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Root-mean-square charge radius of a muonic atom

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In some atomic problems, the two-body muonic ions act as a kind of compound nucleus of a more complicated atomic system (containing a nucleus, a muon, and some electrons). Here we study its root-mean-square charge radius including various relativistic and radiative corrections. The numerical results are given for the muonic ions with the following nuclei: ^{3}He, ^{4}He, ^{12}C, and ^{16}O.

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I. INTRODUCTION

When a muon is captured by an atom, the atomic electrons are kicked out. However, the resulting muonic atoms can have some electrons remaining, partly because some electrons may survive the collision of the initial ordinary atom and the muon and partly because of the subsequent recombination of the electrons. Through a cascade of atomic transitions the muon reaches the ground state. A system with the electrons and with the muon in a low state is very similar to an ordinary [electronic] atom with a compound nucleus, which consists of the "true" nucleus and the muon. The compound nucleus is much larger than the ordinary one. One may wonder what is its size. In the first approximation all the geometrical integrals related to a certain rms radius are proportional to

$$\langle r^2\rangle_{nl} = \int d^3r\,|\psi_{nl}(\mathbf{r})|^2 r^2, \tag{1}$$

where $\psi_{nl}(\mathbf{r})$ is the wave function of a nonrelativistic two-body Coulomb problem for the compound nucleus, i.e., the system of a muon and a true nucleus. Different radii (say, electric or magnetic) have different prefactors. As follows from Eq. (1), the characteristic size of the compound nucleus is of the order of $\hbar/(Z\alpha m_\mu c)$ which is comparable to the Compton wave length of the electron $[\hbar/(Z\alpha m_\mu c) \simeq \hbar/(1.5 Z m_e c)]$.

The normalization of the rms radius depends on what value we are interested in. The value of our particular interest here is the rms electric charge radius, which is important for the development of an accurate theory of a system with the compound nucleus and orbiting electrons. The value of the charge radius is an effective parameter which is important for an accurate theory of the electronic transitions, rather than of the muonic ones. Since the size of the compound nucleus is essentially larger than the size of an ordinary nucleus, the finite-nuclear-size effects are much more important for muonic atoms with electrons than for the related ordinary atoms.

An example of such a system with a bound muon and remaining bound electron(s) is the neutral muonic helium, a three-body system that consists of a nucleus (the α particle for ^{4}He or the helion for ^{3}He), a muon, and an electron. Such a system was successfully created long ago [1] and a transition in the hyperfine structure, caused by the magnetic interaction of the electron and the compound nucleus, was measured [2].

Here we study a general property of such a compound nucleus. The result for the charge radius is of the form

$$\begin{aligned} r^2_{\text{ch:0}} &= \frac{1}{Z-1}\left[-\left(\frac{m_r}{m_\mu}\right)^2 + Z\left(\frac{m_r}{m_N}\right)^2\right]\langle r^2\rangle_{nl} \\ &= -\frac{1}{Z-1}\left[1 - \frac{2m_\mu}{m_N} + (3-Z)\left(\frac{m_\mu}{m_N}\right)^2 + \cdots\right]\langle r^2\rangle_{nl}, \end{aligned} \tag{2}$$

where

$$m_r = \frac{m_\mu m_N}{m_\mu + m_N}$$

is the reduced mass. That is the leading contribution to the rms charge radius of the compound nucleus. The latter is defined here as

$$\int d^3r\,\rho_{\text{ch}}(\mathbf{r})\,r^2,$$

where $\rho_{\text{ch}}(\mathbf{r})$ is the distribution of the charge in the two-body muonic ion. The charge of the compound nucleus is $Z-1$; while the positive charge Z is weakly distributed around the center, the negative charge -1, carried by the muon, creates a relatively large cloud and therefore the muon contribution to $r^2_{\text{ch:0}}$ dominates, which makes the rms charge radius squared negative. The distribution of the central Z charge is due to the fact that the standard value of $\mathbf{r}$ is for the relative motion in the system of the center of the mass. While the position of the muon is $(m_r/m_\mu)\,\mathbf{r}$, the location of the true nucleus is $-(m_r/m_N)\,\mathbf{r}$. The related contributions are the first term (for the muon contribution) and the second term [for the nuclear contribution in the middle part of the identity in Eq. (2)].

