Hydrodynamics of vortices in Bose-Einstein condensates: A defect-gauge field approach

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This work rectifies the hydrodynamic equations commonly used to describe the superfluid velocity field in such a way that vortex dynamics are also taken into account. In the field of quantum turbulence, it is of fundamental importance to know the correct form of the equations which play similar roles to the Navier-Stokes equation in classical turbulence. Here, such equations are obtained by carefully taking into account the frequently overlooked multivalued nature of the $U(1)$ phase field. Such an approach provides exact analytical explanations to some numerically observed features involving the dynamics of quantum vortices in Bose-Einstein condensates, such as the universal $t^{1/2}$ behavior of reconnecting vortex lines. It also expands these results beyond the Gross-Pitaevskii theory so that some features can be generalized to other systems such as superfluid ^{4}He, dipolar condensates, and mixtures of different superfluid systems.

I. INTRODUCTION

Superfluidity is a macroscopic quantum phenomenon that has attracted attention since its discovery in liquid ^{4}He by Kapitza [1] as well as Allen and Misener [2]. London [3] proposed that the superfluidity appearing in ^{4}He was closely related to the existence of a Bose-Einstein condensate (BEC) which can be described by a complex wave function $\psi = \sqrt{\rho}e^{iS}$, where ρ is the condensate density and S is the phase that determines the superfluid velocity, which is usually assumed to be $\mathbf{v} \stackrel{?}{=} (\hbar/m)\nabla S$, where the notation $\stackrel{?}{=}$ represents the fact that this equation is not correct in general and therefore must be modified, as we will see below. The multivalued nature of S implies the quantization of the superfluid vorticity [4,5]. Since S is defined modulo 2π, the velocity circulation $\oint \mathbf{v} \cdot d\mathbf{r}$ must be an integer multiple of $2\pi\hbar/m = h/m$.

Associations between quantum-vortex degrees of freedom and gauge fields have been previously discussed in great detail by Kleinert [6–10], where the concept of defect-gauge fields was introduced. An alternative approach is discussed, for example, by Kozhevnikov [11,12], where vortex gauge fields are introduced as extra terms in the equations of motion for the complex scalar field. An interesting possibility is based on the exploration of approximate boson-vortex dualities as in [13,14], where the continuity equation is used as a basis for the introduction of gauge fields. In Ref. [15], a gauge field which is dual to the velocity field is considered, thus allowing the study of the motion of a two-dimensional (2D) point vortex in inhomogeneous backgrounds. In Ref. [16], Popov's functional integral formalism [17], where a gauge field is introduced in order to enforce the constraint between velocity and vorticity fields, is also applied to the study of 2D vortex motions.

The analogy between quantum and classical hydrodynamics is usually made by using the Gross-Pitaevskii (GP) equation

$$i\partial_t \psi = -\frac{1}{2}\nabla^2\psi + V(\mathbf{r})\psi + g|\psi|^2\psi, \qquad (1)$$

where direct substitution of $\psi = \sqrt{\rho}e^{iS}$ seems to lead to the hydrodynamic equations [4,5]:

$$\partial_t \rho = -\nabla \cdot (\rho\mathbf{v}), \qquad (2)$$

$$\partial_t \mathbf{v} \stackrel{?}{=} \nabla\left[\frac{1}{2}\left(\frac{1}{2\rho}\nabla^2\rho - \frac{1}{4\rho^2}|\nabla\rho|^2 - \frac{v^2}{2}\right) - V - g\rho\right], \qquad (3)$$

where for simplicity the system of units is chosen so that $\hbar = m = 1$. In Eq. (3), a usually unnoticed complication arises: S is a multivalued field and therefore the chain rule of differentiation cannot be applied to e^{iS} [10]. Indeed, by taking the curl in Eq. (3), one would be left with the false statement that vorticity has no dynamics, i.e., $\partial_t\boldsymbol{\omega} = \partial_t(\nabla \times \mathbf{v}) = 0$. Thus, Eq. (3) turns out to be of little use for dealing with situations where the dynamics of vorticity plays an important role, as in the case of quantum turbulence [18–22]. In the latter, it is common to interpret results through an analogy between Eq. (3) and the Navier-Stokes equation, thus establishing a close relationship between quantum and classical turbulence [22,23]. In practice, due to the weaknesses of Eq. (3), studies are normally based on direct numerical simulations of the GP equation, as in [24–26], or the Biot-Savart model, as in [27–29].

