at the Heisenberg Limit, Phys. Rev. Lett. **71**, 1355 (1993); T. Kim, J. Shin, Y. Ha, H. Kim, G. Park, T. G. Noh, and C. K. Hong, The phase-sensitivity of a Mach-Zehnder interferometer for the Fock state inputs, Opt. Commun. **156**, 37 (1998); H. Uys and P. Meystre, Quantum states for Heisenberg-limited interferometry, Phys. Rev. A **76**, 013804 (2007); F. W. Sun, B. H. Liu, Y. X. Gong, Y. F. Huang, Z. Y. Ou, and G. C. Guo, Experimental demonstration of phase measurement precision beating standard quantum limit by projection measurement, Europhys. Lett. **82**, 24001 (2008).

[20] J. P. Dowling, Correlated input-port, matter-wave interferometer: Quantum-noise limits to the atom-laser gyroscope, Phys. Rev. A **57**, 4736 (1998).

[21] N. D. Mermin, Extreme Quantum Entanglement in a Superposition of Macroscopically Distinct States, Phys. Rev. Lett. **65**, 1838 (1990); J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Optimal frequency measurements with maximally correlated states, Phys. Rev. A **54**, R4649 (1996); Ph. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, and A. Zeilinger, De Broglie wavelength of a non-local four-photon state, Nature (London) **429**, 158 (2004); M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, Super-resolving phase measurements with a multiphoton entangled state, *ibid.* **429**, 161 (2004).

[22] O. Giraud, P. Braun, and D. Braun, Quantifying quantumness and the quest for queens of quantum, New J. Phys. **12**, 063005 (2010).