One also has to remember that the value of the rms charge radius r^2_{ch} does not reflect the size of the area where the charge is distributed. The latter is $\sqrt{-(Z-1)r^2_{\text{ch}}}$ rather than $\sqrt{-r^2_{\text{ch}}}$.

The results for $\langle r^2\rangle$ in hydrogenlike systems are well known (see, e.g., [3]),

$$\langle r^2\rangle_{nl} = \frac{n^2}{2}\frac{5n^2+1-3l(l+1)}{(Z\alpha m_r)^2}, \tag{3}$$

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TABLE I. Numerical values of the effective rms charge radius of the hydrogenlike muonic ions in the leading order [see Eqs. (2) and (3)].

Atom	$\mu\,^3$He	$\mu\,^4$He	$\mu\,^{12}$C	$\mu\,^{16}$O
$r^2_{\rm ch:0}(1s)$	$-(221.3\ \text{fm})^2$	$-(221.5\ \text{fm})^2$	$-(33.0\ \text{fm})^2$	$-(20.9\ \text{fm})^2$
$r^2_{\rm ch:0}(2s)$	$-(828.1\ \text{fm})^2$	$-(828.6\ \text{fm})^2$	$-(123.6\ \text{fm})^2$	$-(78.3\ \text{fm})^2$
$r^2_{\rm ch:0}(2p)$	$-(699.9\ \text{fm})^2$	$-(700.3\ \text{fm})^2$	$-(104.5\ \text{fm})^2$	$-(66.2\ \text{fm})^2$

and in particular for the lowest states they are

$$\langle r^2\rangle_{1s} = \frac{3}{(Z\alpha m_r)^2},$$
$$\langle r^2\rangle_{2s} = \frac{42}{(Z\alpha m_r)^2},$$
$$\langle r^2\rangle_{2p} = \frac{30}{(Z\alpha m_r)^2},$$

etc. Here and throughout the paper we use the relativistic units in which $\hbar = c = 1$.

For example, in the case of muonic helium with the muon in the ground state the rms charge radius is about 221 fm (slightly depending on the isotope) [4]. The numerical results for some two-body muonic ions are presented in Table I for the lower states.

In this paper we consider contributions to the rms charge radius and many of them can be expressed as certain corrections to the integral (1) [see also Eq. (3)]. At low Z the dominant correction is due to the Uehling potential. The related contribution to $r^2_{\rm ch}$ is of the order of α in fractional units. In the meantime, the relativistic corrections, being of order $(Z\alpha)^2$, should be important at higher Z.

II. UEHLING CORRECTION TO THE RMS CHARGE RADIUS

The nonrelativistic calculation of the Uehling correction to $\langle r^2\rangle_{nl}$ can be performed for the two-body problem, for which it is sufficient just to use the reduced mass. We use the presentation of the Uehling potential in the form of the integral

$$V_U(r) = -\frac{\alpha(Z\alpha)}{\pi}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{1-v^2}\,\frac{e^{-\lambda r}}{r} \tag{4}$$

over the dispersion variable

$$\lambda = \frac{2m_e}{\sqrt{1-v^2}},$$

which is the effective parameter of the spectral function. The Uehling correction to the rms integral in Eq. (3) appears in the second order of the perturbation theory

$$\Delta\langle r^2\rangle_{nl} = 2\langle nl|r^2(-G'_{nl})V_U|nl\rangle, \tag{5}$$

where

$$G'_{nl}(\mathbf{r},\mathbf{r}') = \sum_{i\neq nl}\frac{\psi_i(\mathbf{r})\psi_i^*(\mathbf{r}')}{E_i-E_n} \tag{6}$$

is the reduced Green's function of the nonrelativistic Coulomb problem for a particle with the reduced mass m_r. Green's function is easy to factorize for the angular and radial parts

$$G'_{nl}(\mathbf{r},\mathbf{r}') = \sum_{lm} G'_{nl}(r,r')Y_{lm}(\Omega)Y^*_{lm}(\Omega'), \tag{7}$$

and angular integrations are rather trivial.