The present work aims to provide a general framework where exact hydrodynamic equations can be obtained for models of superfluidity described by complex fields which have equations of motion of the form

$$\frac{\partial \psi(\mathbf{r},t)}{\partial t} = \mathcal{F}\{\psi^*,\psi\}(\mathbf{r},t), \qquad (4)$$

where $\mathcal{F}\{\psi^*,\psi\}(\mathbf{r},t)$ can be any arbitrary functional of ψ which is local in time (i.e., depends only on ψ at the instant t) and has explicit dependency in $\mathbf{r}$ and t. The Gross-Pitaevskii theory based on Eq. (1) is therefore only one example. This is made possible through a careful analysis of the multivalued nature of the phase field S, following the lines presented in Ref. [10]. The hydrodynamic equation obtained this way makes it possible to derivate the superfluid behavior in its entirety, which includes its vorticity dynamics.

A. Two-dimensional case

For illustrative purposes, let us consider a scalar complex field ψ in two spatial dimensions with its usual definition for

the velocity field [4,5]

$$\mathbf{v} = \frac{\psi^* \nabla \psi - \psi \nabla \psi^*}{2i\, \psi^* \psi}. \tag{5}$$

A common approach at this point would be to consider the Madelung's representation $\psi = \sqrt{\rho} e^{iS}$ and use the chain rule of derivatives in order to obtain the relation between the phase S and velocity field $\mathbf{v}$, as follows:

$$\mathbf{v} = \frac{\psi^* \nabla \psi - \psi \nabla \psi^*}{2i\, \psi^* \psi} = \frac{e^{-iS} \nabla e^{iS} - e^{iS} \nabla e^{-iS}}{2i} \overset{?}{=} \nabla S. \tag{6}$$

However, as pointed out by Kleinert in [10], the chain rule should not be indiscriminately used in the case of multivalued fields. A typical example is that of a 2D isotropic vortex which, in polar coordinates, can be expressed as $\psi(\mathbf{r}) = f(r) e^{i\varphi}$, with $0 \leqslant \varphi < 2\pi$ as in Fig. 1(a). In that case, the field $S = \varphi$ is discontinuous over the cut line [see Fig. 1(a)], thus giving

$$\nabla S = \frac{\hat{\varphi}}{r} - 2\pi \Theta(x) \delta(y) \hat{y}, \tag{7}$$

where $\Theta(x)$ and $\delta(y)$ are the Heaviside and Dirac functions, respectively, while $\hat{\varphi}$ and $\hat{y}$ are the unit vectors corresponding to φ and y. Observe that, in this way, the property $\nabla \times \nabla S = 0$ is preserved as expected. However, a direct calculation of the velocity field gives

$$\mathbf{v} = \frac{\psi^* \nabla \psi - \psi \nabla \psi^*}{2i\, \psi^* \psi} = \frac{\hat{\varphi}}{r}, \tag{8}$$

which means that

$$\nabla S = \mathbf{v} - \mathbf{A}, \tag{9}$$

$$\mathbf{A} = 2\pi \Theta(x) \delta(y) \hat{y}. \tag{10}$$

Therefore, formula (6) for the velocity field must be correctly defined according to

$$\mathbf{v} = \nabla S + \mathbf{A}, \tag{11}$$

where the vector field $\mathbf{A}$ compensates for the discontinuity in S. In addition, all the vorticity of $\mathbf{v}$ is concentrated in the field $\mathbf{A}$, i.e.,

$$\nabla \times \mathbf{v} = \nabla \times \mathbf{A} = 2\pi \delta(\mathbf{r}) \hat{z}. \tag{12}$$

In order to make these results consistent with Eq. (6), the common chain rule of differentiation must be modified [10] according to

$$\nabla e^{iS} = i\mathbf{v} e^{iS} = i(\nabla S + \mathbf{A}) e^{iS}. \tag{13}$$

Due to the $U(1)$ symmetry of ψ, the definition of S can always be modified by adding to it a scalar field Q which assumes values equal to $2\pi l$, with $l \in \mathbb{Z}$, where l can be different for different regions of the plane [see Fig. 1(c)]. Observe also in Figs. 1(b) and 1(c) that cut lines can be moved due to the extra Q field. This way, the field $\mathbf{A}$ also has to change in order not to modify $\mathbf{v} = \nabla S + \mathbf{A}$, therefore $\mathbf{v}$ is invariant under the following gauge transformations

$$S \rightarrow S + Q, \tag{14}$$

$$\mathbf{A} \rightarrow \mathbf{A} - \nabla Q. \tag{15}$$

B. Arbitrary number of dimensions

From now on, the tensor notation with Einstein summation rule will be used, where Greek indices correspond to space-time coordinates and Latin indices correspond to pure spatial coordinates.

The previous analysis can be extended to arbitrary space-time dimensions, where we have the four-velocity field v_μ given by

$$v_\mu = \frac{\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^*}{2i\, \psi^* \psi} = \frac{e^{-iS} \partial_\mu e^{iS} - e^{iS} \partial_\mu e^{-iS}}{2i}$$
$$= \partial_\mu S + A_\mu, \tag{16}$$

where $\partial_\mu e^{iS} = i(\partial_\mu S + A_\mu) e^{iS}$ and the gauge field A_μ must be chosen such that it accounts for any artificial discontinuities from $\partial_\mu S$. This leads to the gauge transformations

$$S \rightarrow S + Q, \tag{17}$$

$$A_\mu \rightarrow A_\mu - \partial_\mu Q. \tag{18}$$

C. Topological conservation laws

In analogy to the electromagnetic theory, we can use the gauge field A_μ to define the force field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$= \partial_\mu v_\nu - \partial_\nu v_\mu, \tag{19}$$

which is invariant under the gauge transformations (17)–(18). Such a definition leads to the conservation laws

$$\text{In } (2+1) \text{ dimensions: } \partial_\mu \left(\tfrac{1}{2} \varepsilon^{\mu\alpha\beta} F_{\alpha\beta} \right) = 0, \tag{20}$$

$$\text{In } (3+1) \text{ dimensions: } \partial_\mu \left(\tfrac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \right) = 0, \tag{21}$$

where ε is the Levi-Civita symbol.

The topological charge density in $(2+1)$ dimensions is the vorticity

$$\Omega^0 = \tfrac{1}{2} \varepsilon^{0ij} F_{ij} = \varepsilon^{ij} \partial_i v_j = \omega, \tag{22}$$

while the vortex-current vector is

$$\Omega^i = \tfrac{1}{2} \varepsilon^{i0j} F_{0j} + \tfrac{1}{2} \varepsilon^{ij0} F_{j0} = -\varepsilon^{ij} F_{0j} = -\varepsilon^{ij} E_j. \tag{23}$$

In $(3+1)$ dimensions the topological charge density is the vorticity vector

$$\Omega^{0i} = \tfrac{1}{2} \varepsilon^{0ijk} F_{jk} = \varepsilon^{ijk} \partial_j v_k = \omega^i, \tag{24}$$

while the vortex-current tensor is

$$\Omega^{ij} = \tfrac{1}{2} \varepsilon^{ij0k} F_{0k} + \tfrac{1}{2} \varepsilon^{ijk0} F_{k0} = -\varepsilon^{ijk} F_{0k} = -\varepsilon^{ijk} E_k. \tag{25}$$

Here, the field E_i is the timelike component of the antisymmetric force-field tensor which in the electromagnetic theory corresponds to the electric field

$$E_i \equiv F_{0i} \equiv \partial_0 A_i - \partial_i A_0. \tag{26}$$

D. Hydrodynamic equations

Now, in order to derive the correct hydrodynamic equations, let us consider the derivative