In our further evaluations we utilize the so-called Sturmian presentation for the radial component of Green's function $G'_{nl}(r,r')$ [5]

$$G'_{nl}(r,r') = \frac{n^2}{(Z\alpha)^2 m_r}\left\{\sum_{\substack{k=l+1\\k\neq n}}^{\infty}\frac{k}{k-n}\times R_{kl}(n;\ r)R_{kl}(n;\ r') + \frac{1}{2}R_{nl}(n;\ r)R_{nl}(n;\ r') + r\,R'_{nl}(n;\ r)R_{nl}(n;\ r') + r'\,R_{nl}(n;\ r)R'_{nl}(n;\ r')\right\}, \tag{8}$$

where the Sturmian basis functions

$$R_{kl}(n;\ r) = \frac{1}{r}\sqrt{\frac{Z\alpha m_r}{kn}}\sqrt{\frac{(k-l-1)!}{(k+l)!}}\,e^{-Z\alpha m_r r/n}\times\left(\frac{2Z\alpha m_r r}{n}\right)^{l+1} L^{2l+1}_{k-l-1}\left(\frac{2Z\alpha m_r r}{n}\right) \tag{9}$$

satisfy the condition

$$\int_0^\infty R_{pl}(p,r)R_{ql}(q,r)r^2\,dr = \delta_{pq}, \tag{10}$$

$L^q_p(x)$ stands for the associated Laguerre polynomials, and

$$R'_{kl}(n;\ r) = \frac{1}{r}\frac{\partial}{\partial r}[r\,R_{kl}(n;\ r)]. \tag{11}$$

The expression for $R_{kl}(n;\ r)$ is similar to that for the standard hydrogenic function $R_{kl}(k;\ r)$ for the bound kl state, except that the latter is expressed in terms of $L^{2l+1}_{k-l-1}(2Z\alpha m_r r/k)\ \exp(-Z\alpha m_r r/k)$, while the Sturmian basis is built of $L^{2l+1}_{k-l-1}(2Z\alpha m_r r/n)\ \exp(-Z\alpha m_r r/n)$. In other words, the Sturmian presentation of Green's function is the sum over the intermediate states with the running k index [cf. Eq. (6)]. The latter determines the number of the state in the sum, but not the argument of the Laguerre polynomials and exponential factors. The parameter n, which determines the argument, depends only on the energy at which Green's function is calculated and therefore it is the same for the whole basis. In contrast to the sum over all the intermediate states with the hydrogenic functions, which involves the continuous and discrete spectrum, the Sturmian sum is the sum over the discrete spectrum only.

TABLE II. Coefficients $C_{nl}(k)$ and $C'_{nl}(n)$, relevant for the calculation of the Uehling correction to the rms charge radius for the lowest states [in units $(Z\alpha m_r)^{-2}$].

	$1s$	$2s$	$2p$
$C_{nl}(1)$	3	−18	
$C_{nl}(2)$	−9/2	42	30
$C_{nl}(3)$	3	−48	$-15\sqrt{6}$
$C_{nl}(4)$	−3/4	27	$9\sqrt{5}$
$C_{nl}(5)$	0	−6	$-3\sqrt{2}$
$C_{nl}(k), k>5$	0	0	0
$C'_{nl}(1)$	−9/2		
$C'_{nl}(2)$		−63	−45