$$\partial_i v_0 = \partial_i (\partial_0 S + A_0) = \partial_0 (\partial_i S + A_i) + \partial_i A_0 - \partial_0 A_i, \tag{27}$$
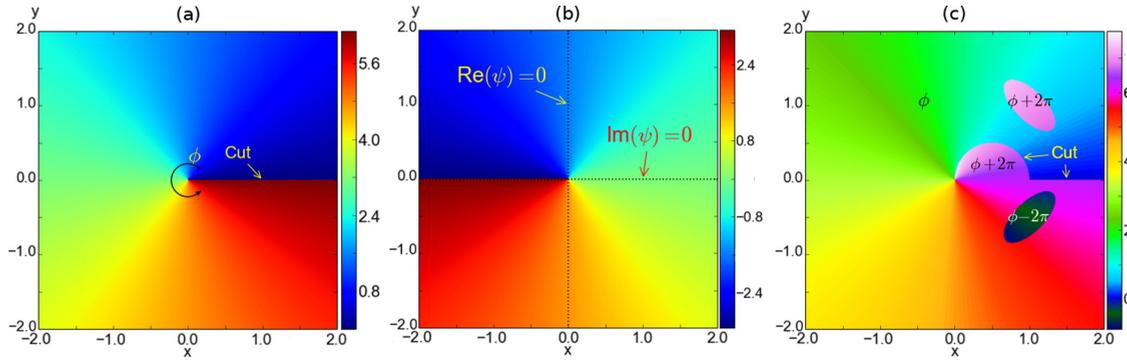
FIG. 1. Different representations of the same multivalued phase corresponding to the velocity field in Eq. (8). In (a), S is defined so that $0 \leqslant S < 2\pi$. In (b), S is defined so that $-\pi \leqslant S < \pi$, and (c) corresponds to a gauge transformation which adds $2\pi l$, with different $l \in \mathbb{Z}$ for different regions of the plane.

which can be rearranged in order to give the time derivative of the spacelike velocity field

$$\partial_t v_i \equiv \partial_0 v_i = E_i + \partial_i v_0, \tag{28}$$

where v_0 is given by (16). Observe that $\partial_0 \psi$ in Eq. (4) can be used for the calculation of v_0 in terms of ρ and v_i. In the case of the GP equation (1), we have

$$v_0 = \frac{1}{2}\left[\frac{1}{2\rho}\boldsymbol{\nabla}^2\rho - \frac{1}{4\rho^2}|\boldsymbol{\nabla}\rho|^2 - \frac{v^2}{2}\right] - V - g\rho. \tag{29}$$

This corrects the usual hydrodynamic equation (3) so that E_i takes into account all possible vorticity effects. Thus, the correct hydrodynamic equation following from GP Eq. (1) differs from the classical Euler equation in two aspects: the quantum pressure and the force E_i. The vorticity equations can be obtained by taking the curl in Eq. (28), which ends up reproducing the vorticity conservation laws already stated in Eqs. (20) and (21). Although it is necessary to correct (3), a straightforward calculation shows that the continuity equation (2) remains valid, despite the discontinuities of S.

E. Explicit form of force fields

In order to construct a full hydrodynamic theory, it is also necessary to express the force fields in Eq. (19) in terms of ρ and v_i. According to (19), $F_{\mu\nu}$ can be obtained straightforwardly once an explicit form of the gauge field A_μ is known. In order to do that, let us consider the phase field S as being restricted to $0 \leqslant S < 2\pi$, in analogy to the two-dimensional example presented earlier. The discontinuities appearing in S must be compensated by the gauge field in order to allow for the correct calculation of the velocity field, as defined in Eq. (16). Considering that R and I are the real and imaginary parts of ψ, respectively, such a convention for S implies that its discontinuities appear when $R \geqslant 0$ and $I = 0$. This means that $\partial_\mu S$ will have discontinuities of the form $-2\pi \Theta(R)\partial_\mu \Theta(I)$. Therefore the gauge field must be given by

$$A_\mu = 2\pi \Theta(R)\partial_\mu \Theta(I). \tag{30}$$

This leads directly to the force field

$$\begin{aligned} F_{\mu\nu} &= 2\pi \delta(R)\delta(I)(\partial_\mu R \partial_\nu I - \partial_\nu R \partial_\mu I) \\ &= i\pi \delta(R)\delta(I)(\partial_\mu \psi \partial_\nu \psi^* - \partial_\nu \psi \partial_\mu \psi^*). \end{aligned} \tag{31}$$

Finally, by using the property $\delta(R)\delta(I) = 2\delta(R^2 + I^2)/\pi$, we get the hydrodynamic form of the force field

$$F_{\mu\nu} = 2\delta(\rho)(\partial_\mu \rho v_\nu - \partial_\nu \rho v_\mu). \tag{32}$$

In such a way, the field $E_i = F_{0i}$, necessary in Eq. (28), is

$$E_i = -2\delta(\rho)[v_i \partial_j (\rho v_j) + v_0 \partial_i \rho], \tag{33}$$

where $\partial_0 \rho$ is obtained from the continuity equation Eq. (2) and v_0 is model specific.

F. Vortex motion

As a testing ground to the validity of the theory presented, let us check whether it is indeed capable of predicting the correct motion of point vortices in 2D and vortex lines in three dimensions (3D). In fact, the motion of vortex lines can be analyzed by looking at the motion of point vortices over planes crossed by the vortex line. Hence this discussion can be reduced to the two-dimensional situation.

In this case, the motion of vortices can be described by Eq. (20), while vortex currents can be directly evaluated from (31). As illustrated in Fig. 1, a singly quantized vortex is always located at the crossing between $R = 0$ and $I = 0$ lines in the xy plane. Let us consider, without loss of generality, that such a crossing happens at the origin. At the vicinity of the crossing point, the δ functions in (31) can then be simplified to

$$\delta(R)\delta(I) = \frac{\delta(x)\delta(y)}{|\varepsilon^{ij}\partial_i R \partial_j I|}. \tag{34}$$

From (22), (32), and (34) we get

$$\begin{aligned} \omega = \Omega^0 &= 2\pi \delta(x)\delta(y)\frac{\epsilon^{ij}\partial_i R \partial_j I}{|\varepsilon^{lm}\partial_l R \partial_m I|} \\ &= 2\pi \, \mathrm{sgn}(\epsilon^{ij}\partial_i R \partial_j I)\delta(x)\delta(y), \end{aligned} \tag{35}$$

where $\mathrm{sgn}(\epsilon^{ij}\partial_i R \partial_j I)$ gives the vortex sign. Now combining (23), (32), and (34), we have the vortex current

$$\Omega^i = -2\pi \delta(x)\delta(y)\epsilon^{ij}\frac{\partial_0 R \partial_j I - \partial_j R \partial_0 I}{|\varepsilon^{lm}\partial_l R \partial_m I|} = \omega w^i, \tag{36}$$

where the vortex velocity w^i is

$$w^i = -\varepsilon^{ij}\frac{\partial_0 R \partial_j I - \partial_j R \partial_0 I}{\varepsilon^{lm}\partial_l R \partial_m I}. \tag{37}$$

Observe that w^i is indeed consistent with the equations describing the motion of the crossing point between the lines $R = 0$ and $I = 0$, as in Ref. [30]:

$$\partial_0 R + w^i \partial_i R = 0, \tag{38}$$

$$\partial_0 I + w^i \partial_i I = 0, \tag{39}$$

whose solution for w_i is given by (37).

An elegant approximation for w_i can be obtained for the case of quasi-isotropic vortices with dynamics given by the GP equation (1), i.e., when

$$\psi \approx [(x - x_0) + i(y - y_0)]\phi, \tag{40}$$

with $\phi = Ae^{i\lambda}$, where both A and λ have their values as well as their first derivatives well defined at $(x, y) = (x_0, y_0)$. By directly substituting (40) into (37) and observing that at the vortex location we have $\partial_0 \psi = i\frac{1}{2}\nabla^2 \psi$, we get

$$w^i = \partial^i \lambda + \varepsilon^{ij} \partial_j \ln(A). \tag{41}$$

This gives a correction to the so-called point-vortex model, where $\partial^i \lambda$ is the velocity field over the vortex core excluding the self-generated velocity field, while $\varepsilon^{ij} \partial_j \ln(A)$ gives a contribution perpendicular to the density gradient. The necessity for this correction has already been observed in the numerical studies of Ref. [31].

Our 2D analysis can be directly generalized to 3D vortex lines by considering a plane crossed by the vortex line. In this case, w^i would describe the motion of the crossing point over the considered plane.