The Uehling correction to the rms radius in the terms of the Sturmian expansion is of the form

$$\Delta\langle r^2\rangle_{nl} = 2\langle nl|r^2(-G'_{nl})V_U|nl\rangle = -2\frac{n^2}{(Z\alpha)^2 m_r}\left\{\sum_{\substack{k=l+1\\k\neq n}}^{\infty}\frac{k}{k-n}C_{nl}(k)A_{nl}(k) + \frac{1}{2}C_{nl}(n)A_{nl}(n) + C'_{nl}(n)A_{nl}(n) + C_{nl}(n)A'_{nl}(n)\right\}, \tag{12}$$

where

$$A_{nl}(k) = \int_0^\infty R_{kl}(n;\ r)V_U(r)R_{nl}(n;\ r)\, r^2dr,$$
$$A'_{nl}(k) = \int_0^\infty r R'_{kl}(n;\ r)V_U(r)R_{nl}(n;\ r)\, r^2dr, \tag{13}$$
$$C_{nl}(k) = \int_0^\infty R_{kl}(n;\ r)r^2R_{nl}(n;\ r)\, r^2dr,$$
$$C'_{nl}(n) = \int_0^\infty r R'_{nl}(n;\ r)r^2R_{nl}(n;\ r)\, r^2dr. \tag{14}$$

The summation over the running index k is in fact finite. One can see that for $k > n+3$ the integral $C_{nl}(k)$ is equal to zero, which simplifies the calculations and makes a result in closed analytic form possible (see below).

All the values of the coefficients C and C' relevant to calculating the corrections for the $1s$, $2s$, and $2p$ states are presented in Table II.

Applying them to identity (12), we obtain

$$\frac{\Delta\langle r^2\rangle_{1s}}{\langle r^2\rangle_{1s}} = \frac{1}{(Z\alpha)^2 m_r}\bigg[2A_{1s}(1)+6A_{1s}(2)-3A_{1s}(3) + \frac{2}{3}A_{1s}(4) - 2A'_{1s}(1)\bigg],$$

$$\frac{\Delta\langle r^2\rangle_{2s}}{\langle r^2\rangle_{2s}} = \frac{8}{21}\frac{1}{(Z\alpha)^2 m_r}\bigg[-9A_{2s}(1)+21A_{2s}(2) + 72A_{2s}(3)-27A_{2s}(4)+5A_{2s}(5)-21A'_{2s}(2)\bigg],$$

$$\frac{\Delta\langle r^2\rangle_{2p}}{\langle r^2\rangle_{2p}} = \frac{1}{(Z\alpha)^2 m_r}\bigg[8A_{2p}(2)+12\sqrt{6}A_{2p}(3) - \frac{24}{\sqrt{5}}A_{2p}(4) + \frac{4\sqrt{2}}{3}A_{2p}(5) - 8A'_{2p}(2)\bigg]. \tag{15}$$

The integrals A and A' required for the low states, such as

$$A_{1s}(k) = -\frac{\alpha(Z\alpha)}{\pi}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{1-v^2}\,\frac{4(Z\alpha m_r)^3}{\lambda^2} \times\left(\frac{\lambda}{2Z\alpha m_r+\lambda}\right)^{k+1} \tag{16}$$

and

$$A'_{1s}(k) = -\frac{\alpha(Z\alpha)}{\pi}\int_0^1 dv\,\frac{v^2(1-v^2/3)}{1-v^2}\,\frac{2(Z\alpha m_r)^5}{\lambda^4} \times\left(-2(k-1)-2(k-1)\frac{\lambda}{Z\alpha m_r}+\frac{\lambda^2}{(Z\alpha m_r)^2}\right) \times\left(\frac{\lambda}{2Z\alpha m_r+\lambda}\right)^{k+2}, \tag{17}$$

can be expressed in the terms of the base integrals [6,7]

$$K_{abc}(\kappa) = \int_0^1 dv\,\frac{v^{2a}}{(1-v^2)^{b/2}}\left(\frac{\kappa\sqrt{1-v^2}}{1+\kappa\sqrt{1-v^2}}\right)^c, \tag{18}$$

where

$$\kappa = \frac{Z\alpha\, m_r}{m_e}.$$

The base integrals in their turn can be expressed in terms of the hypergeometric functions ${}_3F_2$ [8]. There are also numerous recurrent relations, asymptotic expansions, etc., available [6,7,9–11]. In the case of natural c, which is only required for the nonrelativistic calculations, the base integrals can be expressed in terms of the elementary functions [6,7] (cf. [12]).

For instance, the result for the ground state eventually reads

$$\frac{\Delta\langle r^2\rangle_{1s}}{\langle r^2\rangle_{1s}} = \frac{\alpha}{\pi}\frac{1}{108\kappa^3(\kappa^2-1)^3} \times\bigg[\kappa(152\kappa^8-978\kappa^6+2763\kappa^4 - 2924\kappa^2+1032) - 3(48\kappa^{10}-168\kappa^8+780\kappa^6-1505\kappa^4 + 1204\kappa^2-344) \times\frac{\ln(\kappa+\sqrt{\kappa^2-1})}{\sqrt{\kappa^2-1}} + 516\pi(\kappa^2-1)^3\bigg]. \tag{19}$$

The numerical results for the Uehling correction for the low states in some muonic ions are listed in Table III (see also the $\Delta r^2_{\rm ch:Ueh}$ contribution in the summary table).

TABLE III. Uehling correction to the rms radius for the low states in some hydrogenlike muonic ions in the units $\alpha/\pi\,\langle r^2\rangle_{nl}$.

Atom	$\mu\,{}^3$He	$\mu\,{}^4$He	$\mu\,{}^{12}$C	$\mu\,{}^{16}$O
$1s$	− 1.184 61	− 1.193 53	− 2.482 92	− 2.853 15
$2s$	− 0.621 487	− 0.626 485	− 1.489 64	− 1.791 47
$2p$	− 0.149 915	− 0.152 607	− 0.823 521	− 1.104 63

In the case of large κ one can find some useful asymptotic expressions; e.g., for the ground state such an expression is of the form

$$\frac{\Delta\langle r^2\rangle_{1s}}{\langle r^2\rangle_{1s}} = \frac{\alpha}{\pi}\left[-\frac{4}{3}\ln\kappa + \frac{2}{3}C + \frac{56}{27} + \psi\left(\frac{1}{2}\right) - \frac{1}{3}\psi\left(\frac{3}{2}\right) + O\left(\frac{\ln\kappa}{\kappa^2}\right)\right], \tag{20}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ and $C = -\psi(1)$ is Euler's constant. The accuracy in terms of the orders of the expansion is shown in the identity above. In the meanwhile, numerically the asymptotic relation (20) yields −2.44173 and −2.82843 for the $1s$ state in μ^{12}C and μ^{16}O, correspondingly, which pretty well agrees with the exact values (cf. Table III).

III. RELATIVISTIC CORRECTIONS

The result in Eqs. (2) and (3) corresponds to a nonrelativistic calculation with the reduced mass. One can easily find the relativistic contributions in the external field approximation

$$\frac{1}{Z-1}\int d^3r|\Psi_{nlj}(\mathbf{r})|^2\left[Z\left(\frac{m_r}{m_N}\mathbf{r}\right)^2 - \left(\frac{m_r}{m_\mu}\mathbf{r}\right)^2\right] = r^2_{\rm ch:0}(nl) + \Delta r^2_{\rm ch:rel}(nlj), \tag{21}$$

where $\Psi_{nlj}(\mathbf{r})$ are the eigenfunctions of the Dirac-Coulomb equation. Those corrections dominate at high and medium Z. For this expression one may use the same normalization as in Eq. (2) or, alternatively, just put $m_r/m_N = 0$, $m_r/m_\mu = 1$ into Eq. (21). In both cases the further higher-order correction is of the order $(Z\alpha)^2 m_\mu/m_N$ and its complete evaluation requires a separate consideration. We note that this uncalculated correction is always small in contrast to the relativistic correction of the order $(Z\alpha)^2$ which is not small at high Z. The uncalculated relativistic-recoil correction roughly is of order $\alpha(Z\alpha)m_\mu/m_p$, since in the standard atoms $Z/A = O(1)$.