G. Reconnection of lines and creation or annihilation of pairs

An interesting situation occurs when the $R = 0$ and $I = 0$ lines touch each other tangentially at a single point as in Fig. 2(b). Actually, Fig. 2 can illustrate either the situation right at the beginning of a vortex-pair-creation process or at the end of a vortex-pair annihilation process. Indeed, the sequence (a)-(b)-(c) in Fig. 2 exemplifies a vortex-pair annihilation process, while the inverse sequence (c)-(b)-(a) describes a pair-creation process. For simplicity, without loss of generality, one can consider that the lines touch at $x_0 = y_0 = t_0 = 0$ and are tangent to the x axis at this point, i.e., $\partial_{x_0} R = \partial_{x_0} I = 0$. At

the vicinity of the touching point the Taylor expansion can be used:

$$R = \frac{\partial R}{\partial y_0} y + \frac{\partial R}{\partial t_0} t + \frac{1}{2} \frac{\partial^2 R}{\partial x_0^2} x^2 + \cdots, \tag{42}$$

$$I = \frac{\partial I}{\partial y_0} y + \frac{\partial I}{\partial t_0} t + \frac{1}{2} \frac{\partial^2 I}{\partial x_0^2} x^2 + \cdots. \tag{43}$$

Close to the touching point, the curves $R = 0$ and $I = 0$ can then be obtained by considering the dominant terms in (42) and (43), according to

$$y_{\text{Re}} \approx -\frac{1}{2} \frac{\partial^2 R/\partial x_0^2}{\partial R/\partial y_0} x^2 - \frac{\partial R/\partial t_0}{\partial R/\partial y_0} t, \tag{44}$$

$$y_{\text{Im}} \approx -\frac{1}{2} \frac{\partial^2 I/\partial x_0^2}{\partial I/\partial y_0} x^2 - \frac{\partial I/\partial t_0}{\partial I/\partial y_0} t. \tag{45}$$

The crossing points as depicted in Fig. 2(a) are solutions of the condition $y_{\text{Re}} = y_{\text{Im}}$, which are given by

$$x^2 \approx 2t\alpha, \tag{46}$$

$$\alpha = \frac{\partial_t R \partial_y I - \partial_t I \partial_y R}{\partial_x^2 I \partial_y R - \partial_x^2 R \partial_y I}\bigg|_{x=y=t=0}. \tag{47}$$

The sign of α indicates whether there are real solutions for x with $t < 0$ or with $t > 0$, thus determining if it is the case of an annihilation ($\alpha < 0$) or creation ($\alpha > 0$) process. Also from (46), we get the power-law behavior for the creation or annihilation process:

$$x \sim \pm t^{1/2}. \tag{48}$$

Observe that these results can also be directly obtained from (37) by considering the expansions (42) and (43) and neglecting the subdominant terms. This calculation would then
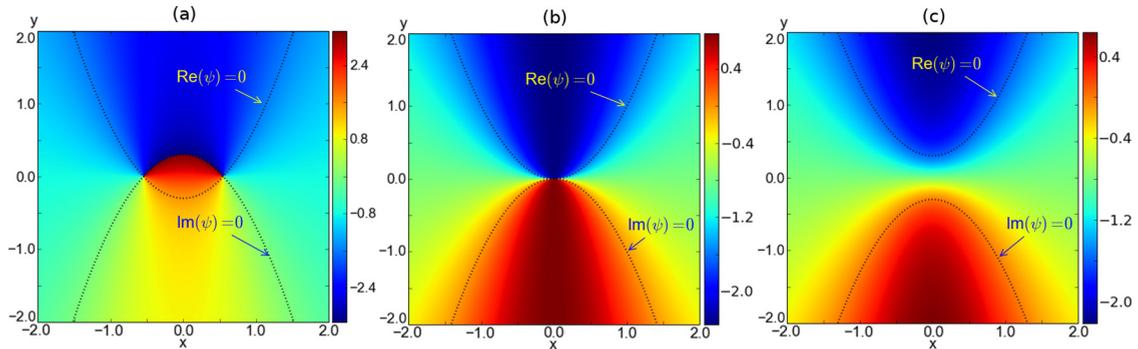