The relativistic correction is found to be

$$\Delta r^2_{\rm ch:rel}(nlj) = (Z\alpha)^2\, r^2_{\rm ch:0}(nl) \times[a_{nl_j} + O(m_\mu/m_N)], \tag{22}$$

where

$$a_{1s} = -\frac{7}{12},\quad a_{2s} = -\frac{113}{168},\quad a_{2p_{1/2}} = -\frac{19}{24},\quad a_{2p_{3/2}} = -\frac{11}{60}. \tag{23}$$

Note that it is not a complete account of the relativistic effects in the order $(Z\alpha)^2$. The correction above considers a nonrelativistic interaction of the muon with the external electromagnetic field, while the relativistic effects affect its distribution within the compound nucleus. The other relativistic correction is due to the relativistic interaction of the muon with the external electric field, related to the Darwin-Foldy term. It does not depend on the state [13]

$$\Delta r^2_{\rm ch:DF} = -\frac{3}{4}\frac{1}{Z-1}\frac{1}{m_\mu^2} \tag{24}$$

being of the same order in $Z\alpha$ as the relativistic correction above. The effect is similar to the presence of a nonzero charge radius of a constituent. For example, the nuclear charge radius produces a correction

$$\Delta r^2_{\rm ch:fns0} = \frac{Z}{Z-1}\, r_N^2. \tag{25}$$

These two contributions have different signs because of the different signs of the related charges. Their numerical values for the ground state in some two-body systems can be found in Table IV. The numerical values of the nuclear radii applied there are 1.973(16) fm (for ^{3}He), 1.681(4) fm (for ^{4}He) [14], 2.4702(22) fm (for ^{12}C), and 2.6991(52) fm (for ^{16}O) [15].

The corrections (24) and (25) look like the leading contributions of their kind. However, they are not the only leading contributions of the related orders.

Acting in the same manner as for the Uehling potential in the previous section [cf. Eq. (5)], we could calculate the perturbation to the radius of the compound system induced by

TABLE IV. Contributions to the mean-square charge radius of the $1s$ state of the compound nuclei. The uncertainty is due to the uncertainty in the determination of the nuclear charge radius [14,15] and due to the estimation of the recoil corrections to the relativistic term.

Atom	$\mu\,{}^3$He	$\mu\,{}^4$He	$\mu\,{}^{12}$C	$\mu\,{}^{16}$O
$r^2_{\rm ch:0}$	−48985.2 fm^2	−49045.3 fm^2	−1091.06 fm^2	−438.433 fm^2
$\Delta r^2_{\rm ch:Ueh}$	−145.5 fm^2	−144.0 fm^2	−32.08 fm^2	−20.637 fm^2
$\Delta r^2_{\rm ch:rel}$	6.1(2) fm^2	6.1(2) fm^2	1.22(1) fm^2	0.872(6) fm^2
$\Delta r^2_{\rm ch:DF}$	−2.6 fm^2	−2.6 fm^2	−0.52 fm^2	−0.374 fm^2
$\Delta r^2_{\rm ch:fns0}$	7.8(1) fm^2	5.65(3) fm^2	7.32(1) fm^2	8.33(3) fm^2
$\Delta r^2_{\rm ch:fns}$	−28.5(5) fm^2	−20.7(1) fm^2	−8.95(2) fm^2	−7.63(3) fm^2
$r^2_{\rm ch:tot}$	−49148.0(6) fm^2	−49200.9(2) fm^2	−1124.07(3) fm^2	−457.878(6) fm^2
$\sqrt{-r^2_{\rm ch:tot}}$	221.693(1) fm	221.813 fm	33.527 fm	21.398 fm

the Darwin-Foldy correction

$$-\frac{2}{Z-1}\langle nl|r^2(-G'_{nl})V_{\rm DF}|nl\rangle, \tag{26}$$

where

$$V_{\rm DF}(r) = \mathcal{C}_{\rm DF}\delta(\mathbf{r}), \tag{27}$$

$$\mathcal{C}_{\rm DF} = -\frac{\pi\alpha}{2m_\mu^2}. \tag{28}$$

However, such a correction has already been taken into account, because it is a part of the relativistic correction.

We similarly find the additional correction due to the finite nuclear size. The related potential is of the form

$$V_{\rm fns}(r) = \mathcal{C}_{\rm fns}\delta(\mathbf{r}), \tag{29}$$

where

$$\mathcal{C}_{\rm fns} = \frac{2\pi Z\alpha r_N^2}{3}. \tag{30}$$

The perturbations are given by the expressions similar to Eq. (15) with coefficients A_{nl} and A'_{nl} replaced by their analogs for the δ-function-like potential [Eq. (27)]

$$D_{nl}(k) = \frac{\mathcal{C}_{\rm fns}}{4\pi} R_{kl}(n;\,0)R_{nl}(n;\,0),$$
$$D'_{nl}(k) = \frac{\mathcal{C}_{\rm fns}}{4\pi} \lim_{r\to 0}[r R'_{kl}(n;\,0)R_{nl}(n;\,0)]. \tag{31}$$

One can see that they do not depend on k:

$$D_{nl}(k) = D'_{nl}(k) = \frac{2(Z\alpha)^4 m_r^3 r_N^2}{3n^3}\,\delta_{l0}. \tag{32}$$

Eventually, we arrive at

$$\frac{\Delta r^2_{\rm ch:fns}(1s)}{\langle r^2\rangle_{1s}} = -\frac{22}{9}\,\frac{(Z\alpha m_r)^2 r_N^2}{Z-1},$$
$$\frac{\Delta r^2_{\rm ch:fns}(2s)}{\langle r^2\rangle_{2s}} = -\frac{82}{63}\,\frac{(Z\alpha m_r)^2 r_N^2}{Z-1}. \tag{33}$$

All the numerical results for the $1s$ state in a few muonic ions are summarized in Table IV. As we mentioned, the largest uncalculated contributions are from the recoil corrections to the relativistic term $\Delta r^2_{\rm ch:rel}$ and they dominate in the uncertainty together with the uncertainty in the determination of the rms nuclear charge radius [14,15].

IV. CONCLUSION

We have studied the rms charge radius of hydrogenlike muonic ions. Such an ion serves as a compound nucleus in the systems which contain a nucleus, a muon, and an electron or a few of them. The results are obtained in the closed analytic form using the nonrelativistic expansion. The numerical results are presented for muonic ^{3}He, ^{4}He, ^{12}C, and ^{16}O. We have studied the Uehling potential, relativistic, and nuclear-finite-size contributions. Some of the corrections reached the few-percent level for ^{16}O. The uncertainty of our calculation is limited by the uncertainty of one of the input parameters, namely, the value of the related rms nuclear charge radius, and by the estimation of the uncalculated terms of the higher orders.

The obtained results may be applied to atomic systems with a nucleus, a muon, and a few electrons, while considering the muon-nucleus two-body system as a compound nucleus surrounded by electrons. In such a case the leading nuclear-finite-size correction is expressed in the terms of the mean-square charge radius of the [compound] nucleus, which is found above. The related coefficient for the contribution in simple atoms can be calculated, while for many-electron systems it may be found by applying the King formula [16] and matching with the available experimental results on ordinary atoms. In the meantime, the higher-order nuclear-structure contributions require the values of other parameters of the compound nucleus.

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