FIG. 2. Annihilation or creation of a vortex pair. The sequence (a)-(b)-(c) corresponds to the annihilation process, while the inverse sequence (c)-(b)-(a) corresponds to the pair-creation process. Such sequences also describe the recombination process of 3D vortex lines, where the depicted plane crosses the recombination point.

lead to

$$w^y \approx -\frac{\partial_t R \partial_x^2 I - \partial_t I \partial_x^2 R}{\partial_x^2 I \partial_y R - \partial_x^2 R \partial_y I}\bigg|_{x=y=t=0}, \tag{49}$$

$$w^x \approx \frac{1}{x}\left(\frac{\partial_t R \partial_y I - \partial_t I \partial_y R}{\partial_x^2 I \partial_y R - \partial_x^2 R \partial_y I}\bigg|_{x=y=t=0}\right) = \frac{\alpha}{x}. \tag{50}$$

Creation and annihilation of vortex pairs also leave their signatures in the hydrodynamic equation (28). A direct evaluation of $F_{\mu\nu}$ at $t = 0$ and around the point $x_0 = y_0 = 0$ can be obtained with the help of Eqs. (42) and (43). It then gives

$$F_{\mu\nu} = 4\pi \delta(x^2)\delta(y)\frac{\partial_\mu R \partial_\nu I - \partial_\nu R \partial_\mu I}{\left|\partial_x^2 R \partial_y I - \partial_x^2 I \partial_y R\right|}. \tag{51}$$

Since $\partial_x R = \partial_x I = 0$ at $x = y = 0$, the vorticity $\omega = \varepsilon^{ij} F_{ij}$ vanishes. However, it does not mean that the vorticity flux vanishes in all directions. Actually, the vorticity flux in the y direction $\Omega^2 = F_{01}$ vanishes, while for the x direction we have

$$\Omega^1 = -F_{02} = -4\pi \delta(x^2)\delta(y)\frac{\partial_t R \partial_y I - \partial_y R \partial_t I}{\left|\partial_x^2 R \partial_y I - \partial_x^2 I \partial_y R\right|}. \tag{52}$$

This reflects the fact that although no vortex actually exists at $t = 0$, a vorticity flux is still necessary to account for the creation and annihilation of vortex pairs occurring in the superfluid. Also, the possibility of having a nonzero $E_i = F_{0i}$, even in the absence of vortices, shows that the hydrodynamic equation (28) is capable of describing the creation and annihilation of vortex pairs.

Again, it should be emphasized that such a two-dimensional analysis can also be directly generalized to the case of recombinations of 3D vortex lines by considering planes crossed by the vortex lines. Indeed, the present analysis demonstrates exactly the $x \sim \pm t^{1/2}$ behavior for the reconnection of vortex lines which was observed experimentally in Ref. [32], numerically in the context of Biot-Savart models in Ref. [33], and analytically in the context of GP equation in Ref. [34]. In

addition, such a $t^{1/2}$ law turns out to be very general in the sense that it is not restricted to any particular superfluid model such as the GP equation. Indeed, this result depends only on the existence of the first time derivative as well as the first and second spacial derivatives of ψ.

II. CONCLUSIONS

This work provides a general framework for the construction of hydrodynamic theories which are capable of correctly including any possible vortex dynamics that may exist in a large set of superfluidity models. By a detailed examination of the role of the multivalued nature of the phase field S in the vortex dynamics, the general hydrodynamic equation (28) was obtained, where all details of a specific model are introduced through the quantity v_0, defined in Eq. (16). Such multivaluedness of S demands the introduction of the gauge field A_μ, where the time-like component $E_i = F_{0i}$ of its force field must be introduced in the hydrodynamic equation (28). The only restriction of this approach is that the equation of motion for the macroscopic wave function ψ must be of first order in time, according to Eq. (4). As a test for the practicality of this approach, the dynamics of 2D point vortices and 3D vortex lines have been considered. It turns out that the numerically observed behavior [31] of point vortices moving over a background density gradient is analytically reproduced in Eq. (41). In addition, the $t^{1/2}$ behavior of creation or annihilation of 2D vortex pairs as well as of 3D vortex line reconnections [32–34] is exactly demonstrated for a large class of superfluid models.

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